MATH7016: 20% Written Assessment 1 SOLUTIONS

Name:

1. Consider the displacement, $x(t)$ (in metres), of a body of mass m (in kg), after t seconds, subject to two forces, gravity, and a drag force, proportional to the velocity. Formulate this problem as a second order initial value problem. Assume that the initial displacement and initial velocity are zero.

[6 Marks]

Solution: This is simply Newton's Second Law $F = ma$ where we use the fact that $a = x''$. We also have to give the initial conditions. If we introduce v we should declare that $v = x'$.

$$m \cdot \frac{d^2x}{dt^2} = mg - \alpha\frac{dx}{dt}, \qquad x(0) = 0, x'(0) = 0.$$

Some students had $-mg$ and that is OK. The above assumes down as positive. Down as negative works too, and gives $-mg$.

2. Calculate the first two non-zero terms of the Maclaurin Series of $y(t) = \sin t$.

[4 Marks]

Solution: Using:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots,$$

we must calculate derivatives and evaluate them at $t = 0$:

$$\begin{aligned} y(0) &= \sin 0 = 0 \\ \implies y'(t) &= \cos t|_{t=0} = 1 \\ \implies y''(t) &= -\sin t|_{t=0} = 0 \\ \implies y'''(t) &= -\cos t|_{t=0} = -1 \\ \implies y(t) &= 0 + 1t + \frac{0}{2}t^2 + \frac{-1}{3!}t^3 + \cdots \\ \implies \sin t &\approx t - \frac{t^3}{6}. \end{aligned}$$

Some students lost a mark because they had the independent variable x rather than t.

3. Consider an initial value problem with solution $y(t)$:

$$\frac{dy}{dt} = F(t, y), \qquad y(t_0) = y_0 \tag{1}$$

(a) Give one example or scenario where it would be necessary to use a *numerical method* in order to approximate $y(t)$.

(b) *Taylor Methods* assume that the solution has a infinite series representation near $t = t_0$:

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \frac{y'''(t_0)}{3!}(t - t_0)^3 + \cdots$$

and uses a *truncation*, only finitely terms of the infinite series, to approximate $y(t)$. For example, the *Three-Term-Taylor Method* uses:

$$y(t) \approx y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2.$$

What is the main disadvantage of using Taylor Methods such as the Three-Term-Taylor Method to solve (1) over a Runge–Kutta Method such as Heun's Method?

(c) Increasing the *order* of Taylor Methods means increasing the number of derivatives used. For example, the Euler Method is a Two-Term-Taylor Method, and it just uses just the first derivative, while the Three-Term-Taylor Method uses the first derivative *and* the second derivative. Increasing the order reduces the error. Runge–Kutta doesn't increase the number of derivatives used. Complete the following:

Increasing the order of Runge–Kutta methods means increasing the number of ________ *used.*

[HINT: There are two reasonable answers. One will do.]

[6 Marks]

Solution:

(a) Two reasonable answers here:
   i. if $F(t, y)$ does not have an elementary antiderivative,
   ii. if $F(t, y)$ is given by data.

(b) With Taylor Methods you have to calculate higher order derivatives, and if $F(t, y)$ depends on y, this requires implicit differentiation.

(c) Answers that were good for full marks: slopes/points/derivatives/lines. Answers that were good for half marks: predictors/steps/calculations/terms/variables.

4. Consider the equation of motion for a *hard* spring:

$$\frac{d^2x}{dt^2} + 0.3\frac{dx}{dt} + x + x^3 = 0, \qquad x(0) = 0,\, x'(0) = 1.$$

Use Euler's Method with a step-size of $h = 0.5$ to approximate $x(1.5)$. Use five significant figures for all calculations.

[16 Marks]

Solution: Students who didn't write this as two first order differential equations got no marks. Whatever you do if you don't do this makes no sense. Let

$$\frac{dx}{dt} =: v \implies \frac{d^2x}{dt^2} = \frac{dv}{dt} \implies \frac{dv}{dt} + 0.3v + x + x^3 = 0,$$

So

$$\begin{cases} \dfrac{dx}{dt} = v, & x(0) = 0 \\ \dfrac{dv}{dt} = -0.3v - x - x^3, & v(0) = 0. \end{cases}$$

Off we go:

$$\begin{aligned} x_1 &= 0 + 0.5[1] = 0.5 \\ v_1 &= 1 + 0.5[-0.3(1) - 0 - 0^3] = 0.85 \\ x_2 &= 0.5 + 0.5[0.85] = 0.925 \\ v_2 &= 0.85 + 0.5[-0.3(0.85) - 0.5 - 0.5^3] = 0.41 \\ x_3 & 0.925 + 0.5[0.41] = 1.13. \end{aligned}$$

Note $x_3 \approx x(1.5)$.

5. Consider an initial value problem

$$\frac{dy}{dx} = F(x)\,; \;\; y(0) = 0.$$

Suppose Euler's Method is to be used to approximate $y(10)$.

(a) Complete the following:

An Euler Method approximation $y_1 \approx y(x_0 + h)$ has large errors when the ____ changes a lot between x_0 and x_1. In other words when the second ____________ is ____

(b) By reducing the step-size, the error in the Euler Method approximation can be reduced. What is the disadvantage of doing this?

(c) The local error in the Euler Method approximation is $\mathcal{O}(h^2)$, which means that for each local error ε_i^{L} there is a constant k_i such that:

$$|\varepsilon_i^{\text{L}}| \leq k_i h^2.$$

Hence show that if we use the Euler Method to approximate $y(10) = y(0 + n \cdot h)$, that the *global error*, $\varepsilon^{\text{G}} = |y(10) - y_n|$, is $\mathcal{O}(h)$.

(d) What is the effect on the global error if we quarter the step-size?

(e) Given the issue with Euler's Method that you identified in part (a), what does Heun's Method take into account to improve on Euler's Method?

(f) Given that Heun's Method takes something into account that Euler's Method does not, how does Heun's Method do this?

[14 Marks]

Solution:

(a) *An Euler Method approximation $y_1 \approx y(x_0 + h)$ has large errors when the slope changes a lot between x_0 and x_1. In other words when the second derivative is large.*

(b) More calculations/longer running time.

(c) *Some students had L, R, T variables without defining what they were. Others did define them, e.g. $L = x_n - x_0$. However, in this case $x_n = 10$ and $x_0 = 0$ and so this $L = 10$.* In the below, $k := \max_i k_i$ and we use the fact that $10 = nh \implies n = 10/h$:

$$\begin{aligned} |\varepsilon^G| &= |\varepsilon_1^L + \varepsilon_2^L + \cdots + \varepsilon_n^L| \\ &\leq |\varepsilon_1^L| + |\varepsilon_2^L| + \cdots + |\varepsilon_n^L| \\ &\leq k_1 h^2 + k_2 h^2 + \cdots + k_n h^2 \\ &\leq \underbrace{kh^2 + kh^2 + \cdots + kh^2}_{n \text{ terms}} = nkh^2 \\ \implies |\varepsilon^G| &\leq \frac{10}{h} \cdot k \cdot h^2 = 10k \cdot h^2, \end{aligned}$$

that is ε^G, the global error, is order h.

(d) It is also quartered. *Why? Because it is order h, $\varepsilon^G \sim kh$. If you change $h \to h/4$ what happens to the global error is $kh \to kh/4$, that is it quarters. The* local *error gets decreased by a factor of 16 because it is order h^2. That is $\varepsilon^L \sim kh^2$ and so changing $h \to h/4$, $kh^2 \to k(h/4)^2 = kh^2/16$.*

(e) It takes into account that the slope changes between x_i and x_{i+1}.

(f) It considers the slope at the next point. It does this with the use of an Euler predictor y_{i+1}^p.

6. Consider the initial value problem:

$$\frac{dy}{dx} = x^2 \cdot y + e^{-y}\,; \qquad y(0) = 0.$$

Use Heun's Method with a step-size of 0.1 to approximate $y(0.2)$. Use five significant figures for all calculations.

[14 Marks]

Solution: Off we go:

$$y_1^p = 0 + 0.1[0^2 \cdot 0 + e^{-0}] = 0.1$$
$$y_1 = 0 + 0.1\frac{[0^2 \cdot 0 + e^{-0} + 0.1^2 \cdot 0.1 + e^{-0.1}]}{2} \approx 0.095292$$
$$y_2^p = 0.095292 + 0.1[0.1^2 \cdot 0.095292 + e^{-0.095292}] \approx 0.18630$$
$$y_2 = 0.095282 + 0.1\frac{[0.1^2 \cdot 0.095292 + e^{-0.95292} + 0.2^2 \cdot 0.18630 + e^{-0.18630}]}{2} \approx 0.18267$$

And $y_2 \approx y(0.2)$. *A number of students weren't careful with previous/current vs predicted/next x.*