Another look at idempotent states (on quantum permutation groups)

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Quantum Groups & Interactions, Glasgow, May 2023

Questions

1. Of philosophy:

   What is a quantum permutation? Is there an intuition for quantum permutations?

2. Of random walk theory:

   What are (necessary and sufficient) conditions on the support projection of a state φ on $C(\mathbb{G})$ for:

   $$\varphi^{\star k} \to h?$$

3. Of quantum permutation groups:

   Is the classical permutation group a maximal quantum subgroup of the quantum permutation group?

The quantum permutation group

Definition (Wang)

$$C(S_N^+) := \mathrm{C}^*(u_{ij} : \; u \text{ an } N \times N \text{ magic unitary}),$$
$$\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}.$$

If $v \in M_N(C(\mathbb{G}))$ is a magic fundamental representation:

$$\pi : C(S_N^+) \to C(\mathbb{G}) \implies \mathbb{G} \subseteq S_N^+.$$

Let $\mathbb{1}_{j\to i}(\sigma) = \delta_{i,\sigma(j)}$: the entries of magic representation

$$v = \left(\mathbb{1}_{j\to i}\right)_{i,j=1}^{N}$$

generate $C(S_N)$, and so $S_N \subseteq S_N^+$.

The Gelfand–Birkhoff picture

Three interpretations for quantum permutations:

1. The Gelfand Picture — S_N^+ is a virtual object, the 'abstract spectrum' of $C(S_N^+)$; quantum permutations don't exist.
2. The QIT Picture — let $X = (V, E)$ be a finite graph:

$$ud_X = d_X u \implies \text{“}u \in G^+(X)\text{”}.$$

3. Gelfand–Birkhoff Picture [M2] —
   - $\varphi \in S_N^+ \iff$ state φ on $C(S_N^+)$ (and therefore $S_N^+ = \mathcal{S}(C(S_N^+))$),
   - $\mathbb{P}[\varphi(j) = i] := \varphi(u_{ij})$, the matrix of which is the *Birkhoff* slice $\Phi(\varphi)$,
   - wave-function collapse $\varphi \mapsto \widetilde{q}\varphi$ follows measurement:

$$\widetilde{q}\varphi(f) = \frac{\omega_\varphi(qfq)}{\omega_\varphi(q)} \qquad (f \in C(S_N^+),\ q \in \mathcal{P}(C(S_N^+)^{**})).$$

Van Daele's Haar existence proof

Proof. ($\mathbb{G} = \mathcal{S}(C(\mathbb{G}))$).

1. Let $\varphi \in \mathbb{G}$: as $\mathbb{G}$ is a non-empty, convex, weak*-compact, and closed under convolution the Cesàro averages of $\{\varphi, \varphi^{\star 2}, \ldots, \varphi^{\star n}\}$ have a limit point ϕ_φ such that $\phi_\varphi \star \varphi = \phi_\varphi = \varphi \star \phi_\varphi$.
2. For each positive linear functional ω on $C(\mathbb{G})$ define:

$$K_\omega = \{\varphi \in \mathbb{G} : \omega \star \varphi = \omega(1_{C(\mathbb{G})})\varphi\}.$$

3. Assume the intersection of all the K_ω is empty.
4. Then the union of the complements $K_\omega^c := \mathbb{G} \backslash K_\omega$ is $\mathbb{G}$.
5. Compactness of $\mathbb{G}$ gives a finite subcover $\{K_{\omega_i}^c\}_{i=1}^n$ of $\mathbb{G}$ and thus the intersection of the K_{ω_i} is empty.
6. But Van Daele showed $K_{\omega_1 + \cdots + \omega_n}$ is in this intersection.
7. The intersection of all the K_ω is non-empty.

Pal sets

Definition

A *Pal set* is a non-empty convex weak*-closed subset $\mathbb{S} \subseteq \mathbb{G}$ that is closed under convolution.

Theorem

A Pal set $\mathbb{S} \subseteq \mathbb{G}$ contains a unique state $\phi_{\mathbb{S}} \in \mathbb{S}$ such that for all $\varphi \in \mathbb{S}$:

$$\phi_{\mathbb{S}} \star \varphi = \phi_{\mathbb{S}} = \varphi \star \phi_{\mathbb{S}}.$$

.

Proof.

This is exactly Van Daele's Haar existence proof, except rather than defining a K_ω for each positive linear functional ω on $C(\mathbb{G})$, they are defined only for each $\omega \in \text{cone}(\mathbb{S})$. □

Non-injectivity

Definition

Let $\mathbb{G}$ be a compact quantum group. Where:

$$\{u_{ij}^{\alpha} : i, j = 1, \ldots, d_{\alpha}, \alpha \in \text{Irr}(\mathbb{G})\}$$

are matrix coefficients of mutually inequivalent irreducible unitary representations, a *central state* $\varphi \in \mathbb{G}$ is one such that for all $\alpha \in \text{Irr}(\mathbb{G})$ there exists $\varphi(\alpha) \in \mathbb{C}$ such that:

$$\varphi(u_{ij}^{\alpha}) = \varphi(\alpha)\delta_{i,j}.$$

Proposition

Let $\mathbb{G}$ be a quantum group. The set of central states $\mathbb{G}_0$ is a Pal set.

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Non-injectivity & fix

But the $\mathbb{G}_0$-invariant idempotent is the Haar state! That is there are multiple Pal sets giving the same idempotent; e.g.

$$\{h\} \subset \mathbb{G}_0 \subseteq \mathbb{G}.$$

The largest Pal set $\mathbb{S} \subseteq \mathbb{G}$ containing $\mathbb{S}$-invariant idempotent ϕ is:

$$\mathbb{S}_\phi := \{\varphi \in \mathbb{G} : \varphi \star \phi = \phi = \phi \star \varphi\}.$$

Definition

A *quasi-subgroup* is a subset of the state space of the form $\mathbb{S}_\phi$ for an idempotent state ϕ on $C(\mathbb{G})$; the *quasi-subgroup generated by* ϕ.

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Example: Stabiliser quasi-subgroups

Given a finite group $G \subseteq S_N$ and a partition $\mathcal{P} = B_1 \sqcup \cdots \sqcup B_k$ of $\{1, \ldots, N\}$:

$$G_{\mathcal{P}} = \{\sigma \in G : \sigma(B_p) = B_p,\ 1 \leq p \leq k\}.$$

Define, for $\mathbb{G} \subseteq S_N^+$:

$$\varphi \in \mathbb{G}_{\mathcal{P}} \iff \Phi(\varphi) = \begin{bmatrix} \Phi_{B_1}(\varphi) & 0 & \cdots & 0 \\ 0 & \Phi_{B_2}(\varphi) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ldots \\ 0 & 0 & \cdots & \Phi_{B_k}(\varphi) \end{bmatrix},$$

where $\Phi_{B_p}(\varphi) = [\varphi(u_{ij})]_{i,j \in B_p}$.

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For any partition $\mathcal{P}$ of $\{1, \ldots, N\}$, $\mathbb{G}_{\mathcal{P}}$ is a quasi-subgroup.

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The bidual

Let ω_φ be extension of $\varphi \in S_N^+$ to a state on $C(S_N^+)^{**}$:

$$\mathfrak{N}_\varphi := \{f \in C(S_N^+)^{**} : \omega_\varphi(|f|^2) = 0\} \implies \mathfrak{N}_\varphi = C(S_N^+)^{**} q_\varphi.$$

Definition

The *support projection* of $\varphi \in S_N^+$ is $p_\varphi := \mathbb{1}_{S_N^+} - q_\varphi$.

Where $\Delta^{**} : C(\mathbb{G})^{**} \to (C(\mathbb{G}) \otimes C(\mathbb{G}))^{**}$:

Definition

A group-like projection $p \in C(\mathbb{G})^{**}$ is a non-zero projection such that:

$$\Delta^{**}(p)(\mathbb{1}_{\mathbb{G}} \otimes p) = p \otimes p.$$

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Group-like projections

Proposition/Theorem/Corollary

1. *If* $\varphi_1, \varphi_2 \in \mathbb{G}$ *are supported on a group-like projection* $p \in C(\mathbb{G})^{**}$*, then so is* $\varphi_1 \star \varphi_2$*.*
2. *Suppose that an idempotent state* $\phi \in \mathbb{G}$ *has group-like support projection* $p \in C(\mathbb{G})^{**}$*. Then the quasi-subgroup generated by* ϕ*:*

$$\mathbb{S}_\phi \subseteq \{\varphi \in \mathbb{G} : \, \omega_\varphi(p) = 1\}.$$

3. *Suppose* $\mathbb{G}$ *is non-coamenable. Then the support projection* $p_h \in C(\mathbb{G})^{**}$ *of the Haar state is not a group-like projection. Furthermore:*

$$\{\varphi \in \mathbb{G} : \, \omega_\varphi(p_h) = 1\} \subsetneq \mathbb{S}_h.$$

Intermediate quasi-subgroups

Is there an *exotic* intermediate quasi-subgroup:

$$S_N \subsetneq \mathbb{S}_N \subsetneq S_N^+?$$

Can associate to each 'genuinely quantum' permutation φ:

$$\varphi|_{J_{\text{comm}}} \neq 0 \quad \rightsquigarrow \quad S_N \subsetneq \mathbb{S} \subseteq S_N^+.$$

Associated idempotent ϕ either:

1. a non-Haar idempotent; or,
2. the Haar idempotent from an exotic quantum subgroup $S_N \subsetneq \mathbb{G}_N \subsetneq S_N^+$ $(N \geq 6)$; or
3. the Haar state on $C(S_N^+)$.

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Supports of characters

Recall $\Phi(\varphi) = [\varphi(u_{ij})]_{i,j=1}^{N}$. In the below, $\sigma, \tau \in S_N$:

Proposition

1. $\varphi \in \hom(C(S_N^+), \mathbb{C}) \iff \Phi(\varphi) = P_\sigma \iff \varphi = \text{ev}_\sigma$;
   ($\text{ev}_\sigma := \delta^\sigma \circ \pi_{ab}$, *where* $\pi_{ab}(u_{ij}) = \mathbb{1}_{j \to i}$),
2. *support of* ev_σ, p_σ, *is central and* $p_\sigma p_\tau = \delta_{\sigma,\tau} p_\sigma$,
3. $p_\sigma = u_{\sigma(1),1} \wedge u_{\sigma(2),2} \wedge \cdots \wedge u_{\sigma(N),N}$,
4. $\Delta^{**}(p_\sigma)(\mathbb{1}_{S_N^+} \otimes p_\tau) = p_{\sigma\tau^{-1}} \otimes p_\tau$.

Theorem

Let $p_{S_N} = \sum_{\sigma \in S_N} p_\sigma$. *Then:*

$$\Delta^{**}(p_{S_N})(\mathbb{1}_{S_N^+} \otimes p_{S_N}) = p_{S_N} \otimes p_{S_N}.$$

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Exotic quasi-subgroups

Corollary

Suppose h_{S_N} is the state on $C(S_N^+)$ defined by $h_{C(S_N)} \circ \pi_{ab}$. Then

$$\varphi \star h_{S_N} = h_{S_N} = h_{S_N} \star \varphi \implies \omega_\varphi(p_{S_N}) = 1.$$

Theorem

Let $\varphi \in S_N^+$ be genuinely quantum, $\omega_\varphi(p_{S_N}) < 1$. Form the idempotent ϕ_φ from the Cesàro means of φ, and then define:

$$\phi := w^*\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (h_{S_N} \star \phi_\varphi)^{\star k}.$$

Then $S_N \subsetneq \mathbb{S}_\phi \subseteq S_N^+$.

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Epilogue

This question brings to the fore something that is fundamental and pervasive: that what we [mathematicians] are doing is finding ways for people to understand and think about mathematics.

William Thurston

Theorem

Suppose that ϕ is an idempotent state on $C(\mathbb{G})$.

(i) *If ϕ is a (universal) Haar idempotent (i.e. from a quantum subgroup $\mathbb{H} \subseteq \mathbb{G}$), then $\mathbb{S}_\phi$ is closed under wave-function collapse.*

(ii) *If ϕ is a non-Haar idempotent with group-like support projection, then $\mathbb{S}_\phi$ is not closed under wave-function collapse.*

Moral: quantum subgroups are the quasi-subgroups that are closed under wave-function collapse.

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Epilogue: Closed under wave-function collapse

Quantum group fixing bottom card:

$$u = \operatorname{diag}(u^{S_{N-1}^+}, \mathbb{1}_{S_{N-1}^+}) \quad \Longrightarrow \quad S_{N-1}^+ \subset S_N^+,$$

with associated idempotent $h_{N\to N} := h_{C(S_{N-1}^+)} \circ \pi$.

If $\varphi \star h_{N\to N} = h_{N\to N} = h_{N\to N} \star \varphi$:

1. $\varphi = \varphi_0 \circ \pi$,
2. $\varphi(u_{NN}) = 1$,
3. if $q \in \mathcal{P}(C(S_N^+)^{**})$ such that $\omega_\varphi(q) > 0$,

$$\widetilde{q}\varphi(u_{NN}) = \frac{\omega_\varphi(q\, u_{NN}\, q)}{\omega_\varphi(u_{NN})} = 1.$$

4. $\widetilde{q}\varphi = \psi \circ \pi$ and $\widetilde{q}\varphi \star h_{N\to N} = h_{N\to N} = h_{N\to N} \star \widetilde{q}\varphi$.

Epilogue: Closed under wave-function collapse

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Epilogue: Not closed under wave-function collapse

Quasi-subgroup fixing bottom card:

$$\mathbb{S}_{N\to N} := \{\varphi \in S_N^+ : \varphi(u_{NN}) = 1\},$$

with associated idempotent

$$\phi_{N\to N} := \frac{h(u_{NN} \cdot u_{NN})}{h(u_{NN})}.$$

But $\mathbb{S}_{N\to N}$ is not closed under wave-function collapse:

$$\widetilde{u_{11}}\phi_{N\to N}(u_{NN}) = \frac{\phi_{N\to N}(u_{11}u_{NN}u_{11})}{\phi_{N\to N}(u_{11})} < 1$$

In fact:

$$S_{N-1}^+ \subsetneq \mathbb{S}_{N\to N} \subsetneq S_N^+.$$

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Random, mixed, & truly quantum permutations

Definition

Let $p_Q := \mathbb{1}_{S_N^+} - p_{S_N}$. Say that $\varphi \in S_N^+$

1. is a *(classically) random permutation* if $\omega_\varphi(p_Q) = 0$,
2. is a *mixed quantum permutation* if $0 < \omega_\varphi(p_Q) < 1$,
3. is a *truly quantum permutation* if $\omega_\varphi(p_Q) = 1$.

If φ is mixed:

$$\varphi = \omega_\varphi(p_{S_N}) \frac{\omega_\varphi(p_{S_N} \cdot p_{S_N})}{\omega_\varphi(p_{S_N})} + \omega_\varphi(p_Q) \frac{\omega_\varphi(p_Q \cdot p_Q)}{\omega_\varphi(p_Q)}.$$

$\star$	r	m	tq
r	r	m	tq
m	m	m	$\neg r$
tq	tq	$\neg r$	-

The Haar state is truly quantum

Theorem

The Haar state is truly quantum.

Proof.

1. Assume $\omega_h(p_{S_N}) > 0$. Then $\omega_h(p_\sigma) > 0$ for $\sigma \in S_N$. Suppose σ has λ fixed points.
2. Where $\text{fix} := \text{Tr}\, u$, it follows that

$$\mathbb{P}[h \text{ has } \lambda \text{ fixed points}] := \omega_h(\mathbb{1}_{\{\lambda\}}(\text{fix})) \geq \omega_h(p_\sigma) > 0.$$

3. But for any $\lambda \in [0, N]$:

$$\mathbb{P}[h \text{ has } \lambda \text{ fixed points}] = \int_{\{\lambda\}} (2\pi)^{-1} \sqrt{4t^{-1} - 1}\, dt = 0.$$

□

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Truly quantum permutations are wild

Let G_0 be the Kac–Paljutkin quantum group with algebra of functions

$$C(G_0) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3 \oplus \mathbb{C}f_4 \oplus M_2(\mathbb{C}),$$

$$u^{G_0} := \begin{bmatrix} f_1+f_2 & f_3+f_4 & p & I_2-p \\ f_3+f_4 & f_1+f_2 & I_2-p & p \\ p^T & I_2-p^T & f_1+f_3 & f_2+f_4 \\ I_2-p^T & p^T & f_2+f_4 & f_1+f_3 \end{bmatrix},$$

$$\implies G_0 \subset S_N^+ \qquad (N \geq 4).$$

$$\omega_{E^{11}\circ\pi_{G_0}}(p_{S_N}) = 0 \implies E^{11}\circ\pi_{G_0} \text{ is truly quantum},$$

As Birkhoff slice is multiplicative, $\Phi(\varphi_1 \star \varphi_2) = \Phi(\varphi_1)\Phi(\varphi_2)$.

$$(\omega_{E^{11}\circ\pi_{G_0}})^{\star 2}(p_{S_N}) = 1 \implies (E^{11}\circ\pi_{G_0})^{\star 2} \text{ is random}.$$

Consider for $0 \leq c \leq 1$:

$$\varphi := \sqrt{1-c}\,(E^{11}\circ\pi_{G_0}) + (1-\sqrt{1-c})\,h \implies \omega_{\varphi^{\star 2}}(p_Q) = c.$$

Dynamics

Definition

A quantum permutation $\varphi \in S_N^+$ is called α-quantum if $\omega_\varphi(p_Q) = \alpha$.

Proposition

If $\varphi \in S_N^+$ is α-quantum and $\rho \in S_N^+$ is β-quantum, then

$$\alpha + \beta - 2\alpha\beta \leq \omega_{\varphi\star\rho}(p_Q) \leq \alpha + \beta - \alpha\beta.$$

This generalises to other quantum subgroups if $\mathbb{G} \subseteq S_N^+$ is such that:

1. $h_{\mathbb{G}} := h_{C(\mathbb{G})} \circ \pi_{C(\mathbb{G})}$ has group-like support projection $p_{\mathbb{G}}$,
2. exists quantum permutations φ_1, φ_2 supported "off" $\mathbb{G}$ $(\omega_{\varphi_i}(p_{\mathbb{G}}) = 0)$ such that:
    1. $\omega_{\varphi_1\star\varphi_1}(p_{\mathbb{G}}) = 0$,
    2. $\omega_{\varphi_2\star\varphi_2}(p_{\mathbb{G}}) = 1$.

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2. exists quantum permutations φ_1, φ_2 supported "off" $\mathbb{G}$ $(\omega_{\varphi_i}(p_{\mathbb{G}}) = 0)$ such that:
    1. $\omega_{\varphi_1\star\varphi_1}(p_{\mathbb{G}}) = 0$,
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Dynamics

Definition

A quantum permutation $\varphi \in S_N^+$ is called α-quantum if $\omega_\varphi(p_Q) = \alpha$.

Proposition

If $\varphi \in S_N^+$ is α-quantum and $\rho \in S_N^+$ is β-quantum, then

$$\alpha + \beta - 2\alpha\beta \leq \omega_{\varphi\star\rho}(p_Q) \leq \alpha + \beta - \alpha\beta.$$

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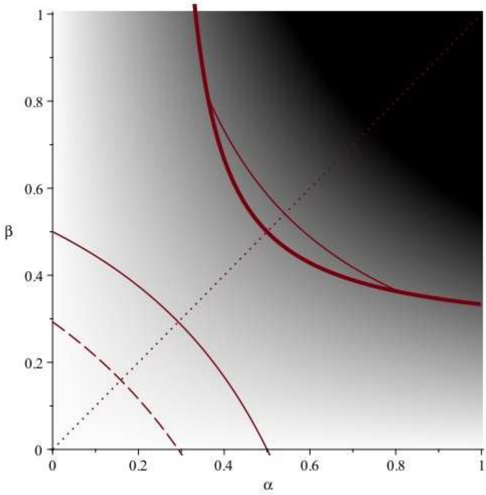
‘Phase’ Diagram



$$Q_{3I} \subset Q_{2I} \subset Q_I; \qquad \partial_W; \qquad Q_{\frac{1}{2}W} \subset Q_W.$$

The quantum part of idempotent states

Corollary

If $\phi \in S_N^+$ is an idempotent state, $\phi \star \phi = \phi$, then

$$\omega_\phi(p_Q) \in \{0\} \cup [1/2, 1].$$

This suggests the following study: consider

$$\chi_N := \{\omega_\phi(p_Q) : \phi \in S_N^+, \phi \star \phi = \phi\}.$$

It is the case that $\chi_N = \{0\}$ for $N \leq 3$, and otherwise not a singleton.

Corollary

A finite quantum permutation group $S_N \subsetneq \mathbb{G} \subsetneq S_N^+$ satisfies:

$$\dim C(\mathbb{G}) \geq 2N!$$

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Some references

[M1] J.P. McCarthy, Analysis for idempotent states on quantum permutation groups, (2023), available at arXiv:2301.13423.

[M2] J.P. McCarthy, A state-space approach to quantum permutations, *Exp. Math.*, 40(3), (2022), 628–664.