# MATH7021: Assignment 2 — Remarks for Poorly Answered Questions

April 28, 2023

## 1 General Remarks

- regardless of whether we are solving an ordinary differential equation (ode) where the independent variable is $t$ (time) or $x$ (distance), Laplace methods only apply to positive values of the independent variable; i.e. $t \geq 0$ or $x \geq 0$. Therefore if you are ever plotting a solution to an ode solved using Laplace, $t < 0$ or $x < 0$ should not be included (even if the formal 'solution' (e.g. $x(t) = \sin t$) makes sense for $t < 0$).

- given that ye had the time, it would have been a good idea to check your answers. This would have saved a lot of us lost marks. You can check your answers by seeing if

  A. they satisfy the differential equation[1],

  B. they satisfy the initial conditions.

  You will probably not have the time to check answers in the final exam.

- We need to be careful when (linearly) taking the Laplace transform of a constant times a derivative. For example, suppose you are taking the Laplace transform of $-6f''(t)$, where $f(0) = -1$ and $f'(0) = 2$, here is good practise

$$\begin{aligned}\mathcal{L}\left\{-6f''(t)\right\} &= -6\mathcal{L}\{f''(t)\} \\ &= -6(s^2F(s) - sf(0) - f'(0)) \\ &= -6(s^2F(s) - s(-1) - 2) \\ &= -6(s^2F(s) + s - 2) \\ &= -6s^2F(s) - 6s + 12.\end{aligned}$$

  Overkill? Well how about:

$$\begin{aligned}\mathcal{L}\left\{-6f''(t)\right\} &= -6(s^2F(s) - sf(0) - f'(0)) \\ &= -6(s^2F(s) - s(-1) - 2) \\ &= -6s^2F(s) - 6s + 12.\end{aligned}$$

  However this is what some of us are doing:

$$\begin{aligned}\mathcal{L}\left\{-6f''(t)\right\} &= -6 \cdot s^2F(s) - sf(0) - f'(0) \\ &= -6 \cdot s^2F(s) - s(-1) - 2 \\ &= -6s^2F(s) - 6s + 12.\end{aligned}$$

[1] Cf. Problem A 3. (b)

This is very dangerous (without the bracket) and you are very likely (and some of us have) done the following:

$$\begin{aligned}\mathcal{L}\left\{-6f''(t)\right\} &= -6\cdot s^2F(s) - sf(0) - f'(0)\\ &= -6\cdot s^2F(s) - s(-1) - 2\\ &= -6s^2F(s) + s - 2,\end{aligned}$$

which is wrong... you need the bracket: technically without it, you are wrong:

$$a\cdot b + c = ab + c \neq ab + ac = a(b+c),$$

although if you end up with the correct answer I leave you off... but many of ye are not ending up with the correct answer so I would like to see the bracket to avoid this.

## Question-Specific Remarks

A. There was a number of references by students in answers to a damping force. There was no damping force in this question just a spring force:

$$\begin{aligned} m\cdot x''(t) + \overbrace{\lambda\, x'(t)}^{\text{damping term}} + k\,x(t) &= 0,\\ \underbrace{m\cdot x''(t) + k\,x(t)}_{\text{no damping term}} &= 0.\end{aligned}$$

1. (b) Show that we have $|x(t)|\leq 1$ for all $t$.

*Remark:* We need to prove the result for *ALL* $t$. Showing the result for a few cases, say $t = 0, 1, 2$, isn't sufficient to prove the result for all possible times, $t$.

For example, suppose you were asked to prove the following 'theorem':

*All Brazilians are good at soccer.*

Pointing out a few Brazilians that are good is not sufficient to prove that *all* Brazilians are good.

The correct answer here is that

$$x(t) = \cos(\omega t),$$

for some $\omega$; and for all $x\in\mathbb{R}$

$$-1\leq \cos x\leq 1;$$

in other words cosine only takes values between $-1$ and 1. Therefore the absolute value of cosine is always between 0 and 1 and so $|x(t)|\leq 1$ for *all* t.

(c) Explain this result $|x(t)|\leq 1$.

What I needed was something like:

*The mass begins from rest at a distance of one from equilibrium.* $|x(t)|\leq 1$ *means that the mass is never more than a distance of one from equilibrium and in fact oscillates between* $x = -1$ *and* $x = 1$.

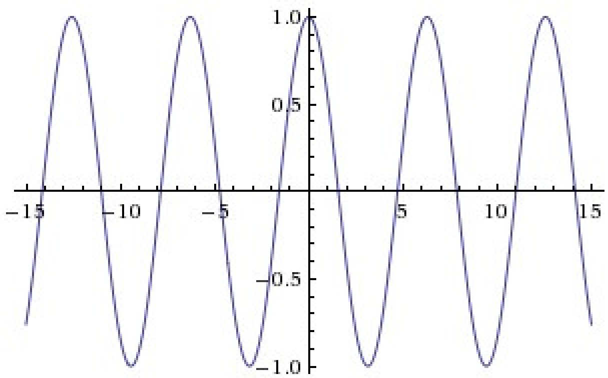


Figure 1: Cosine only takes on values between −1 and 1

Note with no damping there is no energy loss and the oscillations continue forever.

2. (d) What is the behaviour of $x(t)$ for large $t$?

*Remark*: People didn't quite nail this one. If you have oscillations then the amplitude is the magnitude of the crest — in other words the height from the neutral position. With resonance, the amplitude is growing and so $x(t)$ oscillates but reaching greater and greater 'crests' aka with growing amplitude:

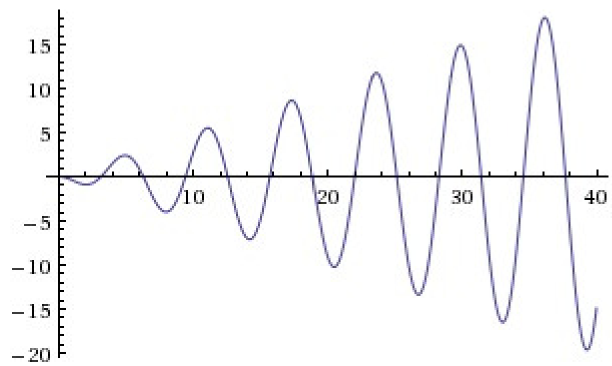


Figure 2: We can see that the amplitude of $x(t)$ is growing — this is *resonance*.

3.

(b) Show that where $x(t)$ is the solution of the previous problem, the function

$$\tilde{x}(t) = x(t) - \frac{mg}{k}$$

satisfies

$$m \cdot \frac{d^2\tilde{x}}{dt^2} + k \cdot \tilde{x}(t) = 0.$$

*Remark*: This question was answered poorly. We had

$$\tilde{x}(t) = A\cos(\omega t),$$

and what we had to do was check that

$$m\tilde{x}''(t) + k\tilde{x}(t) = 0.$$

For $m = 7$ and $k = 6$ here is a proper treatment of this question. From part (a) we would have

$$x(t) = \frac{35}{3} - \frac{32}{3}\cos\left(\sqrt{\frac{6}{7}}\right).$$

Let $\omega = \sqrt{6/7}$ and so

$$\tilde{x}(t) = -\frac{32}{3}\cos(\omega t).$$

Now note

$$\begin{aligned}
\frac{d\tilde{x}}{dt} &= -\frac{32}{3}\left(-\sin(\omega t)\cdot\omega\right) \\
&= \frac{32\omega}{3}\sin(\omega t) \\
\Rightarrow \frac{d^2\tilde{x}}{dt^2} &= \frac{32\omega}{3}\cos(\omega t)\cdot\omega \\
&= \frac{32}{3}\omega^2\cos(\omega t).
\end{aligned}$$

Therefore

$$\begin{aligned}
m\cdot\frac{d^2\tilde{x}}{dt^2} + k\cdot\tilde{x}(t) &= 7\frac{32}{3}\omega^2\cos(\omega t) + 6\cdot\left(-\frac{32}{3}\cos(\omega t)\right) \\
&= 7\cdot\frac{32}{3}\cdot\frac{6}{7}\cos(\omega t) - 6\cdot\frac{32}{3}\cos(\omega t) \\
&= 64\cos(\omega t) - 64\cos(\omega t) = 0,
\end{aligned}$$

therefore $\tilde{x}(t)$ does satisfy the differential equation.

B. 1. Watch the video and write down:

(a) for what range of distances do we have underdamping

(b) for what distance do we have critical damping

(c) for what range of distances do we have overdamping?

Those of us who answered:

(a) 20 to 2.5 mm

(b) 2.5 to 1.5 mm

(c) 1.5 to 0 mm

or even

(a) 20 to 2.5 mm

(b) 2 mm

(c) 1.5 to 0 mm,

did not get the full mark. You need to understand that CRITICAL damping only occurs at one distance and everything else is either overdamping or underdamping (no matter how insignificant); i.e. at 2.25 mm, although not shown in the video, is underdamped.

A good answer is:

(a) any distance $> 2$ mm,

(b) 2 mm

(c) any distance $< 2$ mm,

3. (a) The biggest issue here was student's getting to, say,

$$Y(s) = \frac{\frac{4}{s} + 2s + 2}{(s+4)^2}.$$

People from here went on to try and find the Rule II partial fraction expansion:

$$Y(s) \stackrel{!}{=} \frac{A}{s+4} + \frac{B}{(s+4)^2}.$$

You see on p.106 that to find a partial fraction expansion, that $Y(s)$ must be a *rational* function. This means that

$$Y(s) = \frac{p(s)}{q(s)},$$

with $p$ and $q$ *polynomials*, sums of positive powers of $s$. The top of $Y(s)$, $\frac{4}{s} + 2s + 2$ is *not* a polynomial because it is equal to

$$4s^{-1} + 2s + 2,$$

and the $s^{-1}$ is not a positive power of $s$. There are various ways to avoid this, the easiest being to take the $Y(s)$ above and to multiply above and below by $s$:

$$Y(s) = \frac{\frac{4}{s} + 2s + 2}{(s+4)^2} \cdot \frac{s}{s} = \frac{4 + 2s^2 + 2}{s \cdot (s+4)^2},$$

showing that $Y$ actually requires a Rule I and a Rule II. A Rule I and a Rule II looks like:

$$\frac{p(s)}{s(s+4)^2} \stackrel{!}{=} \frac{A}{s} + \frac{B}{s+4} + \frac{C}{(s+4)^2}.$$

(b) Assuming the wind stays constant, <u>what is the behaviour of $y(t)$ for large $t$?</u>

*Remark:* Most people got an answer of the form

$$y(t) = B + Ce^{-pt} + Dte^{-pt}.$$

Now, as explained in lectures, if $p > 0$,

$$Ce^{-pt} \to 0 \text{ , and } Dte^{-pt} \to 0 \to 0,$$

but the constant just stays as it is and so (Cf. Figure 3)

$$y(t) \to B.$$

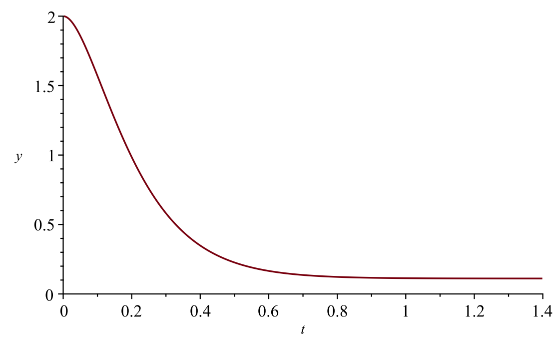


Figure 3: A great gust of wind must have pushed the wind out to $y = 2$... then however the spring force caused the bridge to come back to vertical. However the constant wind of e.g. "$A = 9$" kept the bridge slightly off vertical with $y(t) = \frac{1}{9}$ for large $t$.

(c) What type of damping will the bridge undergo if the wind suddenly stops? If the wind stops then you are left with:

$$\frac{d^2y}{dt^2} + b \cdot \frac{dy}{dt} + c \cdot y(t) = 0,$$

i.e. a damped harmonic oscillator. If we do the $b^2 - 4ac$ analysis:

- $b^2 - 4ac < 0 \Rightarrow$ underdamped
- $b^2 - 4ac = 0 \Rightarrow$ critically damped
- $b^2 - 4ac > 0 \Rightarrow$ overdamped,

we can see the damping is critical.

You can also see this by looking at the solution:

$$y(t) = Ce^{-pt} + Dte^{-pt} + B.$$

There are no oscillations — with no sine nor cosine there are no oscillations — and we don't have a two speed convergence (two exponentials). Therefore it is critically damped.