MATH7016: 20% Written Assessment 1 [Ex 54 Marks]

Name:  

1. Consider the statement:

   There is a need for numerical methods in the study of differential equations that arise in engineering problems.

Do you agree or disagree with this statement?  

Justify your answer.  

**Solution:** I agree. 

Some differential equations may not be solved analytically in terms of elementary functions. Sometimes derivatives are given in terms of data rather than a formula. 

2. Consider the displacement, \( x(t) \) (in metres), of a body of mass \( m \) (in kg), after \( t \) seconds, subject to two forces:

   - a constant force of +10 N, and
   - a damping force, proportional to the velocity.

The initial displacement and initial velocity are both zero. Formulate this problem as a second order initial value problem. 

**Solution:**

\[
m \frac{d^2x}{dt^2} = +10 - \lambda \frac{dx}{dt}, \quad x(0) = 0, \quad x'(0) = 0
\]
3. Calculate the first two non-zero terms of the Maclaurin Series of \( y(t) = \cos t \). [4 Marks]

**Solution:**

\[
y(t) = y(0) + y'(0)t + \frac{y''(0)}{2!} t^2 + \frac{y'''(0)}{3!} t^3
\]

\[
y(0) = \cos 0 = 1 \quad \boxed{1}
\]

\[
y'(t) = -\sin t \implies y'(0) = 0 \quad \boxed{1}
\]

\[
y''(t) = -\cos t \implies y''(0) = -1 \quad \boxed{1}
\]

\[\Rightarrow \cos t \approx 1 + 0t - \frac{1}{2} t^2\]

\[= 1 - \frac{1}{2} t^2 \quad \boxed{1}\]
4. Consider an initial value problem

\[ \frac{dy}{dx} = F(x, y); \quad y(0) = 0. \]

Euler’s Method with step \( h = 0.5 \) uses the tangent to the curve at \( x = 0 \) to approximate the \( y \)-value at \( x = x_1 = 0 + 0.5 = 0.5 \).

(a) Write down, in terms of \( h \) and \( F \), the formula for \( y_1 \), the Euler Method approximation \( y(0.5) \approx y_1 \).

(b) Name two factors which influence the Euler Method local error:

\[ \epsilon^L = |y(0.5) - y_1|. \]

(c) Give two strategies for getting a more accurate approximation to \( y(x_1) \).

(d) For one of these strategies, give one disadvantage of that strategy.

Solution:

a) \[ y_1 = 0 + hF(0, 0) \]

b) Step-size \( h \) and second derivative

c) Smaller step-size or use TTT or Heun

d) Smaller step-size \( \Rightarrow \) more calculating

TTT \( \Rightarrow \) implicit differentiable
5. Consider an initial value problem:

\[ \frac{dy}{dx} = F(x); \quad y(x_0) = y_0. \]

Using the Three Term Taylor Method with a step-size of \( h \), it can be shown that the local error at step \( i \), \( \xi_i \), satisfies:

\[ |\xi_i| \leq \frac{|y'''_{\text{max}}| h^3}{6}, \]

where \( |y'''_{\text{max}}| \) is the maximum of the absolute value of the third derivative between \( x = x_i \) and \( x = x_{i+1} \).

(a) What does it mean to say that the local error is \( O(h^3) \)?

(b) Show that if we use the Three Term Taylor Method to approximate \( y(x_n) = y(x_0 + n \cdot h) \), that the global error is \( O(h^2) \).

(c) What is the effect on the global error if we quarter the step-size?

(d) The local and global errors analysed here are truncation errors; errors arising from deficiencies in the method used. In practical terms, what is the other main source of error when approximating solutions of differential equations?

[10 Marks]

Solution:

a) \( |\xi_i| \leq K_i h^3 \) or (absolute value of) local error is \( |\xi_i| = K_i h^3 \) (less than or equal to a multiple of \( h^3 \))

\[ |\xi_0| = |\xi_1 + \ldots + \xi_n| \leq |\xi_1| + \ldots + |\xi_n| \]

\[ \leq K_1 h^3 + \ldots + K_n h^3 \]

\[ \leq K_{\text{max}} h^3 \]

\[ = n K_{\text{max}} h^3 \]

\[ = \frac{L}{h} K_{\text{max}} h^3 = L K_{\text{max}} h \approx \varepsilon_g e^{O(h^2)} \]

(c) reduced by factors of \( h \)

(d) rounding
6. Consider an initial value problem

\[
\frac{dV}{dx} = -100e^{-x^2}; \quad V(0) = 80
\]

This models the shear force, \( V \) (in kN), at a distance \( x \) (in metres) along a fixed end beam of span 6 m.

Use Heun’s Method with a step-size of 0.1 to approximate \( V(0.3) \). Use five significant figures for all calculations.

\[ V_1 = 80 + 0.1 \left( \frac{-100e^{-0^2} - 100e^{0.1^2}}{2} \right) \approx 67.050 \text{ kN} \]

\[ V_2 = 70.05 + 0.1 \left( \frac{-100e^{-0.1^2} - 100e^{-0.2^2}}{2} \right) \approx 60.296 \text{ kN} \]

\[ V_3 = 60.296 + 0.1 \left( \frac{-100e^{-0.2^2} - 100e^{-0.3^2}}{2} \right) \approx 50.922 \text{ kN} \]
7. Suppose we have telemetric data on the speed of a vehicle, collected at 0.1 s intervals. Suppose the first few data points are given by:

<table>
<thead>
<tr>
<th>t [s]</th>
<th>v [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4</td>
</tr>
<tr>
<td>0.2</td>
<td>1.1</td>
</tr>
<tr>
<td>0.3</td>
<td>2.3</td>
</tr>
<tr>
<td>0.4</td>
<td>3.8</td>
</tr>
<tr>
<td>0.5</td>
<td>5.2</td>
</tr>
</tbody>
</table>

Can we use this data to approximate the distance travelled by the vehicle?

(a) If yes, use a method you studied in MATH7016 to approximate the distance travelled after \( t = 0.5 \) s, given that the distance travelled after \( t = 0 \) s is zero. Use five significant figures for all calculations.

(b) If no, explain why not.

**Solution:**

Yes. \[
\frac{dx}{dt} = v
\]

**Forward Euler**

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 ( (0) )</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4 ( (1) )</td>
</tr>
<tr>
<td>0.2</td>
<td>1.1 ( (2) )</td>
</tr>
<tr>
<td>0.3</td>
<td>2.3 ( (3) )</td>
</tr>
<tr>
<td>0.4</td>
<td>3.8 ( (4) )</td>
</tr>
<tr>
<td>0.5</td>
<td>5.2 ( (5) )</td>
</tr>
</tbody>
</table>

**Heun**

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 ( (0) )</td>
</tr>
<tr>
<td>0.1</td>
<td>0.02 ( (1) )</td>
</tr>
<tr>
<td>0.2</td>
<td>0.095 ( (2) )</td>
</tr>
<tr>
<td>0.3</td>
<td>0.265</td>
</tr>
<tr>
<td>0.4</td>
<td>0.57</td>
</tr>
<tr>
<td>0.5</td>
<td>1.02</td>
</tr>
</tbody>
</table>

[8 Marks]
Rough Work:

\[ x_1 = 0 + 0.1(0) + 0.1^2 \frac{4}{2} = 0.02 \]  
\[ x_2 = 0.02 + 0.1(0.4) + 0.1^2 \frac{[7]}{2} = 0.095 \]  
\[ x_3 = 0.095 + 0.1(1.1) + 0.1^2 \frac{[12]}{2} = 0.265 \]  
\[ x_4 = 0.265 + 0.1(2.3) + 0.1^2 \frac{[15]}{2} = 0.57 \]  
\[ x_5 = 0.57 + 0.1(5.2) + 0.1^2 \frac{[14]}{2} = 1.02 m \]
Useful Formulae

A tables page will also be provided.

\[ y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!} x^2 + \frac{y'''(0)}{3!} x^3 + \ldots \]

\[ y(x) = y(a) + y'(a)(x - a) + \frac{y''(a)}{2!} (x - a)^2 + \frac{y'''(a)}{3!} (x - a)^3 + \ldots \]

\[ y_{i+1} = y_i + h \cdot F(x_i, y_i) \]

\[ y_{i+1} = y_i + h \cdot y_i' + \frac{h^2}{2} y_i'' \]

\[ y_{i+1}^0 = y_i + h \cdot F(x_i, y_i) \]

\[ y_{i+1} = y_i + h \cdot \frac{F(x_i, y_i) + F(x_{i+1}, y_{i+1}^0)}{2} \]

Runge-Kutta Notation

\[ k_1 = F(x_i, y_i) \]

\[ y_{i+1} = y_i + h \cdot k_1 \]

Where

\[ k_1 = F(x_i, y_i) \]  \hspace{1cm} (1) \]

\[ k_2 = F \left( \frac{x_i + h}{2}, \frac{y_i + h \cdot k_1}{2} \right) \]  \hspace{1cm} (2) \]

\[ y_{i+1} = y_i + h \cdot \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) \]