MATH7016: 20% Written Assessment 1 [Ex 54 Marks]

Name: Marking Scheme

1. Consider the statement:

   There is a need for numerical methods in the study of differential equations that arise in engineering problems.

   Do you agree or disagree with this statement?

   [1 Mark]

   Justify your answer.

   [4 Marks]

   Solution: I agree. 1

   Some differential equations may not be solved analytically/in terms of elementary functions. Sometimes derivatives are given in terms of data rather than a formula. 4

2. Consider the displacement, $x(t)$ (in metres), of a body of mass m (in kg), after t seconds, subject to two forces

   - a constant force of +10 N, and
   - a damping force, proportional to the velocity.

   The initial displacement and initial velocity are both zero.

   Formulate this problem as a second order initial value problem.

   [6 Marks]

   Solution:

$$m\frac{d^2x}{dt^2} = +10 - \lambda\frac{dx}{dt} \quad ; \quad x(0)=0, \quad x'(0)=0$$

1 1 1 1 1 1

3. Calculate the first two non-zero terms of the Maclaurin Series of $y(t) = \cos t$. [4 Marks]

Solution:

$$y(t) = y(0) + y'(0)t + \frac{y''(0)}{2}t^2 + \frac{y'''(0)}{3!}t^3$$

$y(0) = \cos 0 = 1$ 1

$y'(t) = -\sin t \Rightarrow y'(0) = 0$ 1

$y''(t) = -\cos t \Rightarrow y''(0) = -1$ 1

$\Rightarrow \cos t \approx 1 + 0t - \frac{1}{2}t^2$

$= 1 - \frac{1}{2}t^2$ 1

t , 3

4. Consider an initial value problem

$$\frac{dy}{dx} = F(x, y)\,; \;\; y(0) = 0.$$

Euler's Method with step $h = 0.5$ uses the tangent to the curve at $x = 0$ to approximate the y-value at $x = x_1 = 0 + 0.5 = 0.5$.

(a) Write down, in terms of h and F, the formula for y_1, the Euler Method approximation $y(0.5) \approx y_1$.

(b) Name two factors which influence the Euler Method local error:

$$\varepsilon^L = |y(0.5) - y_1|.$$

(c) Give two strategies for getting a more accurate approximation to $y(x_1)$.

(d) For one of these strategies, give one disadvantage of that strategy.

[9 Marks]

Solution: 1 1

a) $y_1 = 0 + hF(0,0)$

b) step-size h and 1 second derivative 1 1

c) smaller step-size 1 or use TTT or Heun 1

d) smaller-stepsize ⇒ more calculations 2
TTT ⇒ implicit differentiation

5. Consider an initial value problem:

$$\frac{dy}{dx} = F(x)\,; \quad y(x_0) = y_0.$$

Using the Three Term Taylor Method with a step-size of h, it can be shown that the local error at step i, $\varepsilon_i^{\mathrm{L}}$ satisfies:

$$|\varepsilon_i^{\mathrm{L}}| \le \frac{|y_i'''|_{\max} h^3}{6},$$

where $|y_i'''|_{\max}$ is the maximum of the absolute value of the third derivative between $x = x_i$ and $x = x_{i+1}$.

(a) What does it mean to say that the local error is $\mathcal{O}(h^3)$?

(b) Show that if we use the Three Term Taylor Method to approximate $y(x_n) = y(x_0 + n \cdot h)$, that the *global error* is $\mathcal{O}(h^2)$.

(c) What is the effect on the global error if we quarter the step-size?

(d) The local and global errors analysed here are *truncation errors*; errors arising from deficiencies in the method used. In *practical* terms, what is the other main source of error when approximating solutions of differential equations?

[10 Marks]

Solution:

a) $|\varepsilon_i^L| \le K \cdot h^3$ or (absolute value of) local error is

$|\varepsilon_i^L| = k \cdot h^3$ 1 (less than or) equal to a multiple of h^3

b) let $L = nh \Rightarrow x_n - x_0 = nh = L \Rightarrow n = \frac{L}{h}$

$$|\varepsilon^G| = |\varepsilon_1^L + \dots + \varepsilon_n^L| \le |\varepsilon_1^L| + \dots + |\varepsilon_n^L| \quad 1$$

$\le k_1 h^3 + \dots + k_n h^3$ 1 . let $K_* := \max_i K$

$\le K_* h^3 + \dots + K_* h^3$ 1

$= n K_* h^3$ 1

$= \frac{L}{h} K_* h^3 = L K_* h^2 \Rightarrow \varepsilon^G \in \mathcal{O}(h^2)$ 1 1

c) reduced by factor of 16 1

d) rounding 2

6. Consider an initial value problem

$$\frac{dV}{dx} = -100\, e^{-x^2}\,; \;\; V(0) = 80$$

This models the shear force, V (in kN), at a distance x (in metres) along a fixed end beam of span 6 m.

Use Heun's Method with a step-size of 0.1 to approximate $V(0.3)$. Use five significant figures for all calculations.

[12 Marks]

Solution: $\frac{dV}{dx}$ is independent of V $\Rightarrow$ no need for predictor

$$V_1 = 80 + \frac{0.1}{2}\left[-100 e^{-0^2} - 100 e^{-0.1^2}\right] \simeq 70.050 \text{ kN}$$

$$V_2 = 70.05 + \frac{0.1}{2}\left[-100 e^{-0.1^2} - 100 e^{-0.2^2}\right] \simeq 60.296 \text{ kN}$$

$$V_3 = 60.296 + \frac{0.1}{2}\left[-100 e^{-0.2^2} - 100 e^{-0.3^2}\right] \simeq 50.922 \text{ kN}$$

7. Suppose we have *telemetric* data on the *speed* of a vehicle, collected at 0.1 s intervals. Suppose the first few data points are given by:

t [s]	v [m/s]
0	0
0.1	0.4
0.2	1.1
0.3	2.3
0.4	3.8
0.5	5.2

1

V

4

7

12

15

14

Can we use this data to *approximate* the distance travelled by the vehicle?

(a) If yes, use a method you studied in MATH7016 to approximate the distance travelled after $t = 0.5$ s, given that the distance travelled after $t = 0$ s is zero. Use five significant figures for all calculations.

(b) If no, explain why not.

[8 Marks]

Solution: a) Yes. $\frac{dx}{dt} = v$ 3

Forward Euler
4/5

Euler 1:

$x_1 = 0 + 0.1(0) = 0$ 1

$x_2 = 0 + 0.1(0.4) = 0.04$ 1

$x_3 = 0.04 + 0.1(1.1) = 0.15$

$x_4 = 0.15 + 0.1(2.3) = \cancel{0.78}\ 0.38$

$x_5 = 0.38 + 0.1(3.8) = \cancel{1.16\text{m}}\ 0.76$ 2

Heun 1:

$x_1 = 0 + \frac{0.1}{2}[0 + 0.4] = 0.02$ 1

$x_2 = 0.02 + \frac{0.1}{2}[0.4 + 1.1] = 0.095$ 1

$x_3 = 0.095 + \frac{0.1}{2}[1.1 + 2.3] = 0.265$

$x_4 = 0.265 + \frac{0.1}{2}[2.3 + 3.8] = 0.57$

$x_5 = 0.57 + \frac{0.1}{2}[3.8 + 5.2] = 1.02\text{m}$ 2

Rough Work:

TTT

(1) $x_1 = 0 + 0.1(0) + \frac{0.1^2}{2}4 = 0.02$ (1)

$x_2 = 0.02 + 0.1(0.4) + \frac{0.1^2}{2}[7] = 0.095$

$x_3 = 0.095 + 0.1(1.1) + \frac{0.1^2}{2}[12] = 0.265$

$x_4 = 0.265 + 0.1(2.3) + \frac{0.1^2}{2}[15] = 0.57$

$x_5 = 0.57 + 0.1(5.2) + \frac{0.1^2}{2}[14] = 1.02m$

(2)

Useful Formulae

A tables page will also be provided.

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

$$y(x) = y(a) + y'(a)(x-a) + \frac{y''(a)}{2!}(x-a)^2 + \frac{y'''(a)}{3!}(x-a)^3 + \cdots$$

$$y_{i+1} = y_i + h \cdot F(x_i, y_i)$$

$$y_{i+1} = y_i + h \cdot y_i' + \frac{h^2}{2} \cdot y_i''$$

$$\begin{aligned} y_{i+1}^0 &= y_i + h \cdot F(x_i, y_i) \\ y_{i+1} &= y_i + h \cdot \frac{F(x_i, y_i) + F(x_{i+1}, y_{i+1}^0)}{2} \end{aligned}$$

Runge–Kutta Notation

$$\begin{aligned} k_1 &= F(x_i, y_i) \\ y_{i+1} &= y_i + h \cdot k_1 \end{aligned}$$

Where

$$k_1 = F(x_i, y_i) \qquad (1)$$

$$k_2 = F\left(\underbrace{x_i + h}_{\text{endpoint of } [x_i, x_i]}, \overbrace{y_i + h \cdot k_1}^{\text{Euler prediction for the endpoint}}\right) \qquad (2)$$

$$y_{i+1} = y_i + h \cdot \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)$$