The Frucht property in the quantum group setting

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(joint work with Teo Banica)

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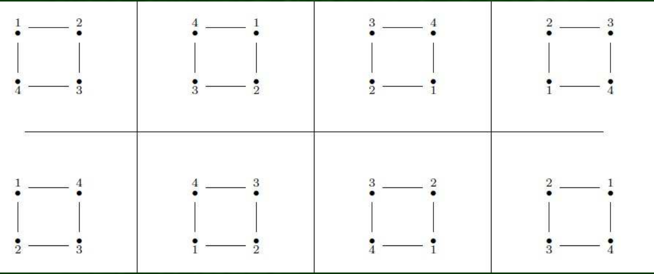

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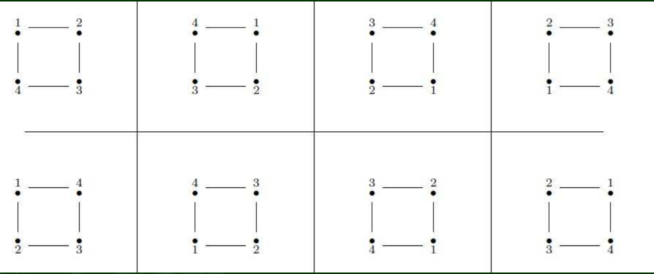

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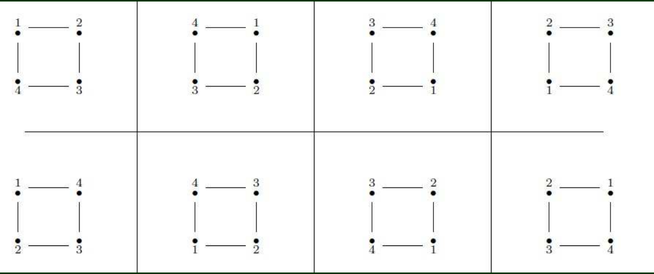

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Is every group a set of symmetries?

Is every (compact) quantum group a... of quantum symmetries?

Quantum Permutation Groups

A magic unitary $u \in M_N(C(\mathbb{X}))$:

$$u_{ij} = u_{ij}^* = u_{ij}^2; \quad \sum_k u_{ik} = \sum_k u_{kj} = \mathbb{1}_{\mathbb{X}}.$$

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Wang's quantum permutation group:

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If $C(\mathbb{G})$ is a unital C^*-algebra generated by a magic unitary $u \in M_N(C(\mathbb{G}))$ such that Δ is a $*$-homomorphism, then

$$\mathbb{G} \subseteq S_N^+;$$

$\mathbb{G}$ is a quantum permutation group, with fundamental magic representation u.

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$$X \to G^+(X) \qquad \text{vs} \qquad \mathbb{G} \to \{X_\alpha : \mathbb{G} \curvearrowright X_\alpha\}.$$

Embedded CMQG vs Abstract CQG

Consider e.g. the Kac–Paljutkin quantum group:

$$C(G_0) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3 \oplus \mathbb{C}f_4 \oplus M_2(\mathbb{C});$$

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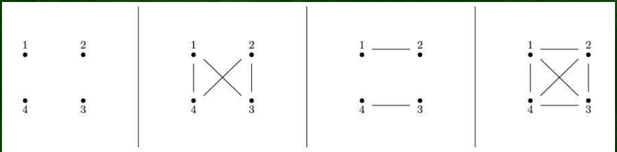

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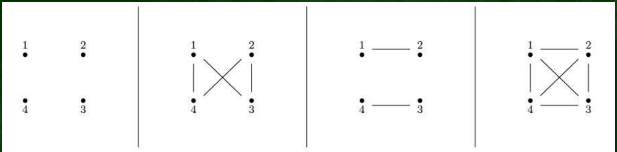

Need to consider (modulo some equivalence) all embeddings of $G_0 \subset S_N^+$.

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1. If $N \leq 3$, $G^+(X) = G(X) \implies G^+(\bullet) = \mathbb{Z}_1$, $G^+(:) = \mathbb{Z}_2$, $G^+(\Delta) = S_3$,
2. $G^+(::) = S_4^+$,
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The first example of a 'genuine' finite quantum group with the Frucht property was exhibited recently (dual of an order 256 non-abelian group) [RS].

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Blocks $u^p \in M_{N_p}(C(\mathbb{G}))$ are transitive magic representations, and each corresponds to a block $V_p^{(m)} \subseteq V$. The entries of $\{u^1, \ldots, u^k\}$ generate $C(\mathbb{G})$.

Group duals $\widehat{\Gamma} \subseteq S_N^+$

<u>Claim</u>: Any such $\widehat{\Gamma}$ comes from $\mathbb{Z}_{N_1} * \cdots * \mathbb{Z}_{N_k} \to \Gamma$ of the form:

$$u = \mathrm{diag}(u^{g_1}, \ldots, u^{g_k}).$$

For each $g_p \in \Gamma$ the magic unitary $u^{g_p} = (u_{kl}^{g_p})$ is as follows, with $w = e^{2\pi i/N_p}$:

$$u_{kl}^{g_p} = \frac{1}{N_p} \sum_{m=1}^{N_p} w^{(k-l)m} g_p^m. \tag{2}$$

These are circulant, and we will see examples on the next page.

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For finite G, the classical version of $\widehat{G}$ has order equal to the number of one dimensional representations of G.

Bichon's Group Dual $\widehat{\mathbb{Z}_2 * \mathbb{Z}_3} \subset S_5^+$

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we have $\widehat{\mathbb{Z}_2 * \mathbb{Z}_3} \subset S_5^+$ via:

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Uncountably many quantum permutation groups

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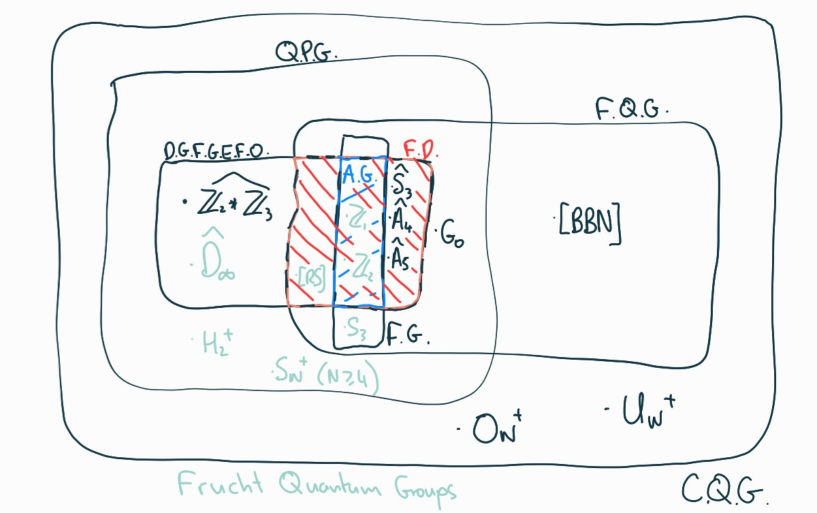
Unsatisfactory.

A map of compact quantum groups



The set of finite quantum groups is countable [S].

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Theorem I: If $\mathbb{G} \curvearrowright X$ with $u \in M_N(C(\mathbb{G}))$ then $\exists A \subseteq O$:

$$E = \bigcup_{o \in A} (o \cup o^{-1}).$$

If $\sigma \in S_V$ and for all $o \cup o^{-1} \in O$

$$(\sigma \times \sigma)(o \cup o^{-1}) = o \cup o^{-1} \implies \sigma \in G(X).$$

Example: a $\widehat{S_3} \subset S_5^+$

Where $\omega = \exp(2\pi i/3)$ and $\sigma = (123)$:

$$u^{(12)} = \frac{1}{2}\begin{pmatrix} e+(12) & e-(12) \\ e-(12) & e+(12) \end{pmatrix}$$

$$u^{(123)} = \frac{1}{3}\begin{pmatrix} e+\sigma+\sigma^2 & e+\omega^2\sigma+\omega\sigma^2 & e+\omega\sigma+\omega^2\sigma^2 \\ e+\omega\sigma+\omega^2\sigma^2 & e+\sigma+\sigma^2 & e+\omega^2\sigma+\omega\sigma^2 \\ e+\omega^2\sigma+\omega\sigma^2 & e+\omega\sigma+\omega^2\sigma^2 & e+\sigma+\sigma^2 \end{pmatrix}$$

$$u := \begin{pmatrix} u^{(12)} & 0 \\ 0 & u^{(123)} \end{pmatrix}$$

The orbitals disjoint of $\Delta_{\{1,\dots,5\}}$ are:

$$o_1 = \{(1,2),(2,1)\}$$

$$o_2 \cup o_2^{-1} = (\{1,2\} \times \{3,4,5\}) \cup (\{3,4,5\} \times \{1,2\})$$

$$o_3 \cup o_3^{-1} = \{(3,4),(4,5),(5,3),(4,3),(5,4),(3,5)\}$$

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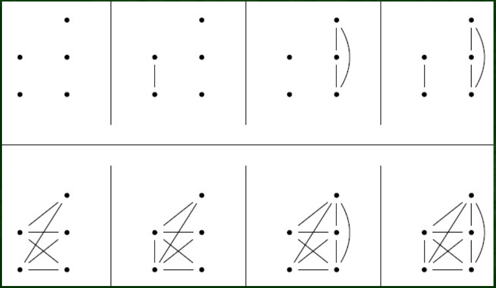
Theorem 1 thus gives eight graphs on five vertices that this $\widehat{S_3} \subset S_5^+$ acts on:



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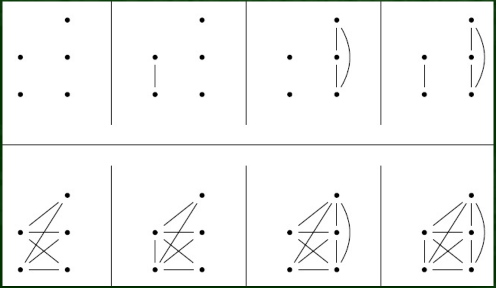

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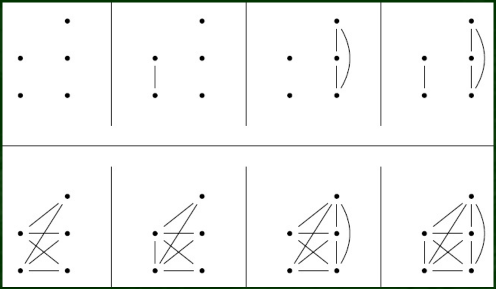

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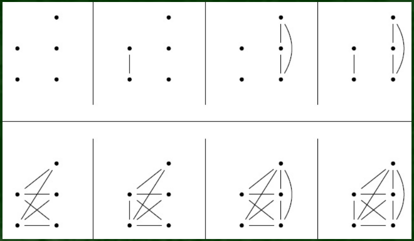
<u>Examples</u>: $\widehat{S_3}, \widehat{A_4}, \widehat{A_5}$.

For total $\widehat{\Gamma}$: $u^{g_p} \not\equiv u^{g_q}$ associated with blocks $V_p, V_q \subset V$:

$$V_p \times V_q,\, V_q \times V_p \in O.$$



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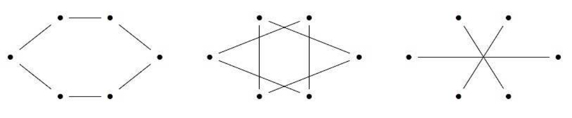
For example, if $N_p = 6$ we have $o \cup o^{-1}$ of the form:



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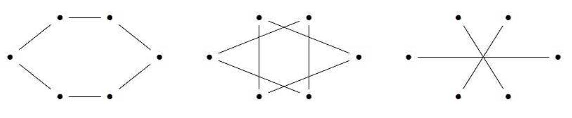

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Total duals with classical versions $\mathbb{Z}_p$

<u>Theorem</u>: Suppose that total $\widehat{\Gamma} \curvearrowright X$. Let $\Gamma_u = \{\gamma_1, \ldots, \gamma_k\}$ be the set of elements of Γ appearing as Fourier-type transitive magic representations in the fundamental representation. Then, where $\gamma_1, \ldots, \gamma_k$ are of order $N_1, \ldots, N_k$,

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Cf. Schmidt, Quantum automorphisms of folded cube graphs, Th.2.2.

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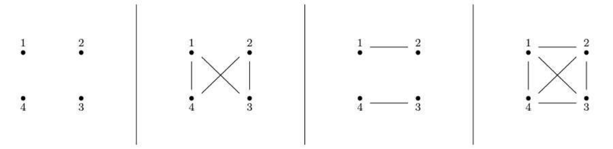

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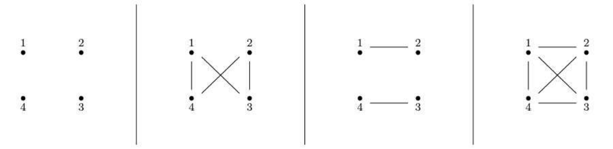

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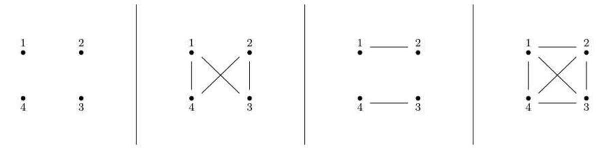

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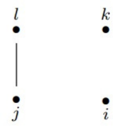

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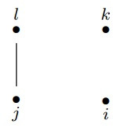

How to think about random automorphisms $\nu \in S(C(G(X)))$?

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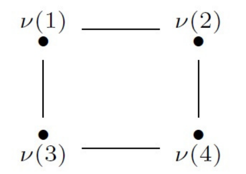

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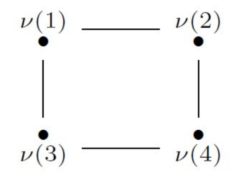

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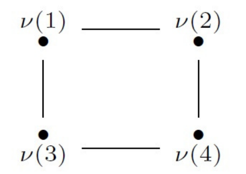

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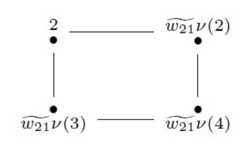
Say $\nu(1) = 2$ then random automorphism collapses to

$$\widetilde{w_{21}}\nu = \frac{\nu(w_{21} \cdot w_{21})}{\nu(w_{21})}:$$



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If you measure $x(j) > x(l)$ with $\varphi \in S(C(G^+(X)))$, and find $[\varphi(j) = i] > [\varphi(l) = k]$, then:

$$\begin{aligned} \mathbb{P}[[\varphi(j) = i] > [\varphi(l) = k]] > 0 &\implies \varphi(u_{kl} u_{ij} u_{kl}) > 0 \\ &\implies u_{ij} u_{kl} \neq 0 \implies [(j,l) \in E \iff (i,k) \in E.] \end{aligned}$$

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Take $G_0 \subseteq G^+(||)$ via $\pi_0 : C(G^+(||)) \to C(G_0)$, $u_{ij} \mapsto u_{ij}^{G_0}$:

$$u^{G_0} := \begin{bmatrix} f_1 + f_2 & f_3 + f_4 & p & I_2 - p \\ f_3 + f_4 & f_1 + f_2 & I_2 - p & p \\ p^T & I_2 - p^T & f_1 + f_3 & f_2 + f_4 \\ I_2 - p^T & p^T & f_2 + f_4 & f_1 + f_3 \end{bmatrix}$$

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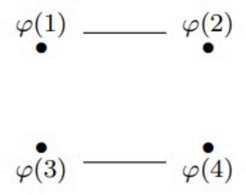
A state $\varphi_0 \in S(C(G_0))$ defines a quantum automorphism of $||$:

$$\varphi := \varphi_0 \circ \pi_0,$$



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If measurement of φ with $x(1)$ yields $\varphi(1) \in \{0,1\}$, centrality of $u_{11}^{G_0}, u_{21}^{G_0}$ implies that $\widehat{u_{\varphi(1),1}}\varphi$ is a random automorphism in $\mathbb{Z}_2 \times \mathbb{Z}_2$.

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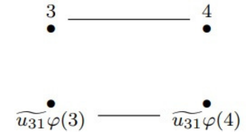

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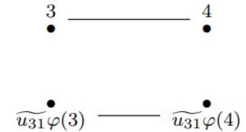

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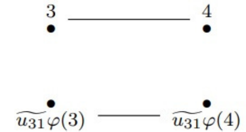

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Can observe: $[\varphi(1) = 4] \succ [\varphi(3) = 1] \succ [\varphi(1) = 3]$.

Epilogue

For $G^+(X) = G_0$, the abelianisation $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset G_0$

$$\begin{bmatrix} f_1+f_2 & f_3+f_4 & p & I_2-p \\ f_3+f_4 & f_1+f_2 & I_2-p & p \\ p^T & I_2-p^T & f_1+f_3 & f_2+f_4 \\ I_2-p^T & p^T & f_2+f_4 & f_1+f_3 \end{bmatrix} \mapsto \begin{bmatrix} f_1+f_2 & f_3+f_4 & 0 & 0 \\ f_3+f_4 & f_1+f_2 & 0 & 0 \\ 0 & 0 & f_1+f_3 & f_2+f_4 \\ 0 & 0 & f_2+f_4 & f_1+f_3 \end{bmatrix},$$

implies, together with e.g. $p(I_2 - p^T) \neq 0$, that classical symmetries are missing.

The centrality of the f_i implies quantumness is missing too.

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