Introduction to Quantum Groups

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UCC-CIT Quantum Probability Seminar

Contents

1 Introduction

The following lectures were given to the UCC-CIT Quantum Probability Seminar. The aim of the notes is to give the reader an appreciation of the motivations behind facts in the study of quantum groups, with an additional focus on analytical and topological concerns. These notes are to be a fuller version of the material presented in the seminar.

We are assuming a basic working knowledge of functional analysis including C*-algebras. We shall be employing what is called the *Gelfand Philosophy*, which is a way of thinking about non-commutative algebras as algebras of functions on quantum spaces. My understanding is that the 'best' categorical equivalence of spaces with algebras requires a careful understanding of what the morphisms are, and also uses opposite categories. We will be a little softer.

Theorem 1.1. *(Gelfand) If A is a non-zero commutative* C*-*algebra, then there exists a locally compact Hausdorff topological space X such that A is isometrically *-isomorphic to $C_0(X)$* •

The Gelfand Philosophy says that noncommutative C*-algebras should be thought of as the algebra of continuous functions on a *quantum space*. Often the blackboard bold $A = C_0(\mathbb{X})$ is used to denote a noncommutative C*-algebra, but I will just write $A = C_0(X)$, and refer to X as a quantum space. We will refer to commutative objects as the *classical case*, and noncommutative objects as the *quantum case*. Classically, if A is unital, $C_0(X) = C(X)$. If in the quantum case, the C*-algebra A is noncommutative we shall write $A = C(X)$ and remark that X is quantum, or $C(X)$ noncommutative. In the classical case, the unit 1_A is equal to the indicator function on the space X, $\mathbb{1}_X$. We will do the same in the quantum case: denote the unit of $C_0(X)$ by $\mathbb{1}_X$. If the algebra is finite dimensional, we will instead write $A = F(X)$.

1.1 Finite Quantum Groups

Quantum groups are defined via their 'algebra of functions', in classes of objects such that whenever an algebra of functions is commutative, it is the algebra of functions on a group.

Let G be a finite group. A finite group is an object in the category of finite sets. There are morphisms:

$$\begin{aligned} m : G \times G &\to G, \\ e : \{\bullet\} &\to G, \\ {}^{-1} : G &\to G, \end{aligned}$$

such that the group axioms can be expressed as commutative diagrams. Let $\mathbb{C} : \mathbf{FinSet} \to \mathbf{FinVec}_{\mathbb{C}}$ be the $\mathbb{C}$-functor. Note that

$$\mathbb{C}m : \mathbb{C}(G \times G) \to \mathbb{C}G,$$

is bilinear. For G finite, $\mathbb{C}(G \times G) \cong \mathbb{C}G \otimes \mathbb{C}G$, and so there is a linear map $\nabla : \mathbb{C}G \otimes \mathbb{C}G \to \mathbb{C}G$. Apply now the dual endofunctor $\mathcal{D}$. Consider a finite group under the functor composition $\mathcal{D} \circ \mathbb{C}$. We see $G \mapsto F(G)$, the algebra of functions on G. The morphisms of group multiplication, inclusion of the identity, and group inverses, get sent to the *comultiplication*, the *counit*, and the *antipode*:

$$\begin{aligned} m &\mapsto \Delta, \\ e &\mapsto \varepsilon, \\ {}^{-1} &\mapsto S. \end{aligned}$$

The algebra of functions on G, $F(G)$, together with the supremum norm, is a C*-algebra. The comultiplication $\Delta : F(G) \to F(G) \otimes F(G)$ is unital $*$-homomorphism ($(f \otimes g)^* = f^* \otimes g^*$). It is given on the basis vectors by

$$\Delta(\delta_s) = \sum_{t \in G} \delta_{st^{-1}} \otimes \delta_t,$$

and the associativity of m, given by the commutative diagram:

$$m \circ (I_G \times m) = m \circ (m \times I_G),$$

under the functor composition, is given by *coassociativity*:

$$(I_{F(G)} \otimes \Delta) \circ \Delta = (\Delta \otimes I_{F(G)}) \circ \Delta.$$

That multiplication by the identity $g \mapsto e \cdot g$ is the identity on G:

$$G \cong \{\bullet\} \times G \underset{e \times I_G}{\to} \{e\} \times G \underset{m}{\to} G,$$

in other words

$$m \circ (e \times I_G) = I_G,$$

under the functor composition, is given by the *counital property*:

$$(\varepsilon \otimes I_{F(G)}) \circ \Delta = I_{F(G)},$$

which is of course also equal to $(I_{F(G)} \otimes \varepsilon) \circ \Delta$, where $\varepsilon(f) = f(e)$. The inverse axiom under the functor composition is given by the *antipodal property*:

$$M \circ (S \otimes I_{F(G)}) \circ \Delta = \eta \circ \varepsilon = M \circ (I_{F(G)} \otimes S) \circ \Delta$$

Here $M : F(G) \otimes F(G) \to F(G)$ is pointwise multiplication,$(Sf)(g) = f(g^{-1})$ the *antipode*, and $\eta : \mathbb{C} \to F(G)$, $\lambda \mapsto \lambda \mathbb{1}_G$ the *unit map*. To see this process in detail see Sections 1.4, 2.1, & 2.2 of arXiv:1709.09357.

Definition A *Hopf *-algebra* is a complex unital *-algebra with a coassociative unital *-homomorphism $\Delta : H \to H \otimes_{\text{alg.}} H$, a counital character $\varepsilon \in H'$, and a linear antipode $S : H \to H$.

Theorem 1.2. *Finite groups G_1 and G_2 are isomorphic if and only if their algebras of functions $F(G_1)$ and $F(G_2)$ are isomorphic as *-Hopf algebras.*

Using the Gelfand philosophy, utilise C*-algebras to define:

Definition An *algebra of functions on a finite quantum group G* is a finite dimensional Hopf *-algebra $F(G)$ such that $F(G)$ is a C*-algebra.

Note that in general a finite dimensional Hopf *-algebra is not considered the algebra of functions on a finite quantum group. This is a long story of choices and considerations, largely covered by the Gelfand Philosophy — the algebra of functions should be a C*-algebra (or a pre-C*-algebra). This means if there is a $a^*a = 0$ for non-zero $a \in H$, it should not be considered an algebra of functions. For example the Sweedler algebra is a four-dimensional Hopf *-algebra H_S generated by g and x such that $g^2 = 1_{H_S}$, $x^2 = 0$ and $gx = -xg$. The comultiplication is given by $\Delta(g) = g \otimes g$ and $\Delta(x) = 1_{H_S} \otimes x + x \otimes g$; the counit by $\varepsilon(g) = 1$ and $\varepsilon(x) = 0$ and the antipode by $S(g) = g$ and $S(x) = gx$. $x^* = -x$ and $g^* = g$ gives H_S the structure of a Hopf*-algebra. Note that $x^*x = 0$.

Are there noncommutative algebras of functions on finite quantum groups? The answer is yes. The image of a non-abelian finite group under the $\mathbb{C}$-functor, the group algebra of G, $\mathbb{C}G$, is the algebra of functions on a quantum group, a quantum group which we denote by $\widehat{G}$. The comultiplication is dual to multiplication in $F(G)$, the counit dual to the unit map in $F(G)$, $\lambda \mapsto \lambda \mathbb{1}_G$, and the antipode dual to antipode in $F(G)$. These maps, defined on the basis elements, are given by:

$$\begin{aligned}
\Delta_{\mathbb{C}G}(\delta^s) &= \delta^s \otimes \delta^s, \\
\varepsilon_{\mathbb{C}G}(\delta^s) &= \delta_{s,e}, \\
S_{\mathbb{C}G}(\delta^s) &= \delta^{s^{-1}}.
\end{aligned}$$

The involution is $(\delta^s)^* = \delta^{s^{-1}}$.

Note that $F(\widehat{G})$ is *cocommutative*, that $\Delta_{\mathbb{C}G} \circ \tau = \Delta_{\mathbb{C}G}$, where $\tau(f \otimes g) = g \otimes f$ is the *flip* map. All commutative algebras of functions on quantum groups are classical, and all cocommutative algebras of functions on quantum groups, $F(C)$, are of the form $F(C) = \mathbb{C}G = F(\widehat{G})$, for a classical group G.

However the algebra of functions $F(\widehat{G})$ is still related to a classical group. Are there quantum groups that are neither classical nor the dual of a classical group? The answer is yes. The smallest such example is the Kac-Paljutkin quantum group $\mathfrak{G}_0$. Its algebra of functions is eight-dimensional and may be concretely realised as

$$F(\mathfrak{G}_0) = \mathbb{C}^4 \oplus M_2(\mathbb{C}).$$

Another is the family of Sekine quantum groups, Y_n, whose algebra of functions has dimension $2n^2$, and may be concretly realised as:

$$F(Y_n) = \mathbb{C}^{n^2} \oplus M_n(\mathbb{C}).$$

The dual $F(G)'$ of an algebra of functions on a finite quantum group is a an algebra of functions on another quantum group, the *dual* quantum group of G. The coalgebra maps on $F(G)'$ are the transposes of the algeba maps on $F(G)$, and the algebra maps on $F(G)'$ the transposes of the coalgebra maps on $F(G)$, and we write:

$$F(G)' = F(\widehat{G}).$$

For a classical finite group G,

$$F(\widehat{G}) \cong \mathbb{C}G \cong F(G)',$$

and $F\left(\widehat{\widehat{G}}\right) \cong F(G)$.

Before thinking about compact quantum groups, which are generalisations of compact groups, consider for a moment $F(S_n)$. Define $u_{ij} \in F(S_n)$ by:

$$u_{ij} = \mathbb{1}_{\{\sigma \in S_n \,:\, \sigma(j)=i\}}.$$

Note that each u_{ij} is a projection. If we form the matrix $u = (u_{ij})_{i,j=1}^n \in M_n(F(S_n))$, both the row and column sums are zero, and also the elements of rows/columns are pairwise orthogonal (as projections). This is to say the rows and sums are *partitions of unity*. A matrix is called a *magic unitary* if all its rows and sums are partitions of unity. It turns out that

$$F(S_n) \cong \mathrm{C}^*_{\mathrm{Comm}}\langle u_{ij} \,:\, u \text{ a magic unitary}\rangle.$$

If we *liberate* the u_{ij} from commutativity, and consider the algebras

$$A_n \cong \mathrm{C}^*\langle u_{ij} \,:\, u \text{ an } n \times n \text{ magic unitary}\rangle,$$

we find that there is an algebra of functions structure in which $A_n \equiv F(S_n)$... but only for $n = 1, 2, 3$. For $n \geqslant 4$, A_n is infinite dimensional. Consider two non-commuting projections p, q and the matrix

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

Then the C*-algebra $\langle p, q\rangle$ is an infinite dimensional subspace of A_4. We will see that a quantum group structure can be put on the A_n and we will write $A_n = C(S_n^+)$, and S_n^+ will be called the *free permutation group*. We see that, perhaps, the appropriate generalisation of a finite group is not a finite quantum group. Also, if we want to talk about infinite groups, we no longer have $\mathbb{C}(G \times G) \cong \mathbb{C}G \otimes \mathbb{C}G$.

There is a lot more that can be said about finite quantum groups but that is enough of an introduction to quantum groups. Let us move to infinite dimensions.

2 Algebraic Compact Quantum Groups

As soon as $|G| = \infty$, the comultiplication on the full algebra of functions no longer maps into the algebraic tensor product. However to keep an eye at this time on motivation, we can first work in a purely algebraic setting, and look at algebras with a *-homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\text{alg}} \mathcal{A}$.

Where μ is the Haar measure on a classical compact group G, consider the functional on $C(G)$ denoted by $\int_G$:

$$f \mapsto \int_G f(t)\, d\mu(t).$$

This is a normalised, positive functional, invariant in the sense that for each $s \in G$,

$$\begin{aligned}
&\int_G f(st)\, d\mu(t) = \int_G f(s)\, d\mu(t) = \int_G f(ts)\, d\mu(t) \\
&\Rightarrow \mathbb{1}_G \cdot \int_G f = \left(I_{C(G)} \otimes \int_G\right) \Delta(f) = \left(\int_G \otimes I_{C(G)}\right) \Delta(f)\,. \qquad (f \in C(G))
\end{aligned} \tag{1}$$

This normalised, invariant, positive functional is called the unique *Haar state.*

If a Hopf *-algebra has a Haar state, denoted $\int_G$, such an object is called an algebra of regular functions on an (algebraic) compact quantum group. Later we will see that algebras of regular functions on algebraic compact quantum groups have various different completions to what are called algebras of continuous functions on compact quantum groups:

$$\mathcal{A} \hookrightarrow C_\alpha(G).$$

The index α refers to the fact that there are various completions. We will learn that starting with what we will call the algebra of continuous functions on a compact quantum group, $C(G)$, we will find a dense subalgebra of regular functions $\mathcal{A} \subset C(G)$. In the commutative case, where G is a compact Hausdorff group, the dense subalgebra is given by $\mathcal{O}(G)$, the algebra of *regular*[1] or *representative functions* on G. Therefore we denote the algebra of regular functions on an algebraic compact quantum group by $\mathcal{O}(G)$ for some (virtual) compact quantum group G. Moreover, given any algebra of continuous functions on a compact quantum group, if we surject onto the dense subalgebra $\mathcal{A}$, complete to $C_\alpha(G)$, and then surject onto the dense subalgebra $\mathcal{A}_\alpha$ associated to $C_\alpha(G)$:

$$C(G) \twoheadrightarrow \mathcal{A} \hookrightarrow C_\alpha(G) \twoheadrightarrow \mathcal{A}_\alpha,$$

it turns out that $\mathcal{A} \cong \mathcal{A}_\alpha$ as algebras of regular functions on algebraic compact quantum groups. In this sense, we can identify compact quantum groups with non-isomorphic algebra of continuous functions $C_\alpha(G)$ and $C_\beta(G)$ if their dense algebra of regular functions are isomorphic.

Definition An *algebraic compact quantum group G* is given by its algebra of regular functions, a Hopf *-algebra $\mathcal{O}(G)$ with a Haar state $\int_G \in \mathcal{S}(\mathcal{O}(G)) =: M_p(G)$.

[1] $f \in \mathcal{O}(G)$ if $f(g) = \phi(\pi(g)v)$ for $\pi : G \to \mathrm{GL}(V)$ a continuous representation, $\phi \in V'$, and $v \in V$

2.1 Representations of Algebraic Compact Quantum Groups

2.1.1 Motivation/Introduction

Let V be a vector space and for motivation consider a finite group G. A *representation* of G on V is a right linear group action $\Phi : V \times G \to V$. Define a *representation matrix* $\rho : G \to GL(V)$ by

$$\Phi(v, s) = \rho(s^{-1})v,$$

which is a group homomorphism $\rho : G \to GL(V)$. Linearly extending Φ to $V \times \mathbb{C}G$ (using the embedding of G into $\mathbb{C}G$, $s \hookrightarrow \delta^s$) gives the bilinear map:

$$\overline{\Phi} : V \times \mathbb{C}G \to V\,,\ \ (v, \delta^s) \mapsto \rho(\delta^{s^{-1}})v$$

and then the linear map

$$\tilde{\Phi} : V \otimes \mathbb{C}G \to V\,,\ \ v \otimes \delta^s \mapsto \rho(\delta^{s^{-1}})v.$$

The properties that make Φ an action are encoded by two relations involving $\tilde{\Phi}$, ∇ and $\eta_{\mathbb{C}G}$. The first is compatibility:

$$\tilde{\Phi} \circ (I_V \otimes \nabla)(v \otimes \delta^s \otimes \delta^h) = \tilde{\Phi} \circ (\tilde{\Phi} \otimes I_{\mathbb{C}G})(v \otimes \delta^s \otimes \delta^h).$$

The relation for identity emanates from

$$\tilde{\Phi} \circ (I_V \otimes \eta_{\mathbb{C}G})v \cong \tilde{\Phi} \circ (I_V \otimes \eta_{\mathbb{C}G})(v \otimes 1_{\mathbb{C}}) = \tilde{\Phi}(v \otimes \delta^e) = I_V v,$$

i.e. $\tilde{\Phi} \circ (I_V \otimes \eta_{\mathbb{C}G}) = I_V$. Fix a basis $\{e_i\}$ of V and let $\{e^i\}$ be the basis of V^* dual to this basis. Now apply the dual functor to this:

$$\kappa_\rho := \mathcal{D}(\tilde{\Phi}) : V^* \to V^* \otimes F(G)\,,\ \ \kappa_\rho(e^i) = e^i \circ \tilde{\Phi}.$$

Together with the dual statements for compatibility (mpatibility!) and identity:

$$(I_{V^*} \otimes \Delta) \circ \kappa_\rho = (\kappa_\rho \otimes I_{F(G)}) \circ \kappa_\rho \tag{2}$$

$$(I_{V^*} \otimes \varepsilon) \circ \kappa_\rho = I_{V^*}, \tag{3}$$

this motivates the definition of a corepresentation of the algebra of regular functions on an algebraic compact quantum group on a complex vector space. Recall that $\Phi(u, s) = \rho(s^{-1})u$ and so

$$\kappa_\rho(v) = \sum_{t \in G} \Phi(v, t) \otimes \delta_{t^{-1}}.$$

Proposition 2.1. *For any group homomorphism $\rho : G \to GL(V)$ on a finite group, the map κ_ρ,* induced by the representation ρ, *given by*

$$\kappa_\rho(v) = \sum_{t \in G} \rho(t)v \otimes \delta_t$$

is a corepresentation of $F(G)$ on V.

On the other hand, suppose that κ_α is a corepresentation of $F(G)$ on V_α (with basis $\{e_i\}$) given by

$$\kappa_\alpha(e_j) = \sum_{i=1}^{d_\alpha} e_i \otimes \rho_{ij}.$$

Then the map $\rho_\alpha : G \to GL(V_\alpha)$, $s \mapsto (\rho_{ij}(s))$ is a group homomorphism •

Definition A *corepresentation* of the algebra of regular functions on an algebraic compact quantum group G on a complex vector space is a linear map $\kappa : V \to V \otimes \mathcal{O}(G)$ that satisfies:

$$(\kappa \otimes I_{\mathcal{O}(G)}) \circ \kappa = (I_V \otimes \Delta) \circ \kappa \qquad \text{and} \qquad (I_V \otimes \varepsilon) \circ \kappa = I_V.$$

Therefore, using the Gelfand philosophy, a corepresentation of the algebra of regular functions on an algebraic compact quantum group $\mathcal{O}(G)$ may be called a *representation* of the quantum group G.

Suppose that V is an inner product space. Define an $\mathcal{O}(G)$*-valued sesquilinear* form on $V \otimes \mathcal{O}(G)$ by:

$$\langle v \otimes f \,|\, w \otimes g \rangle_{\mathcal{O}(G)} = \langle v, w \rangle f^* g.$$

Definition Let $\kappa : V \to V \otimes \mathcal{O}(G)$ be a representation of G.

- a subspace $W \subset V$ is *invariant* with respect to κ if $\kappa(W) \subset W \otimes C(G)$
- κ is *irreducible* if V has no non-trivial invariant subspaces
- κ is *unitary* if $\langle \kappa(v) \,|\, \kappa(w) \rangle_{\mathcal{O}(G)} = \langle v, w \rangle \mathbb{1}_G$
- when a linear map $T : V \to V_0$ satisfies

  $$\kappa_0 \circ T = (T \otimes I_{\mathcal{O}(G)}) \circ \kappa,$$

  for a representation κ_0 of G on a vector space V_0, it is said to *intertwine* κ and κ_0 and be an *intertwiner* from κ to κ_0
- if T is invertible then κ and κ_0 are *equivalent*

Proposition 2.2. *Let V be a vector space with basis $(e_i)_i$ and $\kappa : V \to V \otimes \mathcal{O}(G)$ a linear map. Define $\rho = (\rho_{ij})_{i,j} \subset \mathcal{O}(G)$ by*

$$\kappa(e_j) = \sum_i e_i \otimes \rho_{ij}.$$

The following are equivalent:

(a1) $(\kappa \otimes I_{\mathcal{O}(G)}) \circ \kappa = (I_V \otimes \Delta) \circ \kappa$

(a2) $\Delta(\rho_{ij}) = \sum_k \rho_{ik} \otimes \rho_{kj}$

Assume that (a1), (a2) hold. The following are equivalent:

(b1) $(I_V \otimes \varepsilon) \circ \kappa = I_V$

(b2) $\varepsilon(\rho_{ij}) = \delta_{i,j}$

Assume that these four conditions hold so that κ is a represention. Suppose further that V is an inner product space and $(e_i)_i$ orthonormal. The following are equivalent:

(c1) κ *is unitary*

(c2) $S(\rho_{ij}) = \rho_{ji}^*$

(c3) $\sum_k \rho_{ki}^* \rho_{kj} = \delta_{i,j}$

Proof. Looking at (a) and (b):

(a1)⇔(a2) Write $\kappa(e_j) = \sum_k e_k \otimes \rho_{kj}$:

$$(\kappa \otimes I_{\mathcal{O}(G)}) \circ \kappa(ej) = \sum_{i,k} e_i \otimes \rho_{ik} \otimes \rho_{kj} = \sum_i e_i \otimes \left(\sum_k \rho_{ik} \otimes \rho_{kj} \right)$$

$$(I_V \otimes \Delta) \circ \kappa(e_j) = \sum_i e_i \otimes \Delta(\rho_{ij}).$$

(b1)⇔(b2) Note

$$(I_V \otimes \varepsilon) \circ \kappa(e_j) = \sum_i e_i \varepsilon(\rho_{ij}) \stackrel{!}{=} e_j \Leftrightarrow \varepsilon(\rho_{ij}) = \delta_{i,j} \quad \bullet$$

2.1.2 Decomposition into Irreducible Representations

The below is a generalisation of results true for classical compact groups. The proofs presented by Timmermann uses the map $\mathcal{O}(G)' \ni \nu \mapsto \breve{\nu}(\kappa) \in L(V)$, where the V is associated with a representation of G, $\kappa : V \to V \otimes \mathcal{O}(G)$. The map $\breve{\nu}(\kappa)$ is given by:

$$\breve{\nu}(\kappa)(v) = (I_V \otimes \nu)\kappa(v).$$

The map, $\breve{\mathcal{F}}_\kappa : \mathcal{O}(G)' \to L(V)$, $\nu \mapsto \breve{\nu}(\kappa)$ is a unital algebra *-homomorphism if κ is unitary. A subspace $W \subset V$ is invariant for κ if and only if it is invariant for each $\breve{\mathcal{F}}_\kappa(\nu)$, for $\nu \in \mathcal{O}(G)'$. Furthermore an intertwiner of κ and κ_0 will also intertwine $\breve{\mathcal{F}}_\kappa$ and $\breve{\mathcal{F}}_{\kappa_0}$.

Theorem 2.3. *Let κ be a representation of a compact group G on a vector space V.*

i. κ is equivalent to a unitary representation.

ii. if κ is unitary, and $W \subset V$ invariant, then the orthogonal complement $W^\perp \subset V$ is also invariant

iii. every $v \in V$ is contained in some finite-dimensional invariant subspace of V. In particular, V has finite dimension if κ is irreducible

iv. κ is equivalent to a direct sum of finite-dimensional irreducible unitary representations

Proof. i. We find an inner product on V such that κ is unitary with respect to it. Let $\langle , \rangle_V$ be an inner product on V. Define an inner product on $V \otimes \mathcal{O}(G)$

$$\langle v \otimes f, w \otimes g \rangle_{V \otimes \mathcal{O}(G)} = \langle v, w \rangle_V \int_G f^* g.$$

On the other hand, it can be shown that κ is injective and thus we have an alternative inner product on V:

$$\langle v, w \rangle_\kappa = \langle \kappa(v), \kappa(w) \rangle_{V \otimes \mathcal{O}(G)}.$$

This is the averaging trick. Using this inner product we can define an $\mathcal{O}(G)$-valued sesquilinear form:

$$\langle v \otimes f \,|\, w \otimes g \rangle_1 := \langle v, w \rangle_\kappa f^* g.$$

Exercise: Use the right invariance of $\int_G$ to show that calculating $\left(\int \otimes I_{\mathcal{O}(G)} \right) \Delta(\rho^*_{mi} \rho_{nj})$ gives:

$$\sum_{r,s} \left(\int_G \rho^*_{mr} \rho_{ns} \right) \rho^*_{ri} \rho_{sj} = \left(\int_G \rho^*_{mi} \rho_{nj} \right) \mathbb{1}_G \tag{4}$$

Hence show that, where $\kappa : e_j \mapsto \sum_i e_i \otimes \rho_{ij}$:

$$\langle \kappa(e_i) | \kappa(e_j) \rangle_1 = \langle e_i, e_j \rangle_\kappa \mathbb{1}_G.$$

Therefore with respect to the inner product $\langle , \rangle_\kappa$, κ is unitary.

ii. The invariance of W under κ gives invariance for each $\check{\mathcal{F}}_\kappa(\nu)$. Let $\langle , \rangle$ be an inner product on W and suppose that there exists a $u \in W^\perp$ such that there is an $\varphi \in \mathcal{O}(G)'$ such that $\check{\mathcal{F}}_\kappa(\omega)u \notin W^\perp$. We know that for all $w \in W$, and because $\check{\mathcal{F}}_\kappa : \mathcal{O}(G)' \to L(V)$ is a *-homomorphism:

$$\begin{aligned}\langle w, \check{\mathcal{F}}_\kappa(\omega)u \rangle &= \langle w, \check{\mathcal{F}}_\kappa(\omega^*)^* u \rangle \\ &= \langle \check{\mathcal{F}}_\kappa(\omega^*) w, u \rangle = 0,\end{aligned}$$

so that $\check{\mathcal{F}}_\kappa(\omega)u \in W^\perp$.

iii. Note that because $\check{\mathcal{F}}_\kappa(\varepsilon)v = v$, $\check{\mathcal{F}}_\kappa(\mathcal{O}(G)')v$ contains v. As $\check{\mathcal{F}}_\kappa(\nu)\check{\mathcal{F}}_\kappa(\mu) = \check{\mathcal{F}}_\kappa(\nu \star \mu)$, $\check{\mathcal{F}}_\kappa(\mathcal{O}(G)')v$ is invariant for $\check{\mathcal{F}}_\kappa$, and so also for κ. If $\kappa(v) = \sum_i v_i \otimes \rho_i$, with $v_i \in V$, and $\rho_i \in \mathcal{O}(G)$, then

$$\check{\mathcal{F}}_\kappa(\nu)(v) = \sum_i \nu(\rho_i) v_i,$$

and so $\check{\mathcal{F}}_\kappa(\mathcal{O}(G)')v$ is contained in the linear span of the v_i, and so finite dimensional.

iv. Presumably very similar to the classical case, via Zorn's Lemma •

Proposition 2.4. *(Schur's Lemma) Let κ_V, κ_W be representations on vector spaces V and W.*

i. For every intertwiner from κ_V to κ_W, the subspaces $\ker T$ resp. $\operatorname{Im} T$ are invariant for κ_V resp. κ_W.

ii. Assume that κ_V and κ_W are irreducible. Either the representations are inequivalent and $\hom(\kappa_V, \kappa_W) = \{0\}$, or the interwiner is a scalar.

Proof. i. Let $\check{\mathcal{F}}_V := \check{\mathcal{F}}_{\kappa_V}$ and similar for $\check{\mathcal{F}}_W$. The intertwiner T also intertwines $\check{\mathcal{F}}_V$ and $\check{\mathcal{F}}_W$ so that:

$$\check{\mathcal{F}}_W(\mathcal{O}(G)')T = T\check{\mathcal{F}}_V(\mathcal{O}(G)').$$

Let $v \in \ker T$ so that $Tv = 0$ and so

$$\check{\mathcal{F}}_W(\mathcal{O}(G)')Tv = T\check{\mathcal{F}}_V(\mathcal{O}(G)')v = 0,$$

so that $\check{\mathcal{F}}_V(\mathcal{O}(G)')v \in \ker T$.

Let $w \in W = Tv \in \operatorname{Im} T$:

$$\check{\mathcal{F}}_W(\mathcal{O}(G)') \underbrace{w}_{=Tv} = T\check{\mathcal{F}}_V(\mathcal{O}(G))v \in \operatorname{Im} T.$$

ii. By irreducibility, the kernel and image of T is either trivial or the whole space. If T is non-zero, $\ker T = \{0\}$, $\operatorname{Im} T = W$ and so T is a linear isomorphism, so that $\kappa_V \cong \kappa_W =: \kappa$. As T

is a linear isomorphism between isomorphic finite dimensional vector spaces, $V \cong W$, it has a non-zero eigenvalue $\lambda \in \mathbb{C}$ with associated eigenvector v_λ. Consider $T_\lambda := T - \lambda I$:

$$\begin{aligned}\check{\mathcal{F}}_\kappa(\mathcal{O}(G)')T_\lambda(v) &= \check{\mathcal{F}}_\kappa(\mathcal{O}(G)')Tv - \check{\mathcal{F}}_\kappa(\mathcal{O}(G)')\lambda v \\ &= T\check{\mathcal{F}}_\kappa(\mathcal{O}(G)')v - \lambda I\check{\mathcal{F}}_\kappa(\mathcal{O}(G))v \\ &= (T-\lambda I)\check{\mathcal{F}}_\kappa(\mathcal{O}(G)')v = T_\lambda\check{\mathcal{F}}_\kappa(\mathcal{O}(G)')v,\end{aligned}$$

so that T_λ is an intertwiner from κ to itself. As $v_\lambda \in \ker T_\lambda$, $\ker T_\lambda = V$ and so $T_\lambda = 0$ •

Let $(\kappa_\alpha)_\alpha$ be a family of representations of an algebraic compact quantum group on vector spaces (V_α)

Theorem 2.5. *Let $\mathcal{O}(G)$ be the algebra of functions on a compact quantum group and $(\kappa_\alpha)_{\alpha\in\mathrm{Irr}(G)}$ a maximal family of pairwise inequivalent irreducible representations of G. Every representation κ of G is equivalent to a direct sum $\boxplus_k \kappa_{\alpha_k}$, where $(\alpha_k)_k$ is some family of elements of* $\mathrm{Irr}(G)$ •

2.2 Orthogonality Relations

A *corepresentation matrix* is a matrix $\rho = (\rho_{ij}) \in M_n(\mathcal{O}(G))$ such that:

$$\Delta(\rho_{ij}) = \sum_{k=1}^{n} \rho_{ik} \otimes \rho_{kj}.$$

In the obvious way it gives a representation of G on $\mathbb{C}^n$, $\kappa_\rho : e_j \mapsto \sum_{i=1}^n e_i \otimes \rho_{ij}$, and we say that ρ is irreducible if κ_ρ is. The corepresentation is unitary if and only if $\rho^*\rho = I_{M_n(\mathcal{O}(G))}$, i.e. $\rho^*\rho = \mathrm{diag}(\mathbb{1}_G, \dots, \mathbb{1}_G)$. A (co)representation of G given by $(\rho_{ij})_{i,j=1}^n$ has an associated *corepresentation operator*:

$$X_\rho = \sum_{i,j=1}^{n} E_{i,j} \otimes \rho_{ij}.$$

The following requires some Hopf*-algebra preliminaries. The antipode $S : \mathcal{O}(G) \to \mathcal{O}(G)$ is bijective, and if (ρ_{ij}) is the corepresentation matrix of a unitary representation of G then $S^{-1}(\rho_{ji}^*) = \rho_{ij}$. Furthermore we have the following:

Lemma 2.6. *(Lemma 2.2.12,[Tim08]) For f_1, $f_2 \in \mathcal{O}(G)$:*

$$\left(\int_G \otimes S\right)((f_1 \otimes \mathbb{1}_G)\Delta(f_2)) = \left(\int_G \otimes I_{\mathcal{O}(G)}\right)(\Delta(f_1)(f_2 \otimes \mathbb{1}_G)) \tag{5}$$

$$\left(S \otimes \int_G\right)(\Delta(f_1)(\mathbb{1}_G \otimes f_2)) = \left(I_{\mathcal{O}(G)} \otimes \int_G\right)((1 \otimes f_1)\Delta(f_2) \quad \bullet \tag{6}$$

This gives us a more pedestrian but easier proof than Timmermann of:

Proposition 2.7. *Let κ_V and κ_W be inequivalent irreducible unitary corepresentations with corepresentation matrices (ρ_{ij}^V) and (ρ_{ij}^W) on vector spaces V and W. Then for any matrix elements:*

$$\int_G ((\rho_{ij}^W)^* \rho_{k\ell}^V) = 0 = \int_G \rho_{ij}^W (\rho_{k\ell}^V)^*.$$

Proof. Consider for a fixed $1 \leqslant i \leqslant \dim W$ and $1 \leqslant k \leqslant \dim V$, and (v_i), (w_i) bases of V and W, the linear map $T_{ik} : V \to W$ given by:

$$T_{ik}(v_\ell) = \sum_{j=1}^{\dim W} \left[\int_G \left(\rho_{ij}^W\right)^* \rho_{k\ell}^V \right] w_j.$$

The claim is that this is an intertwiner from κ_V to κ_W.

Exercise: Recall that $T : V \to W$ is an interwiner if

$$\kappa_W(T_1(v_\ell)) = (T_1 \otimes I_{\mathcal{O}(G)})\kappa_V(v_\ell).$$

Show that T is an intertwiner if, for each $1 \leqslant c \leqslant \dim W$ and $1 \leqslant \ell \leqslant \dim V$

$$\sum_{j=1}^{\dim W} \left[\int_G \left(\rho_{ij}^W\right)^* \rho_{k\ell}^V \right] \rho_{cj}^W = \sum_{j=1}^{\dim V} \left[\int_G \left(\rho_{ic}^W\right)^* \rho_{kj}^V \right] \rho_{j\ell}^V. \tag{7}$$

By Lemma 2.6,

$$\left(\int_G \otimes S \right) \left((\rho_{ic}^W)^* \otimes \mathbb{1}_G \right) \Delta(\rho_{k\ell}^V) = \left(\int_G \otimes I_{\mathcal{O}(G)} \right) \Delta(\left(\rho_{ic}^W\right)^*)(\rho_{k\ell}^V \otimes \mathbb{1}_G).$$

Exercise: Develop these equations until the application of S^{-1} to both sides gives (7).

Now T_{ik} is an intertwiner between inequivalent representations and so by Schur's Lemma 2.4, T_{ik} must be zero. Half the result follows. The other half follows in a similar manner •

Corollary 2.8. *For any non-trivial irreducible representation* κ_α, $\int_G \rho_{ij}^\alpha = 0$.

Proof. Consider the trivial matrix element $\mathbb{1}_G$ together with ρ_{ij}^α:

$$\int_G \mathbb{1}_G^* \rho_{ij}^\alpha = 0 \qquad \bullet$$

Given an irreducible unitary representation given by (ρ_{ij}), the transpose need not be a representation matrix. However, it can be shown that its *inverse* say $\left(\rho^t\right)^{-1}$ *is* a corepresentation matrix, and is equivalent to (ρ_{ij}^*).

Proposition 2.9. *Let $\mathcal{O}(G)$ be the algebra of regular functions on an algebraic compact quantum group, and let $\rho = (\rho_{ij}) \in M_n(\mathcal{O}(G))$ be an irreducible corepresentation matrix.*

i. The matrix $\rho^t := (\rho_{ji})$ is invertible, and its inverse $\rho^{-t} = (\rho^t)^{-1}$ is a corepresentation matrix.

ii. There exists a unique intertwiner $F \in \hom(\overline{\rho}, \rho^{-t})$ such that $\operatorname{Tr}(F) = \operatorname{Tr}(F^{-1}) > 0$ and this intertwiner is invertible and positive definite.

iii. For all i, j, k, l,

$$\int_G (S(\rho_{ji})\rho_{kl}) = \int_G (\rho_{ij}^* \rho_{kl}) = \frac{\delta_{j,l}}{\operatorname{Tr}(F^{-1})} (F^{-1})_{ik},$$
$$\int_G (\rho_{ij} S(\rho_{lk})) = \int_G (\rho_{ij} \rho_{kl}^*) = \frac{\delta_{i,k}}{\operatorname{Tr}(F)} F_{jl}$$

iv. The ρ_{ij} are linearly independent.

Proof. i. By Theorem 2.3 i., $\overline{\rho} = (\rho_{ij}^*)$ is equivalent to a unitary corepresentation matrix,, so there exists $T \in \mathrm{GL}(\mathbb{C}^n)$ such that:

$$\varrho := T\overline{\rho}T^{-1}$$

is unitary, and thus invertible with $\varrho^{-1} = \varrho^* = (T^{-1})^*\rho^t T^*$. Note that $\overline{\rho} = T^{-1}\varrho T$ and so

$$\rho^t = \overline{\rho}^* = T^*\varrho^*(T^{-1})^*$$

is invertible, with inverse

$$\rho^{-t} = T^*\varrho(T^*)^{-1} = T^*T\overline{\rho}T^{-1}(T^*)^{-1} = |T|^2\overline{\rho}|T|^{-2},$$

so that ρ^{-t} is intertwined with $\overline{\rho}$ with an invertible intertwiner. Thus ρ^{-t} is a corepresentation matrix.

ii. $F := |T|^2$ is an intertwiner of $\overline{\rho}$ and ρ^{-t}. There are some things to check, such as that $\overline{\rho}$ is irreducible as ρ is, and equivalent representations ($\overline{\rho}$ and ρ^{-t}) are either both reducible or both irreducible.
Exercise: Show that if

$$c := \sqrt{\frac{\mathrm{Tr}(|T|^{-2})}{\mathrm{Tr}(|T|^2)}},$$

then $F := c|T|^2$ has the desired properties.

iii. For $i, k \in \{1, \ldots, n\}$ define

$$T_{ik}(e_\ell) = \sum_{j=1}^n \left[\int_G \rho_{ij}^*\rho_{k\ell}\right] e_j.$$

Using similar calculations to those used in the proof of Proposition 2.7, it can be shown that T_{ik} is an intertwiner from κ_V to itself, and so by Schur's Lemma it is a multiple of the identity, $T_{ik} = \lambda_{ik}I_n$.

Let $\theta = (\theta_{ij})_{i,j=1}^n$ be the representation matrix of $(\rho^t)^{-1}$. As a representation matrix, θ is invertible with inverse $(S(\theta_{ij}))$, and of course this is also equal to ρ^t so $S(\theta_{ij}) = \rho_{ji}$.
Consider the map $R_{j\ell} : V \to \overline{V}$ defined by:

$$R_{j\ell}(e_k) = \sum_{i=1}^n \left[\int_G \rho_{ij}^*\rho_{k\ell}\right] \overline{e_i}.$$

Exercise: Show that $R_{j\ell} \in \hom(\kappa_\theta, \overline{\kappa})$ if for all $i, s \in \{1, \ldots, n\}$:

$$\sum_{r=1}^n \left[\int_G \rho_{ij}^*\rho_{s\ell}\right] \theta_{rs} = \sum_{r=1}^n \left[\int_G \rho_{rj}^*\rho_{s\ell}\right] \rho_{ir}^*. \tag{8}$$

By applying the antipode to both sides of (8), and rewriting $\rho_{ij}^* = S(\rho_{ji})$, etc., $R_{j\ell}$ is an intertwiner if:

$$\sum_{r=1}^n \left[S(\rho_{ji})\rho_{s\ell}\right] \rho_{sr} = \sum_{r=1}^n \left[\int_G S(\rho_{jr})\rho_{s\ell}\right] S(\rho_{ir}^*).$$

Exercise: Apply (6) with $f_1 = S(\rho_{ji})$ and $f_2 = \rho_{s\ell}$ to prove that $R_{j\ell}$ is an interwiner.

By 2.4, the space of intertwiners from $\overline{\rho}$ to ρ^{-t} is one dimensional and so $R_{j\ell}$ is a constant $\mu_{k\ell}$ times F^{-1}.

Hence

$$\begin{aligned}\mu_{j\ell}[F^{-1}]_{ik} = [R_{j\ell}]_{ik} &= \langle \overline{e_i} \mid R_{j\ell} e_k \rangle \\ &= \int_G \rho^*_{ij}\rho_{k\ell} = \langle e_j \mid T_{ik} e_\ell \rangle = \lambda_{ik}\delta_{j,\ell}\end{aligned}$$

and so $\mu_{k\ell} = 0$ for $j \neq \ell$ and $\mu := \mu_{jj}$ is independent of j. Note

$$\begin{aligned}\mu \operatorname{Tr}(F^{-1}) = \sum_{i=1}^n \mu [F^{-1}]_{ii} &= \sum_{i=1}^n \langle \overline{e_i} \mid R_{jj} e_i \rangle \\ &= \sum_{i=1}^n \langle e_j \mid T_{ii} e_j \rangle = \sum_{i=1}^n \int_G \rho^*_{ij}\rho_{ij} \\ &= \int_G \sum_{i=1} \rho^*_{ij}\rho_{ij} = \int_G \mathbb{1}_G = 1\end{aligned}$$

by Proposition 2.2 (c). Thus $\mu = \frac{1}{\operatorname{Tr}(F^{-1})}$. The other result follows in a similar manner.

iv. Assume that there exist $(\lambda_{k\ell}) \subset \mathbb{C}$ such that:

$$\begin{aligned}&\sum_{k,\ell} \lambda_{k\ell}\rho_{k\ell} = 0 \\ \underset{\times \rho^*_{ij}}{\Rightarrow} &\sum_{k,\ell} \lambda_{k\ell}\rho^*_{ij}\rho_{k\ell} = 0 \\ \underset{\int_G}{\Rightarrow} &\sum_{k,\ell} \lambda_{k\ell} \int_G \rho^*_{ij}\rho_{k\ell} = 0 \\ \Rightarrow \frac{1}{\operatorname{Tr}(F^{-1})} &\sum_k (F^{-1})_{ik}\lambda_{kj} = 0,\end{aligned}$$

but columns of invertible matrices are linearly independent so the $\lambda_{kj} = 0$ •

The comultiplication is a representation of G on $\mathcal{O}(G)$, and is a direct sum of finite dimensional irreducible unitary representations. Let (f_i) be a basis of $\mathcal{O}(G)$ so that the matrix elements of Δ are f_{ij} via:

$$\Delta(f_i) = \sum_j f_j \otimes f_{ij}.$$

Note that $(\varepsilon \otimes I_{\mathcal{O}(G)}) \left(\sum_j f_j \otimes f_{ij} \right) \in \operatorname{span}(f_{ij})$, however by the counital property: $(\varepsilon \otimes I_{\mathcal{O}(G)})\Delta(f_i) = f_i \in \operatorname{span}(f_{ij})$. That is the matrix elements of Δ span $\mathcal{O}(G)$. Therefore the matrix elements of irreducible finite dimensional unitary representations span $\mathcal{O}(G)$, and so generated by them as an algebra. A little care shows that, where $\operatorname{Irr}(G)$ is a maximal family of pairwise inequivalent irreducible unitary representations,

$$\{\rho^\alpha_{ij} \mid i,j = 1,\dots,\dim V_\alpha,\ \alpha \in \operatorname{Irr}(G)\}$$

is a basis of $\mathcal{O}(G)$.

2.3 Modular Properties of the Haar State

The Haar state is not in general a trace, and how badly it fails to be a trace is captured by the *Wororonwicz characters* $\varphi_z : \mathcal{O}(G) \to \mathbb{C}$. It is useful at this time to note that the dimension of a representation d_α can differ from what is known as the *quantum dimension* $n_\alpha = \text{Tr}(Q_\alpha)$. All of these technicalities disappear as soon as $S^2 = I_{\mathcal{O}(G)}$ which is the *Kac condition*. In this case the Woronowicz characters are trivial.

Proposition 2.10. *Let $\rho \in M_n(\mathcal{O}(G))$ be an irreducible unitary representation. Then $S^2(\rho)$ is an irreducible representation and there exists a unique invertible $Q \in \text{hom}(\rho, S^2(\rho))$ such that $\text{Tr}(Q) = \text{Tr}(Q^{-1}) > 0$. This Q is positive definite, and the $F \in \text{hom}(\overline{\rho}, \rho^{-t})$ constructed in Proposition 2.9 is equal to $\overline{Q}$.*

Proof. $\overline{\rho}$ is equivalent to a unitary representation ϱ so there exists an invertible $T \in M_n(\mathbb{C})$ such that $\varrho := T\overline{\rho}T^{-1}$ is unitary. As ρ is unitary, $\overline{\rho} = S(\rho)^t$. Note

$$\begin{aligned} \varrho^t &= T^{-t}\overline{\rho}^t T^t \\ \Rightarrow S(\varrho^t) &= T^{-t}S(\overline{\rho}^t)T^t \end{aligned}$$

as S is a linear map. As ϱ is unitary, $S(\varrho^t) = S(\varrho)^t$. So applying conjugation to $\varrho = T\overline{\rho}T^{-1}$:

$$\begin{aligned} \overline{T}\rho\overline{T}^{-1} &= \overline{\varrho} = S(\varrho)^t \\ &= S(\varrho^t) = T^{-t}S(\overline{\rho}^t)T^t \\ &= T^{-t}S(\big(S(\rho)^t\big)^t)T^t = T^{-t}S^2(\rho)T^t \\ \Rightarrow \rho &= (\overline{T}^{-1}T^{-t})S^2(\rho)T^t\overline{T} = (T^t\overline{T})^{-1}S^2(\rho)(T^t\overline{T}) \end{aligned}$$

Similarly to Proposition 2.9, this implies that $S^2(\rho)$ is an irreducible corepresentation matrix and Q is unique, and equal to $\lambda T^t\overline{T} = \lambda\overline{T}^*\overline{T}$ for some $\lambda > 0$. The rest can be seen by way of comparison with elements of the proof of Proposition 2.9 •

Following [Tim08], Section 1.3.2, there are convolution maps $\mathcal{O}(G) \times \mathcal{O}(G)' \to \mathcal{O}(G)$ and $\mathcal{O}(G)' \times \mathcal{O}(G) \to \mathcal{O}(G)$, that is associative with respect to $\mathcal{O}(G)' \times \mathcal{O}(G) \times \mathcal{O}(G)' \to \mathcal{O}(G)$. These convolution maps hit the $\mathcal{O}(G)$ part with the comultiplication, and then wrong-sided-ly hit with the functional, in other words:

$$\varphi \star f = (I_{\mathcal{O}(G)} \otimes \varphi)\Delta(f) \text{ and } f \star \varphi = (\varphi \otimes I_{\mathcal{O}(G)})\Delta(f).$$

Note by the counital property that $\varepsilon \star f = f = f \star \varepsilon$. The convolution of functionals $\varphi_1, \varphi_2 \in \mathcal{O}(G)'$ is given by:

$$\varphi_1 \star \varphi_2 = (\varphi_1 \otimes \varphi_2)\Delta.$$

Theorem 2.11. *Let G be an algebraic compact quantum group. Then there exists a family $(\varphi_z)_{z\in\mathbb{C}}$ of characters on $\mathcal{O}(G)$ such that:*

1. $\varphi_0 = \varepsilon$ *and* $\varphi_w \star \varphi_z = \varphi_{w+z}$.
2. $\varphi_z(\mathbb{1}_G) = 1$ *and* $\varphi_z(S(f)) = \varphi_{-z}(f)$
3. $S^2(f) = \varphi_{-1} \star f$
4. $\int_G fg = \int_G g(\varphi_1 \star f \star \varphi_{-1})$

Proof. For an irreducible representation ρ^α, let $Q_\alpha \in \mathrm{GL}_{d_\rho}(\mathbb{C})$ be the positive definite intertwiner from ρ^α to $S^2(\rho^\alpha)$. As the spectrum of $Q_\alpha \in (0,\infty)$, its natural logarithm can be defined using functional calculus, and the complex power:

$$Q_\alpha^z = e^{z \ln Q_\alpha},$$

is well defined. Define for $\rho_{ij}^\alpha \in \mathcal{O}(G)$:

$$\varphi_z(\rho_{ij}^\alpha) = Q[^z_\alpha]_{ij} \qquad \bullet$$

3 Compact Quantum Groups

Using the Gelfand philosophy, we should really be working inasmuch as possible with C*-algebras, but the algebras considered in the last section are not necessarily complete.

3.1 Classical Version

Lemma 3.1. *If X and Y are compact topological spaces, then $C(X \times Y) \cong C(X) \underset{\min}{\otimes} C(Y)$* $\bullet$

Let G be a compact group. The transpose of the multiplication gives a compultiplication:

$$\Delta : C(G) \to C(G \times G) \cong C(G) \underset{\min}{\otimes} C(G).$$

Proposition 3.2. *Let G be a compact group. Then both*

$$A_R := \Delta(C(G))(\mathbb{1}_G \otimes C(G)) \text{ and } A_L := \Delta(C(G))(C(G) \otimes \mathbb{1}_G)$$

are dense in $C(G \times G)$.

Proof. By Stone-Weierstrass, if $A_R \subset C(G \times G)$ separates points then it is dense. We will show this for A_R (similar for A_L) Let $s := (s_1, s_2) \in G \times G$, and $t := (t_1, t_2) \in G \times G$. Assume that $s \neq t$ cannot be separated by any element of A_R. This implies that for any pair $f, g \in C(G)$,

$$f(s_1 s_2) g(s_2) = f(t_1 t_2) g(t_2).$$

Take $f = \mathbb{1}_G$ so that for all $g \in C(G)$

$$g(s_2) = g(t_2).$$

However $C(G)$ separates points so $s_2 = t_2 = r$. Choose a g such that $g(r) \neq 0$ so that for all $f \in C(G)$:

$$f(s_1 r) = f(t_1 r).$$

However again $C(G)$ separates points so that:

$$s_1 r = t_1 r \Rightarrow s_1 = t_1 \Rightarrow s = t \qquad \bullet$$

3.2 Universal C*-Algebras

A universal C*-algebra is given by a set of generators and relations between them (given by non-commutative polynomials in the generators). Not all such universal C*-algebras exist — at the very least the presentation must give a uniform bound on the generators. A C*-seminorm on a C*-algebra A is a seminorm $p : A \to \mathbb{R}^+$ that satisfies:

$$p(xy) \leqslant p(x,y) \text{ and } p(x^*x) = p(x)^2.$$

Following Weber [cite], we have the following elements:

- Let $E = \{x_i : i \in \mathcal{I}\}$, the set of *generators*.
- Let $P(E)$ be the set of non-commutative polynomials in x_i and x_i^*.
- Let $R \subseteq P(E)$ be the set of *relations*, $p_j = 0$, that the generators satisfy e.g. $x_1^*x_1 + x_2^*x_2 - 1 = 0$.
- Let $I(R) \subseteq P(E)$ be the ideal generated by R.
- Let $A(E,R) = P(E)/I(R)$; the *universal $*$-algebra generated by E and R*
- Let
$$\|x\| = \sup\{p(x) : p \text{ a C}^*\text{-seminorm on } A(E,R)\}$$
- If for all $x \in A(E,R)$ we have $\|x\| < \infty$, then where $N_{\|\cdot\|} = \{x \in A(E,R) : \|x\| = 0\}$, we define $\mathrm{C}^*(E,R)$ the *universal* C*-*algebra generated by E and R* as the completion of the quotient $A(E,R)/N_{\|\cdot\|}$ with respect to $\|\cdot\|$.

If B is another C*-algebra such that elements $\{y_i \in B : i \in \mathcal{I}\}$ satisfy its relations R, then the universal property of $\mathrm{C}^*(E,R)$ is that there is a *-homomorphism $\mathrm{C}^*(E,R) \to B$ such that $x_i \mapsto y_i$.

There seem to be various incarnations of what a universal C*-algebra is. My 'soft' understanding is that a universal C*-algebra A can be thought of the direct sum of $B(H_\alpha)$ such that there exists operators $\{x_j\}_{j=1}^{|E|}$ such that the operators satisfy R:

$$A \subset \bigoplus_{\alpha \in \mathcal{I}} B(H_\alpha),$$

and that the projection maps $\pi_\alpha : A \to B(H_\alpha)$ are *-homomorphisms and that is how this all works. This should explain ALSO why these maps are surjections, and why the factors could be considered subgroups.

MY OTHER UNDERSTANDING IS THAT IF A UNiversal C*-algebra can be shown to be simple, then A is a subset of a unique Hilbert space. My understanding is that this is what happens for some famous

3.3 Group Algebras

Let G be a discrete group and $\mathbb{C}G$ its group algebra:

$$\mathbb{C}G = \left\{ \sum_{g \in G} \alpha_g \delta^g : \alpha \in \mathbb{C}^G \text{ with finite support} \right\}.$$

The multiplication is the linear extension of $\delta^g \otimes \delta^h \mapsto \delta^{gh}$, and $\delta^{g*} = \delta^{g^{-1}}$. A unitary representation of G on a Hilbert space H extend to (non-degenerate) representations of $\mathbb{C}G$. We can use these representations to define a norm on $\mathbb{C}G$:

$$\|f\| = \sup \{\|\pi(f)\| \colon \pi \text{ a representation of } \mathbb{C}G\}.$$

We denote the completion of $\mathbb{C}G$ in this norm by $\mathrm{C}^*(G)$, the group C^*-algebra of G.

Alternatively, the group C^*-algebra of G can be given by the universal C^*-algebra:

$$\mathrm{C}^*(u_g \colon g \in G,\, u_g \text{ unitary},\, u_g u_h = u_{gh},\, u_g^* = u_{g^{-1}}).$$

The map:

$$\Delta(u_g) = u_g \otimes_{\min} u_g$$

is a bounded densely defined *-isomorphism, as the minimal/spatial tensor norm satisfies $\|x \otimes_{\min} y\| = \|x\|\|y\|$ and thus

$$\|\Delta(u_g)\|_{\min} = \|u_g \otimes_{\min} u_g\| = \|u_g\|\|u_g\|,$$

and thus extends to $\mathrm{C}^*(G)$.

The left regular representation of G,

$$\lambda : \mathbb{C}G \to B(\ell^2(G))$$

is defined by the linear extension of

$$\lambda(g)e_h := e_{gh},$$

where $(e_h)_{h\in G}$ is an onb of the Hilbert space $\ell^2(G)$. As λ is faithful on $\mathbb{C}G$, we know that $\mathbb{C}G \cong \lambda(\mathbb{C}G) \subseteq B(\ell^2(G))$. Define

$$\mathrm{C}_r^*(G) := \overline{\lambda(\mathbb{C}G)} \subseteq B(\ell^2(G)).$$

MORE TO DO HERE. TIMM USES LOCALLY COMPACT...

3.4 Compact Quantum Groups

Here we give the definition of a Woronowicz C^*-algebra. We will motivate the definition after the fact, noting at this point that the definition does not include neither a counit nor an antipode. Note the stress on *an*: as per Remark **??**, a compact quantum group can have a family of algebra of continuous functions on it: $C_\alpha(G)$.

Definition *An* (Woronowicz C^*-) algebra of continuous functions on a compact quantum group G is a unital C^*-algebra $C(G)$ together with a C^*-morphism $\Delta : C(G) \to C(G) \otimes C(G)$ that satisfies coassociativity and *Woronowicz cancellation*:

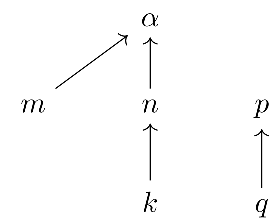
$$\overline{\Delta(C(G))(\mathbb{1}_G \otimes C(G))} = \overline{\Delta(C(G))(C(G) \otimes \mathbb{1}_G)} = C(G) \otimes C(G).$$

The *Woronowicz cancellation* property is inspired by:

Theorem 3.3. *A compact semigroup S with cancellation is a group.*

Proof. For $g \in S$, let L be the closed set of limit points of the sequence $(g^n)_{n\in\mathbb{N}}$. For each $s \in L$, $sL \subset L$.

The author is a wallflower when it comes to subsequences and their notation. Increasing functions m, n, k, $q : \mathbb{N} \to \mathbb{N}$ will be implicitly defined, with $m_\alpha := m(\alpha)$ and $n_\alpha := n(\alpha)$. Thus $(m_\alpha)_{\alpha\in\mathbb{N}}$, $(n_\alpha)_{\alpha\in\mathbb{N}}$ are subsequences of $(\alpha)_{\alpha\in\mathbb{N}}$. Define a subsequence of $(n_\alpha)_{\alpha\in\mathbb{N}}$ by $k_\alpha := k(n_\alpha)$. Define $p_\alpha := k_\alpha - m_\alpha$, and its subsequence $q_\alpha := k_{q(\alpha)} - m_{q(\alpha)}$. The following Hasse diagram helps see the various sequences:



Choose $t = \lim g^{n_\alpha} \in L$. Let $s = \lim g^{m_\alpha}$ be another element of L. By taking a subsequence of n_α, say k_α, one can assume that

$$p_\alpha := k_\alpha - m_\alpha \to \infty.$$

As S is compact, the sequence g^{p_α} has a convergent subsequence, say

$$g^{q_\alpha} \to h \in L.$$

Note

$$\begin{aligned} sh &= \left(\lim g^{m_{q(\alpha)}}\right)\left(\lim g^{q_\alpha}\right) \\ &= \left(\lim g^{m_{q(\alpha)}}\right)\left(\lim g^{k_{q(\alpha)} - m_{q(\alpha)}}\right) \\ &= \lim g^{k_{q(\alpha)}} = t, \end{aligned}$$

as $k_{q(\alpha)}$ is a subsequence of n_α and $t = \lim g^{n_\alpha}$.

This implies that for each $s \in L$, $L \subset sL$ and so $sL = L$.

Let $f \in L$ such that $sf = s$. Then, for all $u \in S$:

$$\begin{aligned} &\Rightarrow sfu = su \Rightarrow fu = u \\ &\Rightarrow ufu = uu \Rightarrow uf = u. \end{aligned}$$

Therefore $f =: e \in L$ is a unit element.

Note also that $gL = L$, and so there exists $r \in L$ such that $gr = e$, that is $r = g^{-1} \in L \subset G$ •

Proposition 3.4. *Let S be a compact semigroup and $\Delta : C(S) \to C(S) \otimes C(S) \cong C(S \times S)$ be defined by:*

$$\Delta(f)(p, q) = f(pq).$$

If S has Woronowicz cancellation it has cancellation.

Proof. Assume $\Delta(C(S))(\mathbb{1}_S \otimes C(S))$ is dense in $C(S) \otimes C(S)$. Take $p, q, r \in S$ and assume $pr = qr$. Then, for all $f, g \in C(S)$,

$$\begin{aligned} (\Delta(f)(\mathbb{1}_S \otimes g))(p, r) &= f(pr)g(r) \\ (\Delta(f)(\mathbb{1}_S \otimes g))(q, r) &= f(qr)g(r), \end{aligned}$$

and by assumption these are equal. Since this is true for all $f, g \in C(S)$, this implies that for all $h \in C(S) \otimes C(S)$

$$h(p, r) = h(q, r).$$

This implies that $p = q$ so that S has right cancellation. In a similar way we can show that S has left cancellation •

Let S be a compact semigroup with Woronowicz cancellation. By Proposition 3.4 it has cancellation. By Theorem 3.3 S is a compact group.

On the other hand, by Proposition 3.2, if G is a compact group, then it is a semigroup with Woronowicz cancellation.

Lemma 3.5. *Where X, Y are compact Hausdorff, if $\pi : C(X) \to C(Y)$ is a unital *-homomorphism, then there is a continuous map $\phi : Y \to X$ such that $\pi(f) = f \circ \phi$* •

Proposition 3.6. *Let A be a commutative Woronowicz* C**-algebra. Then $A = C(G)$ for G a compact group.*

Proof. Where $\Omega(A)$ is the set of characters $A \to \mathbb{C}$, the Gelfand transform $\varphi : A \to C(\Omega(A))$ is an isometric $*$-isomorphism from A to the algebra of continuous functions on a Hausdorff compact set. Therefore, $A \otimes A \cong C(\Omega(A)) \otimes C(\Omega(A)) \cong C(\Omega(A) \times \Omega(A))$. The map $\Delta : C(\Omega(A)) \to C(\Omega(A) \times \Omega(A))$ is a *-homomorphism and hence by there exists a continuous map $m : \Omega(A) \times \Omega(A) \to \Omega(A)$.

Let $(\chi_1, \chi_2, \chi_3) \in \Omega(A)^3$ and note for all $f \in C(\Omega(A))$:

$$\begin{aligned}(\Delta \otimes I_{C(\Omega(A))}) \circ \Delta(f)(\chi_1, \chi_2, \chi_3) &= (I_{C(\Omega(A))} \otimes \Delta) \circ \Delta(f)(\chi_1, \chi_2, \chi_3) \\ \Rightarrow \sum \Delta(f_{(1)}) \otimes f_{(2)}(\chi_1, \chi_2, \chi_3) &= \sum f_{(1)} \otimes \Delta(f_{(2)})(\chi_1, \chi_2, \chi_3) \\ \Rightarrow \sum f_{(1)}(\chi_1\chi_2) f_{(2)}(\chi_3) &= \sum f_{(1)}(\chi_1) f_{(2)}(\chi_2\chi_3) \\ \Rightarrow \Delta f(\chi_1\chi_2, \chi_3) &= \Delta(f)(\chi_1, \chi_2\chi_3) \\ \Rightarrow f \circ m(\chi_1\chi_2, \chi_3) &= f \circ m(\chi_1, \chi_2\chi_3) \\ \Rightarrow f((\chi_1\chi_2)\chi_3) &= f(\chi_1(\chi_2\chi_3))\end{aligned}$$

However $C(\Omega(A))$ separates points, and so m is associative, and so $(\Omega(A), m)$ is a compact semigroup with Woronowicz cancellation and thus a group •

Lemma 3.7. *For every $\rho \in M_p(G)$, there exists a state $h_\rho \in M_p(G)$ such that $h_\rho \star \rho = \rho \star h_\rho = h_\rho$.*

Proof. Consider

$$\rho_n = \frac{1}{n} \sum_{k=1}^{n} \rho^{\star k}.$$

The unit ball of $C(G)'$ is weak-*-compact, and so $(\rho_n)_{n \geqslant 1}$ has a weak-*-accumulation point, h_ρ. Note that:

$$\rho \star \rho_n = \rho_n \star \rho = \rho_n = \frac{1}{n}(\rho^{\star n+1} - \rho).$$

Thinking about large ρ_n approximating h_ρ, we have

$$\rho \star h_\rho = h_\rho \star \rho = h_\rho \qquad \bullet$$

Lemma 3.8. *Let h, $\rho \in M_p(G)$ such that $h \star \rho = \rho \star h = h$. If $\omega \in C(G)'$ satisfies $0 \leqslant \omega \leqslant \rho$, then $h \star \omega = \omega \star h = \omega(\mathbb{1}_G)h$.*

Theorem 3.9. *Every compact quantum group has a unique Haar measure.*

Proof. For each $\omega \in C(G)'$, define $K_\omega \subset M_p(G)$ be the set of states invariant with respect to ω in the sense that

$$h \star \omega = \omega \star h = \omega(\mathbb{1}_G)h.$$

It is closed (Why? Apparently sets defined by a formula almost always are), and thus compact with respect to the weak-*-topology ($M_p(G)$ is compact), and non-empty by Lemma 3.7. Let $h \in K_{\omega_1+\omega_2}$. Note that both ω_1, $\omega_2 \leqslant \omega_1 + \omega_2$, and so by Lemma 3.8, $h \in K_{\omega_1} \cap K_{\omega_2}$ so that:

$$K_{\omega_1+\omega_2} \subseteq K_{\omega_1} \cap K_{\omega_2}.$$

Assume that the intersection of all K_ω is empty. Thus, where the complement is with respect to $M_p(G)$:

$$\bigcup_{\omega \in C(G)'} K_\omega^c = M_p(G),$$

is an open cover of a compact set, and thus admits a finite subcover $\{K_{\omega_i}^c : i = 1, \ldots, n\}$ such that:

$$\bigcup_{i=1}^n K_{\omega_i}^c = M_p(G) \Rightarrow \bigcap_{i=1}^n K_{\omega_i} = \varnothing.$$

Let $\psi = \sum_{i=1}^n \omega_i \in C(G)'$. We know K_ψ is non-empty and:

$$K_\psi \subseteq \bigcap_{i=1}^n K_{\omega_i} = \varnothing,$$

an absurdity, and so the intersection of all K_ω is non-empty and contains a state h left- and right-invariant for all states. The standard argument shows that h is unique •

Definition The unique left- and right-invariant state on a $C(G)$ is called its *Haar* state and denoted by $\int_G$. The algebra of functions on a compact quantum group is called *reduced* if $\int_G$ is faithful.

3.5 The Quantum Permutation Group

As we saw earlier, there exists a compact quantum group S_n^+ such that $F(S_n)$ is a commutative proper subalgebra $C(S_n^+)$. The (perhaps) easier way to see this is to construct a commutative universal C*-algebra isomorphic to $F(S_n)$, and then drop the commutativity condition. The following approach follows Moritz Weber (and aspects of the original paper of Wang). The functor composition $\mathcal{D} \circ \mathbb{C}$ approach, $X \mapsto C(X)$, runs into problems. What was the $\mathbb{C}$ functor needs to map to something like the Radon measures, and I do not believe we in general have a statement like $\mathcal{M}(X \times Y) \cong \mathcal{M}(X) \otimes \mathcal{M}(Y)$. This means the functor composition approach is merely motivation for the finite case, and cannot be employed when talking about compact groups.

A right action of a compact group on a compact topological space is a continuous map:

$$\alpha : X \times G \to G,$$

satisfying, via $(xg)h = x(gh)$, and $xe = x$:

$$\alpha \circ (\alpha \times I_G) = \alpha \circ (I_X \times m) \text{ and } \alpha \circ (I_X \times e) \cong I_X. \tag{9}$$

The transpose of the action, a unital *-homomorphism, $\kappa_\alpha : C(X) \to C(X \times G) \cong C(X) \otimes C(G)$,

$$\kappa_\alpha(f) = f \circ \alpha,$$

inherits the following from (9):

$$(\kappa_\alpha \otimes I_{C(G)}) \circ \kappa_\alpha = (I_{C(X)} \otimes \Delta) \circ \kappa_\alpha \text{ and } (I_{C(X)} \otimes \varepsilon) \circ \kappa_\alpha = I_{C(X)},$$

and we might call κ_α a coaction of $C(G)$ on $C(X)$, or an action of G on X. Furthermore κ_α is a *-homomorphism.

Remark: When problems are met with the counit (later), the identity condition is replaced by a density condition (that is equivalent to the identity condition if G is classical. One can also ask that a coaction preserves a state. For example, when we replace the set $\{1, 2, \ldots, n\}$ with its quantum version $\mathbb{C}^n$, we might want an action to preserve the normalised trace, $\text{tr} \in M_p(X)$, in the sense that:

$$(\text{tr} \otimes I_{C(G)})\kappa(v) = \text{tr}(v)\mathbb{1}_G.$$

We could call the map κ_α a corepresentation (later), but a coaction must be a *-homomorphism (and possibly preserve a state).

Let $X_n = \{x_1, \ldots, x_n\}$. Note that $\text{Aut}(X_n) \cong S_n$. We can dualise X_n using the functor composition:

$$X_n \to \mathbb{C}X_n \to F(X_n) \cong \text{C}^*_{\text{comm}}\left(e_i : e_i \text{ a proj. } : \sum_{i=1}^{n} e_i = \mathbb{1}_{X_n}\right).$$

These e_i form a basis, an action is of the form $\kappa : F(X_n) \to F(X_n) \otimes C(\text{qAut}\, X_n)$:

$$\kappa(e_j) = \sum_{i=1}^{n} e_i \otimes \rho_{ij}.$$

Proposition 3.10. *The elements ρ_{ij} are:*

1. *self-adjoint*
2. *idempotent*
3. *rows $(\rho_{ij})_{j \geqslant 1}$ are partitions of unity*
4. $\Delta(\rho_{ij}) = \sum_k \rho_{ik} \otimes \rho_{kj}$.
5. *columns $(\rho_{ij})_{i \geqslant 1}$ are partitions of unity*

Proof.
1. $\kappa(e_j^*) = \kappa(e_j)^*$
2. $\kappa(e_j^2) = \kappa(e_j)^2$
3. $\kappa(\sum_j e_j) = \mathbb{1}_{X_n} \otimes \mathbb{1}_{\text{QAut}\, X_n}$
4. $(I_{F(X_n)} \otimes \Delta) \circ \kappa(e_k) = (\kappa \otimes I_{C(\text{QAut}\, X_n)}) \circ \kappa(e_k)$
5. If we insist that the coaction must preserve the normalised trace in the sense that:

$$(\text{Tr} \otimes I_{C(\text{QAut}\, X_n)}) \circ \kappa(f) = \text{Tr}(f)\mathbb{1}_{\text{QAut}\, X_n},$$

hit e_j with both sides to show that $\sum_i \rho_{ij} = \mathbb{1}_{\text{QAut}\, X_n}$ •

Denote by $\mathcal{O}(S_n^+)$ the unital *-algebra generated by these n^2 elements (ρ_{ij}). Consider the universal C*-algebra generated by these (ρ_{ij}). This is the algebra of continuous functions, $C(S_n^+)$, on the quantum permutation group S_n^+. Both the counit and the comultiplication extend from $\mathcal{O}(S_n^+)$. TO DO THIS.

Coming from the other direction, this led Wang [cite] in 1998 to the definition:

$$C(S_n^+) := \mathrm{C}^*\left(u_{ij},\, i,j \in \{1,\dots,n\} \colon u_{ij} \text{ projections}, \sum_k u_{ik} = \sum_k u_{kj} = 1\right).$$

Consider

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

Proposition 3.11. *This map is a coassociative unital *-homomorphism.*

Proof. Recall that from the universal property of universal C*-algebras that if some (v_{ij}) satisfy the relations of $C(S_n^+)$, then $u_{ij} \mapsto v_{ij}$ is a *-homomorphism. We will show that $\Delta(u_{ij}) \in C(S_n^+) \otimes_{\min} C(S_n^+)$ satisfy the relations so that $\Delta : C(S_n^+) \to C(S_n^+) \otimes_{\min} C(S_n^+)$ is a *-homomorphism. Note as the (u_{ij}) are self-adjoint (and $(a \otimes b)^* = a^* \otimes b^*$)

$$\Delta(u_{ij}) = \Delta(u_{ij})^*$$

Using the orthogonality $u_{ik}u_{i\ell} = \delta_{k,\ell}u_{ik}$ and $u_{kj}u_{\ell j} = \delta_{k,\ell}u_{kj}$:

$$\Delta(u_{ij})^2 = \Delta(u_{ij}).$$

Note

$$\begin{aligned}\sum_{k=1}^n \Delta(u_{ik}) &= \sum_k \sum_\ell u_{i\ell} \otimes u_{\ell k} \\ &= \sum_\ell u_{i\ell} \otimes \sum_k u_{\ell k} = \mathbb{1}_{S_n^+} \otimes \mathbb{1}_{S_n^+}\end{aligned}$$

Similarly $\sum_{k=1}^n \Delta(u_{kj}) = 1_{C(S_n^+)\otimes_{\min} C(S_n^+)}$ and thus Δ is a *-homomorphism. By considering

$$\Delta\left(\sum_{k=1}^n u_{ik}\right)$$

it is unital, while coassociativity is straightforward.

To prove S_n^+ is a compact quantum group we must show that:

Proposition 3.12. *$C(S_n^+)$ has Woronowicz cancellation.*

Proof. We want to show that a general $u_{ab} \otimes u_{cd} \in \Delta(C(S_n^+))(\mathbb{1}_{S_n^+} \otimes C(S_n^+))$ (the other orientation is similar). Note

$$\Delta(u_{aj})(\mathbb{1}_{S_n^+} \otimes u_{bj}) = \sum_{k=1}^n u_{ak} \otimes u_{kj}u_{bj} = u_{ab} \otimes u_{bj},$$

by orthogonality. This yields

$$u_{ab} \otimes \mathbb{1}_{S_n^+} = \sum_j u_{ab} \otimes u_{bj} = \sum_j \Delta(u_{aj})(\mathbb{1}_{S_n^+} \otimes u_{bj}) \in \Delta(C(S_n^+))(\mathbb{1}_{S_n^+} \otimes C(S_n^+)).$$

Multiply both sides on the right by $\mathbb{1}_{S_n^+} \otimes u_{cd}$:

$$u_{ab} \otimes u_{cd} = \sum_{j=1}^n \Delta(u_{aj})(\mathbb{1}_{S_n^+} \otimes u_{bj}u_{cd}) \in \Delta(C(S_n^+))(\mathbb{1}_{S_n^+} \otimes C(S_n^+)) \qquad \bullet$$

Let us explore the relationship between S_n and S_n^+. Define $\mathbb{1}_{j\to i} \in F(S_n)$ by:

$$\mathbb{1}_{j\to i} := \mathbb{1}_{\{\sigma\in S_n:\, \sigma(j)=i\}}.$$

Proposition 3.13. *The map $C(S_n^+) \to F(S_n)$, $u_{ij} \mapsto \mathbb{1}_{j\to i}$ is a *-homomorphism.*

Proof. The result follows if $(\mathbb{1}_{j\to i})_{i,j=1}^n$ satisfy the relations of $C(S_n^+)$. The $\mathbb{1}_{j\to i}$ are clearly projections. For $i \in \mathbb{N}$, every $\varrho \in S_n$ sends exactly *one* $j \in \mathbb{N}$ to i and so:

$$\sum_{j=1}^n \mathbb{1}_{j\to i}(\varrho) = 1 \Rightarrow \sum_j \mathbb{1}_{j\to i} = \mathbb{1}_{S_n}.$$

Similarly $\sum_i \mathbb{1}_{j\to i} = \mathbb{1}_{S_n}$ •

If we consider the universal *commutative* C*-algebra A^c given by the same relations as $C(S_n^+)$, we get the that $A^c = C(X)$, for X the characters of A^c. Consider the matrix:

$$u^c := \begin{bmatrix} u_{11}^c & u_{12}^c & \cdots & u_{1n}^c \\ u_{21}^c & u_{22}^c & \cdots & u_{2n}^c \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1}^c & u_{n2}^c & \cdots & u_{nn}^c \end{bmatrix}$$

We know by Proposition 3.6 that the characters form a group. We know also that for any $\chi \in X$ $u^c(\chi) := (u_{ij}^c(\chi))_{i,j=1}^n \in M_n(\mathbb{C})$ is a permutation matrix, and so

$$u^c : X \to S_n.$$

It remains to show that u^c is a bijective homomorphism. By the universal property there is a *-homomorphism from the generators $u_{ij}^c \mapsto \mathbb{1}_{j\to i}$. STRUGGLING HERE. MOVE ON.

Consider the universal C*-algebra B generated by non-commuting projections p, q. Define $v \in M_4(B)$:

$$v = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}.$$

The entries $(v_{ij})_{i,j=1}^4$ satisfy the relations of $C(S_4^+)$ and thus by the universal property there is a *-homomorphism $\varphi(u_{ij}) = v_{ij}$. However,

$$\varphi(u_{11}u_{33}) = pq \neq qp = \varphi(u_{33}u_{11}),$$

and thus $C(S_4^+)$ is non-commutative. By considering $v \oplus I_{n-4}$ we have $C(S_n^+)$ is noncommutative for all $n \geqslant 4$. IS IT COCOMMUTATIVE?

A subgroup (H, m_H) of a classical group (G, m) is a classical group together with a monomorphism/injection $\iota : H \to G$ that satisfies:

$$\iota \circ m_H = m \circ (\iota \times \iota).$$

Via the functor composition mentioned earlier, this motivates the (standard) definition:

Definition If G and H are compact quantum groups and $\pi : C(G) \to C(H)$ is a surjective unital *-homomorphism such that

$$\Delta_{C(H)} \circ \pi = (\pi \otimes \pi) \circ \Delta_{C(G)},$$

then H is called a *subgroup* of G.

Consider the commutative C*-algebra $F(S_n)$ and take an element $f \in F(S_n)$:

$$f = \sum_{\sigma \in S_n} f(\sigma)\delta_\sigma.$$

We can write:

$$\delta_\varrho = \prod_{j=1}^{n} \mathbb{1}_{j\to\varrho(j)},$$

and thus we have a *-homomorphism $C(S_n^+) \to F(S_n)$, $\pi(u_{ij}) = \mathbb{1}_{j\to i}$, surjective because:

$$\pi\left(\prod_{j=1}^{n} u_{j\varrho(j)}\right) = \delta_\varrho.$$

For this to satisfy the subgroup property, we require:

$$\Delta(\mathbb{1}_{j\to i}) = \sum_{k=1}^{n} \mathbb{1}_{k\to i} \otimes \mathbb{1}_{j\to k}.$$

Via $F(S_n) \otimes F(S_n) \cong F(S_n \times S_n)$ we can compare:

$$\Delta(\mathbb{1}_{j\to i})(\sigma_1, \sigma_2) = \mathbb{1}_{j\to i}(\sigma_1\sigma_2) = \begin{cases} 1, & \text{if } \exists k \in \{1, \dots, n\} : \sigma_2(j) = k \text{ and } \sigma_1(k) = i \\ 0, & \text{otherwise.} \end{cases},$$

which is equal to

$$\left(\sum_{k=1}^{n} \mathbb{1}_{k\to i} \otimes \mathbb{1}_{j\to k}\right)(\delta_1^\sigma \otimes \delta_2^\sigma),$$

and therefore $S_n \subset S_n^+$, with the inclusion proper for $n \geqslant 4$. WHAT ABOUT SMALL?

3.6 Compact Matrix Quantum Groups

If a compact quantum group G is such that

- $C(G)$ generated (as a C*-algebra) by the entries of a matrix $u = (u_{ij})_{i,j=1}^n \in M_n(C(G))$, and
- u and u^T are invertible, and
- $\Delta : C(G) \to C(G) \otimes_{\min} C(G)$, $u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}$ is a *-homomorphism,

then we say that G is a compact matrix quantum group.

In fact, we do not need assume that $C(G)$ is the algebra of continuous functions on a compact quantum group, only that it is a unital C*-algebra. However all compact matrix quantum groups are compact quantum groups so here I include it in the definition. Compact Matrix Quantum Groups were introduced by Woronowicz in the 1987, where they were called 'compact matrix *pseudo*-groups', before he defined compact quantum groups.

3.6.1 Free Orthogonal Quantum Groups

Let $O_n \leqslant \mathrm{GL}(\mathbb{R}^n) \subset M_n(\mathbb{C})$ be the group of matrices that preserve the Euclidean norm, and thus the isometry group of the real sphere S^{n-1}, in other words, via $Q^* = Q^T$ for $Q \in \mathrm{GL}(\mathbb{R}^n)$:

$$O_n := \{Q \in \mathrm{GL}(\mathbb{R}^n) : Q^T Q = QQ^T = I_n\}.$$

Exercise: Prove that:

$$C(O_n) = \mathrm{C}^*_{\text{comm}}(u_{ij}, i, j = 1, \ldots, n \colon u_{ij} = u_{ij}^*,\ u = (u_{ij}) \text{ satisfies } u^T u = u u^T = 1).$$

In 1995 Wang [cite] defined the free orthogonal group O_n^+ by:

$$C(O_n^+) := \mathrm{C}^*(u_{ij},\ i, j = 1, \ldots, n \colon u_{ij} = u_{ij}^*, u = (u_{ij}) \text{ satisfies } u^T u = u u^T = 1).$$

One can show that O_n^+ is the quantum isometry group of the 'quantum (real) sphere', S_+^{n-1}:

$$C(S_+^{n-1}) = \mathrm{C}^* \left(x_1, \ldots, x_n \colon x_i = x_i^*, \sum_i x_i^2 = 1 \right),$$

and we have $S_n < S_n^+ < O_n^+$ and $O_n < O_n^+$.

3.6.2 Free Unitary Quantum Groups

We can also define a free unitary quantum group U_n^+:

$$C(U_n^+) = \mathrm{C}^* \left(u = (u_{ij})_{i,j \in [n]} \colon u, u^T \text{ unitary} \right).$$

The condition u^T unitary may be replaced by $\overline{u}$ unitary, but it must be imposed, as for such abstract matrices u unitary does not imply u^T unitary.

3.7 Representation Theory

A finite-dimensional, unitary, continuous representation of a compact group G is a continuous homomorphism $\rho : G \to U(n)$. Viewing the matrix entries as continuous functions on G, $\rho(g) = (\rho_{ij}(g))_{i,j=1}^n$, a representation may be seen as an element of $M_n(C(G))$. Consider

$$\begin{aligned} \rho(gh) &= \rho(g)\rho(h) \\ \underset{\forall i,j=1,\ldots,n}{\Rightarrow} \rho_{ij}(gh) &= \sum_{k=1}^n \rho_{ik}(g)\rho_{kj}(h) \\ \Rightarrow \rho_{ij}(m(\delta^g \otimes \delta^h)) &= \sum_{k=1}^n (\rho_{ik} \otimes \rho_{kj})(\delta^g \otimes \delta^h) \\ \Rightarrow \Delta(\rho_{ij}) &= \sum_{k=1}^n \rho_{ik} \otimes \rho_{kj}. \end{aligned}$$

Definition A *(finite-dimensional, unitary, continuous) representation* of a compact quantum group G is a unitary $\rho = [\rho_{ij}]_{i,j=1}^n \in M_n(G)$ such that

$$\Delta(\rho_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

The *dimension* of ρ, n_ρ, is this n.

The representation can also be viewed via $M_n(C(G)) \cong M_n(\mathbb{C}) \otimes C(G)$ as:

$$\rho = \sum_{i,j=1}^n E_{i,j} \otimes \rho_{ij},$$

where $E_{i,j}$ is the matrix unit $E_{i,j}e_k = \delta_{j,k}e_i$. Two representations ρ^1, ρ^2 are *equivalent* if there exists an intertwiner $T \in \mathrm{GL}(\mathbb{C}^n)$ such that:

$$\sum_{i,j=1}^{n} TE_{ij} \otimes \rho^1_{ij} = \sum_{i,j=1}^{n} E_{ij}T \otimes \rho^2_{ij}.$$

If ρ is merely invertible rather than unitary, there still exists a unitary representation ρ_0 equivalent to ρ.

We can form the direct sum $\rho^1 \oplus \rho^2 \in M_{n_1+n_2}(C(G))$ using block matrices, and the tensor product $\rho^1 \otimes \rho^2 \in M_{n_1 n_2}(C(G))$ with entries:

$$[\rho^1 \otimes \rho^2]_{(i,j)(k,l)} = \rho^1_{ij}\rho^2_{kl}.$$

The *adjoint* of ρ is $\overline{\rho} \in M_n(C(G))$ with $\overline{\rho}_{ij} = \rho^*_{ij}$.

A representation is irreducible if it cannot be non-trivially be decomposed as a direct sum of other representations. Let $\mathrm{Irr}(G)$ denote the set of all equivalence classes of irreducible representations if G, and represent each class by a unitary representation:

$$\rho^\alpha = [\rho^\alpha_{ij}]^{n_\alpha}_{i,j=1}.$$

Theorem 3.14. *(Peter-Weyl Theorem for Compact Quantum Groups (Woronowicz)) Any unitary representation of G decomposes as a direct sum of irreducible ones. The subalgebra generated by the matrix elements of all the representations of G forms a dense unital *-subalgebra of $C(G)$, denoted by $\mathcal{O}(G)$. Furthermore, with $\varepsilon : \mathcal{O}(G) \to \mathbb{C}$ and $S : \mathcal{O}(G) \to \mathcal{O}(G)$ defined by:*

$$\varepsilon(\rho^\alpha_{ij}) = \delta_{i,j}, \text{ and } S(\rho^\alpha_{ij}) = {\rho^\alpha_{ij}}^*,$$

*the *-algebra $\mathcal{O}(G)$ is a Hopf*-algebra* •

The comultiplication is the restriction of Δ to $\mathcal{O}(G)$, and maps into the algebraic tensor product $\mathcal{O}(G) \otimes \mathcal{O}(G)$. The Haar state on $\mathcal{O}(G)$ is the restriction of $\int_G$, and is faithful. Neither ε nor S necessarily extend to $C(G)$. If the coefficients of a representation ρ generate $C(G)$, then ρ is called *fundamental*.

3.7.1 An Unbounded Antipode

The compact Lie Group SU(2) consists of all unitary complex, determinant one, 2×2 matrices of the form:

$$g_{(z_1,z_2)} = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}.$$

Define $f_i \in C(\mathrm{SU}(2))$ by: $f_i(g_{(z_1,z_2)}) = z_i$. Note

$$|f_1|^2 + |f_2|^2 = \mathbb{1}_{\mathrm{SU}(2)}.$$

Indeed,

$$C(\mathrm{SU}(2)) \cong \mathrm{C}^*_{\mathrm{comm}}\langle f_1, f_2 \colon |f_1|^2 + |f_2|^2 = 1\rangle,$$

or, equivalently, the universal commutative C*-algebra generated by f_1, f_2 such that:

$$u := \begin{pmatrix} f_1 & -f_2^* \\ f_2 & f_1^* \end{pmatrix}$$

is a unitary.

Woronowicz [cite] introduced, for each $q \in [-1,1]$, a "twisted" SU(2), a compact quantum group $\mathrm{SU}_q(2)$, the universal C*-algebra generated by elements f_1, f_2 such that the following matrix is a unitary:

$$u := \begin{pmatrix} f_1 & -qf_2^* \\ f_2 & f_1^* \end{pmatrix}$$

Let $|q| < 1$. There is a faithful representation of $C(\mathrm{SU}_q(2))$ as the set of bounded operators on the Hilbert space $\ell^2(\mathbb{Z}_+ \times \mathbb{Z})$, and, where $(e_{n,k})_{n\geqslant 0,k\in\mathbb{Z}}$ is an orthonormal basis:

$$f_2^* e_{n,k} = q^n e_{n,k-1}.$$

Therefore $\|f_2^*\| = 1$ and, for $r \in \mathbb{N}$, $\|(f_2^*)^r\| \leqslant 1$, and we can see that by acting on $e_{0,0}$ that in fact $\|(f_2^*)^r\| = 1$.

Considering just the *-Hopf algebra generated by f_1 and f_2, say $H(\mathrm{SU}_q(2))$, Woronowicz showed that the antipode on f_2^* is given by:

$$S(f_2^*) = -\frac{1}{q} f_2^*.$$

The antipode is an anti-morphism and so:

$$\|S((f_2^*)^r)\| = \|S(f_2^*)^r\| = \left\| \frac{(-1)^r}{q^r}(f_2^*)^r \right\| = \frac{1}{q^r} \to \infty,$$

and so S is unbounded. It can be shown that $H(\mathrm{SU}_q(2)) \cong \mathcal{O}(\mathrm{SU}_q(2))$, a dense subalgebra of $C(\mathrm{SU}_q(2))$, but as S is not bounded it does not extend to a linear operator on $C(\mathrm{SU}_q(2))$.

3.8 From Compact Quantum Groups to Hopf*-algebras

Note that in general a Hopf *-algebra (or rather a multiplier Hopf*-algebra) is not considered the algebra of functions on a quantum group. Usually the existence of a Haar state is assumed, and such an object is called the algebra of functions on an (algebraic) compact quantum group. For the purpose of this note we will abuse convention and assume that Hopf*-algebras have Haar states.

Theorem 3.15. *Let G be a compact quantum group. Then $\mathcal{O}(G)$ is dense in $C(G)$. Furthermore, if A_0 is another Hopf*-algebra dense in $C(G)$, with the comultiplication got by restriction, then $A_0 \cong \mathcal{O}(G)$, and so we can talk about $\mathcal{O}(G)$ as* the *Hopf*-algebra associated with G* •

3.9 From Hopf*-Algebras to Compact Quantum Groups

A Hopf*-algebra can have several different completions.

REPRESENTATION THEORY OF HOPF *

3.10 Reduced Version of a Compact Quantum Group

Let G be a compact quantum group with Haar state $\int_G$. As someone who frequently forgets their GNS construction, allow me to write the null-space:

$$N_{\int_G} = \left\{ f \in C(G) \colon \int_G |f|^2 = 0 \right\},$$

note that it is a closed left ideal, and on the quotient $C(G)/N_{\int_G}$, the map

$$\left(f_1 + N_{\int_G}, f_2 + N_{\int_G} \right) = \int_G f_2^* f_1,$$

is an inner product, and we complete to form the Hilbert space $H_{\int_G}$. For any $f \in C(G)$, we define $\tilde{\pi}_{\int_G}(f) \in B(C(G)/N_{\int_G})$ by:

$$\tilde{\pi}_{\int_G}(f)\left(g + N_{\int_G}\right) = fg + N_{\int_G}.$$

It is bounded and densely defined and so has a unique extension to $\pi_{\int_G} \in B(H_{\int_G})$, and the map $f \mapsto \pi_{\int_G}(f)$ is a *-homomorphism $C(G) \to B(H_{\int_G})$, aka a *representation*.

The algebra:

$$C_r(G) := \pi_{\int_G}(C(G)) \subset B(H_{\int_G}).$$

Exercise aka I cannot show it: Where we have GNS representations

$$\pi_{\int_G \otimes \int_G} : C(G) \otimes C(G) \to B\left(\overline{C(G) \otimes C(G)/N_{\int_G \otimes \int_G}}\right),$$

and $\pi_{\int_G} : C(G) \otimes B(H_{\int_G})$, show that:

$$\ker \pi_{\int_G \otimes \int G} \subseteq \ker(\pi_{\int_G} \otimes \int_G).$$

Denote $\pi := \pi_{\int_G}$ and $N := N_{\int_G}$ for the following.

Proposition 3.16. *The map*

$$\Delta_r(\pi(f)) = (\pi \otimes \pi)\Delta(f)$$

is a compact quantum group comultiplication so that G_r defined by $C(G_r) := C_r(G)$ is a compact quantum group. The Haar state is given by:

$$\int_{G_r} \pi_{\int_G}(f) = \int_G f,$$

and is faithful.

Proof. To show that Δ_r is well defined we must show that whenever:

$$\pi(f_1) = \pi(f_2) \Rightarrow \Delta_r(\pi(f_1)) = \Delta_r(\pi(f_2)).$$

This amounts to showing if $f \in \ker \pi$, then $\Delta(f) \in \ker(\pi \otimes \pi)$. That $f \in \ker \pi$ is to say that for all $g + N \in C(G)/N$. Choosing $g = \mathbb{1}_G$ shows $f \in N \Rightarrow \int_G |f|^2 = 0$.

4 Locally Compact Quantum Groups

5 Appendices

References

[1] McCarthy, J.P., *Random Walks on Finite Quantum Groups: Diaconis-Shahshahani Theory for Quantum Groups*, University College Cork, 2017.

[Tim08] Timmermann, T., *An Invitation to Quantum Groups and Duality*, Eur. Math. Soc., 2008.