A STATE-SPACE APPROACH TO QUANTUM PERMUTATIONS

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ABSTRACT. An exposition of quantum permutation groups where an alternative to the 'Gelfand picture' of compact quantum groups is proposed. This point of view is inspired by algebraic quantum mechanics and posits that states on the algebra of continuous functions on a quantum permutation group can be interpreted as quantum permutations. This interpretation allows talk of an *element* of a compact quantum permutation group, and allows a clear understanding of the difference between deterministic, random, and quantum permutations. The interpretation is illustrated with the Kac-Paljutkin quantum group, the duals of finite groups, as well as by other finite quantum group phenomena.

INTRODUCTION

Of the world of noncommutative mathematics, C*-algebraic compact quantum groups as defined by Woronowicz [49] have been around now for about 35 years. No different to other topics in modern mathematics, it is difficult to have a happy introduction to the world of these quantum groups without a certain level of mathematical maturity, not only in terms of technical knowledge, but also the marrying of this technical knowledge with an easiness with abstraction (a rudimentary acquaintance with quantum physics won't hinder either). When the conceptual leap is made, from classical compact groups with commutative algebras of continuous functions, to compact quantum groups with *noncommutative* algebras of continuous functions, and it understood for the first time that a genuine[1] compact quantum group is "virtual", not a set at all (let alone a group), compact quantum groups can be enjoyed as beautiful, intriguing and mysterious entities.

With for example the quantum Peter-Weyl theorem and the Haar state, compact quantum groups are beautiful in how elegantly their theory generalises that of compact groups. With the emergence of "infinite" quantum generalisations S_N^+ of S_N for $N \geq 4$, and non-Kac compact quantum groups where the "inverse" appears non-involutive, intriguing in how the theory differs. The theory has matured greatly; today a lot of new results for compact quantum groups are written in the more technically demanding language of the locally compact quantum groups of Kustermans and Vaes [26], and the theory is in the foothills of having applications to quantum information by way of the theory of quantum automorphism groups of graphs (see for example [30]).

Through all of this progress, the mystery of the virtual nature of quantum groups remains. The first strike against this conceptual barrier is to deploy a formal notation, the *Gelfand picture*, that makes sense whenever the object in question is classical. Rather than talking

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[1]the theory of compact matrix quantum groups includes compact matrix groups which are not considered "genuine" quantum groups

about a quantum group A, or, slightly better, an algebra of continuous functions on a compact quantum group A, talk instead about an algebra of continuous functions, $C(\mathbb{G})$, on a compact quantum group $\mathbb{G}$. Or the algebra of regular functions $\mathcal{O}(\mathbb{G})$, or the algebra of (essentially) bounded measurable functions $L^\infty(\mathbb{G})$.

Is it possible to interpret compact quantum groups as being like compact groups beyond the correct but staid: "they are generalisations of compact groups in the sense that compact quantum groups with commutative algebras of functions can be identified with compact groups"? Do quantum group theorists hide behind their beautiful and intriguing results about quantum groups an even deeper intuition for what these objects are? If these deeper intuitions and interpretations exist, the author has not seen them written down anywhere in any great detail.

It could be argued that, like the notation emanating from the Gelfand picture, good intuition (certainly) helps the beginner, and (possibly) helps the expert. The aim of this work is to present a good intuition/interpretation for the class of compact quantum groups known as *quantum permutation groups*. Unconventionally, it identifies a *set*, the set of states on the algebra of functions, as the set of quantum permutations/elements of a quantum group.

This *Gelfand–Birkhoff picture* requires a leap to be made before ever 'going quantum'. Pick up a fresh deck of N cards in some known order and "randomly" shuffle the deck. The shuffle is distributed according to some probability ν in the set of probabilities on S_N, $M_p(S_N)$. Without turning over the cards, i.e. making some measurements, it can not be said exactly what permutation it was. The leap here is to not just consider as permutations the *deterministic permutations* in S_N, but also the *random permutations* in $M_p(S_N)$ (which includes via the Dirac measures the deterministic permutations). Bilinearly extending the group law to $M_p(S_N)$, gives the *random group law*, the convolution

$$(\nu_2 \star \nu_1)(\{\sigma\}) = \sum_{\varrho \in S_N} \nu_2(\{\sigma\varrho^{-1}\})\nu_1(\{\varrho\}),$$

for which the Dirac measure ev_e is an identity. Furthermore the inverse can be extended to a map $\mathrm{inv} : M_p(S_N) \to M_p(S_N)$, which gives the inverse of a Dirac measure for the random group law. Precisely because the map inv is not an inverse on the whole of $M_p(S_N)$, the set of random permutations $M_p(S_N)$ does not form a group, but it is nonetheless a monoid whose elements can be well-interpreted, understood, and studied in their own right. Note that where $F(S_N)$ is the algebra of complex-valued functions on S_N, $M_p(S_N)$ is the subset of positive functionals of norm one on $F(S_N)$: the set of *states* of $F(S_N)$.

Once this leap is made, that the elements of $M_p(S_N)$ rather than of S_N can be studied as "permutations", it isn't so difficult to leap to quantum permutation groups, where a state on an algebra defining the quantum permutation group can be interpreted as a permutation, a *quantum permutation*. What makes the interpretation cogent is the choice to interpret the generators $u_{ij} \in C(S_N^+)$ of the quantum permutation groups S_N^+ as relating with the states on $C(S_N^+)$ precisely as the generators $\mathbb{1}_{j\to i}(\sigma) = \delta_{i,\sigma(j)}$ of the complex-valued functions on S_N relate to states on $F(S_N)$. That is, as Bernoulli random variables with distribution

$$\mathbb{P}[\mathbb{1}_{j\to i} = 1 \,|\, \nu] = \nu(\mathbb{1}_{j\to i}) \qquad (\nu \in M_p(S_N)).$$

Furthermore, given a random permutation $\nu \in M_p(S_N)$, this is precisely the probability that it maps $j \to i$. Use the notation thus:

$$\mathbb{P}[\nu(j) = i] := \nu(\mathbb{1}_{j \to i}); \tag{0.1}$$

with some probability, the random permutation ν maps $j \to i$.

A big question here: does an exposition of an interpretation comprise mathematics? It can be claimed that it is at least *of* mathematics; quoting William Thurston [43]:

> *This question brings to the fore something that is fundamental and pervasive: that what we* [mathematicians] *are doing is finding ways for people to understand and think about mathematics.*

Please note that no claim of originality is made: the work is exposition of well-established theory from a certain point of view. Neither does the work comprise a survey (comprehensive or otherwise). Those interested in learning more about compact quantum groups in general can consult the original papers of Woronowicz [48, 49], with exposition/survey well served by the lecture notes (see the web) of Banica, Franz–Skalski–Sołtan, Freslon, Skalski, Weber, and Vergnioux. Overarching references are Timmermann [44] and Neshveyev–Tuset [38]. For those interested in quantum permutations specifically, see the original paper of Wang [46], the survey of Banica–Bichon–Collins [6], and the tome of Banica [4]. It would be remiss not to give a few references that experts have said came to mind when it was communicated that an expository piece on intuition/interpretation for quantum permutation groups was being worked on: [14, 16, 17, 30, 37, 41, 42].

The paper is organised as follows. In Section 1 the conventional Gelfand picture of C*-algebras is outlined together with a very brief overview of the theory of Woronowicz compact quantum groups. Section 2 introduces the state-space-as-quantum-space *Gelfand–Birkhoff picture*, and motivates this by introducing the rudiments of quantum probability and measurement, including the *Born rule*, *sequential measurement*, and *wave function collapse*. In Section 3, following a layperson's motivation of what a quantum permutation should be, the quantum permutation groups of Wang and their subgroups are introduced, along with some basic properties. Section 4 starts with more focussed discussion of the Gelfand–Birkhoff picture, and makes this cogent by introducing the *Birkhoff slice* (essentially the extension of (0.1) to *quantum permutations*). This gives enough intuition to inspire the "simplest yet" proof of no quantum permutations on three symbols, as well as a lucid understanding of how deterministic and random permutations sit in a quantum permutation group. In Section 5 the convolution of states is defined as the quantum group law, and the counit and antipode understood on this level. The duals of discrete groups that are finitely-generated by elements of finite order, understood as *abelian* with respect to the quantum group law, are studied in more depth as quantum permutation groups. Finally in Section 6 some intrigue: an exploration of some of the phenomena that occur once the commutative world of classical groups is left.

1. COMPACT QUANTUM GROUPS

In 1995, Alain Connes posed the question: *What is the quantum automorphism group of a space?* For the case of finite spaces, this question was answered in 1998 by Shuzou Wang [46]. There are two main ways of defining this quantum automorphism group but first some noncommutative terminology/philosophy is required.

1.1. **The Gelfand picture.** The prevailing point of view in the study of compact quantum groups is to employ what could be called the *Gelfand picture*. This starts with a categorical equivalence given by Gelfand's Theorem:

$$\text{compact Hausdorff spaces} \simeq (\text{unital commutative C*-algebras})^{\text{op}}.$$

Starting with a compact Hausdorff space X, the algebra of continuous functions on X, $C(X)$, is a unital commutative C*-algebra; and starting with a unital commutative C*-algebra A, the spectrum, $\Omega(A)$, the set of *characters*, non-zero homomorphisms $A \to \mathbb{C}$, is a compact Hausdorff space such that $A \cong C(\Omega(A))$. Therefore a general unital commutative C*-algebra can be denoted $A = C(X)$. Inspired by this, one can define the category of 'compact quantum spaces' as

$$\text{compact quantum spaces} :\simeq (\text{unital C*-algebras})^{\text{op}}.$$

In analogy with the commutative case, a general not-necessarily commutative unital C*-algebra can be denoted $A = C(\mathbb{X})$, the unit $I_A =: \mathbb{1}_{\mathbb{X}}$, $C(\mathbb{X})$ be called an algebra of continuous functions on the quantum space $\mathbb{X}$, and $\mathbb{X}$ referred to as the spectrum of A. However, in the Gelfand picture, $\mathbb{X}$ is not a set any more but a so-called *virtual* object, only spoken about via its algebra of continuous functions.

Some basic knowledge of C*-algebras is required here. See [36] for further details on the below. The set of states $\mathcal{S}(C(\mathbb{X}))$, is the set of positive linear functionals $C(\mathbb{X}) \to \mathbb{C}$ of norm one. A state $\varphi \in \mathcal{S}(C(\mathbb{X}))$ is *pure* if it has the property that whenever ρ is a positive linear functional such that $\rho \leq \varphi$, necessarily there exists $t \in [0,1]$ such that $\rho = t\varphi$. Otherwise φ is *mixed*. Elements of the form $g^*g \in C(\mathbb{X})$ are *positive*, and for positive $f \in C(\mathbb{X})$ there exists a (pure) state φ such $\varphi(f) = \|f\|$. Let $\pi(C(\mathbb{X})) \subset B(\mathsf{H})$ be a unital representation. For non-zero $\xi \in \mathsf{H}$ and $\hat{\xi} = \xi/\|\xi\|$,

$$\varphi_{\xi}(f) = \langle \hat{\xi}, \pi(f)\hat{\xi}\rangle,$$

defines a state on $C(\mathbb{X})$ called a *vector state*. The *GNS construction* $\pi_{\varphi}(C(\mathbb{X})) \subset B(\mathsf{H}_{\varphi})$ gives norm one $\xi_{\varphi} \in \mathsf{H}_{\varphi}$ such that

$$\varphi(f) = \langle \xi_{\varphi}, \pi_{\varphi}(f)\xi_{\varphi}\rangle. \tag{1.1}$$

Therefore all states are vector states for some representation.

An element $p \in C(\mathbb{X})$ is a *projection* if $p = p^* = p^2$. For $f \in C(\mathbb{X})$ define $|f|^2 = f^*f$. If $p_1, p_2, \ldots, p_n \in C(\mathbb{X})$ are projections,

$$|p_n p_{n-1} \cdots p_2 p_1|^2 = p_1 p_2 \cdots p_{n-1} p_n p_{n-1} \cdots p_2 p_1. \tag{1.2}$$

This work will consider (finite) *partitions of unity*, (finite) sets of projections $\{p_i\}_{i=1}^n \subset C(\mathbb{X})$ such that

$$\sum_{i=1}^n p_i = \mathbb{1}_{\mathbb{X}}.$$

Necessarily elements of partitions of unity are pairwise orthogonal, $p_i p_j = \delta_{i,j} p_i$. If $C(\mathbb{X})$ is finite dimensional it will be denoted by $F(\mathbb{X})$, the algebra of functions on a finite quantum space $\mathbb{X}$, in this case isomorphic to a multi-matrix algebra:

$$F(\mathbb{X}) = \bigoplus_{i=1}^m M_{N_i}(\mathbb{C}).$$

If $F(\mathbb{X})$ is commutative, then $\mathbb{X} := X := \{x_1, x_2, \ldots, x_N\}$ is a finite set, $F(X)$ is the algebra of all complex valued functions on it, isomorphic to the diagonal subalgebra of $M_N(\mathbb{C})$.

1.2. **Compact Quantum Groups.** In the well-established setting of C*-algebraic compact quantum groups as defined by Woronowicz [49], it is through the Gelfand picture that one speaks of a quantum group $\mathbb{G}$, through a unital noncommutative C*-algebra A that is considered an algebra of continuous functions on it, $A = C(\mathbb{G})$. If X and Y are compact topological spaces, then, where $\otimes$ is the minimal tensor product:

$$C(X \times Y) \cong C(X) \otimes C(Y).$$

Let S be a compact semigroup. The transpose of the continuous multiplication $m : S \times S \to S$ is a $*$-homomorphism, the *compultiplication*:

$$\Delta : C(S) \to C(S \times S) \cong C(S) \otimes C(S).$$

The associativity of the multiplication gives *coassociativity* to the comultiplication:

$$(\Delta \otimes I_{C(S)}) \circ \Delta = (I_{C(S)} \otimes \Delta) \circ \Delta. \tag{1.3}$$

If $C(S)$ satisfies *Woronowicz cancellation*

$$\overline{\Delta(C(S))(\mathbb{1}_S \otimes C(S))} = \overline{\Delta(C(S))(C(S) \otimes \mathbb{1}_S)} = C(S) \otimes C(S);$$

then it has cancellation, and a compact semigroup with cancellation is a group. In this sense Woronowicz cancellation is a $C(G)$-analogue of cancellation. This inspires:

DEFINITION 1.1. **An** (Woronowicz C*-) *algebra of continuous functions on a compact quantum group* $\mathbb{G}$ is a unital C*-algebra $C(\mathbb{G})$ together with a unital $*$-morphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ that satisfies coassociativity and *Woronowicz cancellation*:

$$\overline{\Delta(C(\mathbb{G}))(\mathbb{1}_\mathbb{G} \otimes C(\mathbb{G}))} = \overline{\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes \mathbb{1}_\mathbb{G})} = C(\mathbb{G}) \otimes C(\mathbb{G}).$$

If $C(\mathbb{G})$ is finite dimensional, $\mathbb{G}$ is said to be a *finite quantum group*, and $F(\mathbb{G})$ written for $C(\mathbb{G})$.

Algebras of continuous functions on compact quantum groups come with a dense Hopf $*$-algebra $\mathcal{O}(\mathbb{G})$ of *regular functions on a/the algebraic compact quantum group* $\mathbb{G}$. Hopf $*$-algebras are $*$-algebras which satisfy axioms which are precisely $F(G)$-analogues of the (finite) group axioms. The $F(G)$-analogue of the group law, is *comultiplication* $\Delta : F(G) \to F(G) \otimes_{\text{alg.}} F(G)$; an $F(G)$-analogue of the (inclusion of the) identity is the *counit* $\varepsilon : F(G) \to \mathbb{C}$, $\varepsilon(f) := \text{ev}_e(f) = f(e)$; and an $F(G)$-analogue of the inverse is an antihomomorphism called the *antipode*, $S : F(G) \to F(G)$, $Sf(\sigma) = f(\sigma^{-1})$. A most leisurely introduction to how these maps, and the $F(G)$-analogues of associativity (*coassociativity*, (1.3)), of the identity axiom (the *counital property*), and of the inverse axiom (the *antipodal property*), are $F(G)$-analogues of the (finite) group axioms is given in Section 1.1, [33].

Note the stress on *an*: the algebra of regular functions can have more than one completion. There is a maximal, *universal* completion $C_u(\mathbb{G})$, and a minimal, *reduced* completion $C_r(\mathbb{G})$. However all of these completions give the same dense Hopf $*$-algebra of regular functions in the following sense: given any algebra of continuous functions on a compact quantum group, surject onto the dense subalgebra $\mathcal{O}(\mathbb{G})$, complete to $C_\alpha(\mathbb{G})$, and then surject onto the dense subalgebra $\mathcal{O}_\alpha(\mathbb{G})$ associated to $C_\alpha(\mathbb{G})$:

$$C(\mathbb{G}) \twoheadrightarrow \mathcal{O}(\mathbb{G}) \hookrightarrow C_\alpha(\mathbb{G}) \twoheadrightarrow \mathcal{O}_\alpha(\mathbb{G}),$$

it turns out that $\mathcal{O}(\mathbb{G}) \cong \mathcal{O}_\alpha(\mathbb{G})$ as algebras of regular functions on *algebraic compact quantum groups* [44]. In this sense, a compact quantum group can be identified with non-isomorphic algebras of continuous functions $C_\alpha(\mathbb{G})$ and $C_\beta(\mathbb{G})$ if their dense algebras of regular functions are isomorphic. Non-isomorphic algebras of continuous functions can arise for the dual of discrete group Γ. The dual $\widehat{\Gamma}$ is a compact quantum group, with algebra of regular functions given by the group ring, $\mathcal{O}(\widehat{\Gamma}) := \mathbb{C}\Gamma$. All the completions of $\mathcal{O}(\widehat{\Gamma})$ are canonically isomorphic exactly when Γ is amenable: when all the completions of the algebra of regular functions on a compact quantum group $\mathcal{O}(\mathbb{G})$ are canonically isomorphic, in particular $C_u(\mathbb{G}) \cong C_r(\mathbb{G})$, the compact quantum group is said to be *coamenable*.

What makes a compact quantum group a generalisation of a compact group? Consider a commutative Woronowicz C*-algebra A. Gelfand's Theorem states $A \cong C(\Omega(A))$, the unital *-homomorphism $\Delta : C(\Omega(A)) \to C(\Omega(A) \times \Omega(A))$ gives a continuous map $m : \Omega(A) \times \Omega(A) \to \Omega(A)$, the coassociativity of Δ gives associativity to m, and so $(\Omega(A), m)$ is a compact semigroup with Woronowicz cancellation, and so a group. That compact quantum groups with commutative algebras of continuous functions are in fact (classical) compact groups is Gelfand duality: that the virtual quantum objects are still studied through their algebra of functions is the essence of the Gelfand picture.

A particular class of compact quantum group, earlier defined by Woronowicz [48], is a *compact matrix quantum group*.

DEFINITION 1.2. If a compact quantum group $\mathbb{G}$ is such that

- $C(\mathbb{G})$ generated (as a C*-algebra) by the entries of a unitary matrix $u = (u_{ij})_{i,j=1}^N \in M_N(C(\mathbb{G}))$, and
- u and u^t are invertible, and
- $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$, $u_{ij} \mapsto \sum_{k=1}^N u_{ik} \otimes u_{kj}$ is a *-homomorphism,

then $\mathbb{G}$ is a *compact matrix quantum group* with *fundamental representation* $u \in M_N(C(\mathbb{G}))$.

THEOREM 1.3. *(Woronowicz) If a compact matrix quantum group G has commutative algebra of functions $C(G)$, then G is homeomorphic to a closed compact subgroup of the unitary group,* $U_N(\mathbb{C})$ ●

The quantum groups studied in this work are all compact matrix quantum groups such that $u \in M_N(C(\mathbb{G}))$, the *fundamental representation*, is a *magic unitary*. That is the rows and columns of u are partitions of unity:

$$\sum_{k=1}^N u_{ik} = \mathbb{1}_\mathbb{G} = \sum_{k=1}^N u_{kj}.$$

Such compact quantum groups are called *quantum permutation groups*. There are finite quantum groups which are not quantum permutation groups [7].

2. QUANTUM MATHEMATICS

The preceding section outlines the conventional view of compact quantum groups. The Gelfand picture allows nominal talk about a compact quantum group as an object, but in general does not permit the consideration of an *element* of a compact quantum group. Aspects

of quantum mechanics can be used to inspire a way of doing this for quantum permutation groups.

In his book Weaver [47] states and argues the point that:

> *The fundamental idea of mathematical quantisation is that sets are replaced by Hilbert spaces... [and] the quantum version of a [real]-valued function on a set is a [self-adjoint] operator on a Hilbert space.*

Weaver attributes the Hilbert-space-as-set point of view to Birkhoff and von Neumann [12], and the operator-as-function point of view to Mackey [32]. In this picture, the elements of the projective version of a Hilbert space $P(\mathsf{H})$ form a quantum space, and the self-adjoint operators are random variables $P(\mathsf{H}) \to \mathbb{R}$, with the Born rule providing probability, and spectral projections providing wave function collapse. Call this the *Birkhoff picture*.

Inspired by algebraic quantum mechanics [27], and its descendent quantum probability [31], this work will push on slightly, and instead define the compact quantum space associated to a C*-algebra to be the *set* of *states* on the algebra. Call this the *Gelfand–Birkhoff picture*. This is a well worn path outside the field of compact quantum groups, see [21, 28] for discussion and further references.

To illustrate, consider a state φ on a unital C*-algebra $C(\mathbb{X})$, the self-adjoint elements of which are called *observables*. Consider a projection, a *Bernoulli observable*, $p \in C(\mathbb{X})$. Associated to p are two events: $p = 1$ given by $p^1 := p$, and $p = 0$ given by $p^0 := \mathbb{1}_{\mathbb{X}} - p$. The distribution of p given the state φ is given by (essentially the Born rule by (1.1):

$$\mathbb{P}[p = \theta \,|\, \varphi] := \varphi(p^\theta).$$

If the event $p = \theta$ is non-null, $\mathbb{P}[p = \theta \,|\, \varphi] = \varphi(p^\theta) > 0$, and the measurement of φ with p gives $p = \theta$, the state transitions $\varphi \mapsto \widetilde{p^\theta}\varphi$ where for $f \in C(\mathbb{X})$:

$$\widetilde{p^\theta}\varphi(f) := \frac{\varphi(p^\theta f p^\theta)}{\varphi(p^\theta)}.$$

This is *wave function collapse/state conditioning*. If φ_ξ is a vector state given by a representation $\pi(C(\mathbb{X})) \subset B(\mathsf{H})$, and $\varphi_\xi(p^\theta) > 0$ (so that moreover $\pi(p^\theta) \neq 0$) then $\widetilde{p^\theta}\varphi_\xi$ is also a vector state, given by $\varphi_{\pi(p^\theta)\xi}$. To see this use the fact that $\pi(p^\theta)$ is a projection:

$$\widetilde{p^\theta}\varphi_\xi(f) = \frac{\varphi_\xi(p^\theta f p^\theta)}{\varphi_\xi(p^\theta)} = \frac{\langle \xi, \pi(p^\theta f p^\theta)\xi \rangle}{\langle \xi, \pi(p^\theta)\xi \rangle} = \frac{\langle \pi(p^\theta)\xi, \pi(f)\pi(p^\theta)\xi \rangle}{\langle \pi(p^\theta)\xi, \pi(p^\theta)\xi \rangle}$$
$$= \frac{\langle \pi(p^\theta)\xi, \pi(f)\pi(p^\theta)\xi \rangle}{\|\pi(p^\theta)\xi\|^2} = \left\langle \frac{\pi(p^\theta)\xi}{\|\pi(p^\theta)\xi\|}, \pi(f)\frac{\pi(p^\theta)\xi}{\|\pi(p^\theta)\xi\|} \right\rangle = \varphi_{\pi(p^\theta)\xi}(f).$$

Take another projection $q \in C(\mathbb{X})$. Suppose that the event $p = \theta_1$ has been observed so that the state is now $\widetilde{p^{\theta_1}}\varphi$. The probability that measurement *now* produces $q = \theta_2$, and $\widetilde{p^{\theta_1}}\varphi \mapsto \widetilde{q^{\theta_2}}\widetilde{p^{\theta_1}}\varphi$, is:

$$\mathbb{P}[q = \theta_2 \,|\, \widetilde{p^{\theta_1}}\varphi] := \widetilde{p^{\theta_1}}\varphi(q^{\theta_2}) = \frac{\varphi(p^{\theta_1} q^{\theta_2} p^{\theta_1})}{\varphi(p^{\theta_1})}.$$

Define now the event $((q = \theta_2) \succ (p = \theta_1) \,|\, \varphi)$, said 'given the state φ, q is measured to be θ_2 *after* p is measured to be θ_1'. Using the expression above a probability can be ascribed to this event:

$$\mathbb{P}\left[(q = \theta_2) \succ (p = \theta_1) \,|\, \varphi\right] := \mathbb{P}[q = \theta_2 \,|\, \widetilde{p^{\theta_1}}(\varphi)] \cdot \mathbb{P}[p = \theta_1 \,|\, \varphi]$$

$$= \frac{\varphi(p^{\theta_1} q^{\theta_2} p^{\theta_1})}{\varphi(p^{\theta_1})} \cdot \varphi(p^{\theta_1}) \cdot = \varphi(p^{\theta_1} q^{\theta_2} p^{\theta_1}) = \varphi(|q^{\theta_2} p^{\theta_1}|^2).$$

Inductively, for a finite sequence of projections $(p_i)_{i=1}^n$, and $\theta_i \in \{0,1\}$:

$$\mathbb{P}\left[(p_n = \theta_n) \succ \cdots \succ (p_1 = \theta_1) \,|\, \varphi\right] = \varphi(|p_n^{\theta_n} \cdots p_1^{\theta_1}|^2).$$

It is worth noting that

$$\mathbb{P}[(p_2 = \theta_2) \succ (p_1 = \theta_1)|\varphi] \le \mathbb{P}[p_1 = \theta_1|\varphi], \tag{2.1}$$

so that in particular if $\mathbb{P}[(p_2 = \theta_2) \succ (p_1 = \theta_1)|\varphi] > 0$ then $\mathbb{P}[p_1 = \theta_1|\varphi] > 0$.

In general, $pq \ne qp$ and so

$$\mathbb{P}\left[(q = \theta_2) \succ (p = \theta_1) \,|\, \varphi\right] \ne \mathbb{P}\left[(p = \theta_1) \succ (q = \theta_1) \,|\, \varphi\right],$$

and this is to be interpreted that q and p are not simultaneously observable. However the *sequential projection measurement* $q \succ p$ is an 'observable' in the sense that it is a random variable with values in $\{0,1\}^2$. Inductively the sequential projection measurement $p_n \succ \cdots \succ p_1$ is an $\{0,1\}^n$-valued random variable.

If p and q *do* commute, then the distributions of $q \succ p$ and $p \succ q$ are equal in the sense that

$$\mathbb{P}\left[(q = \theta_2) \succ (p = \theta_1) \,|\, \varphi\right] = \varphi(|q^{\theta_2} p^{\theta_1}|^2) = \mathbb{P}\left[(p = \theta_1) \succ (q = \theta_2) \,|\, \varphi\right],$$

it doesn't matter what order they are measured in, the outputs of the measurements can be multiplied together, and this observable can be called $pq = qp$.

Measurement with a projection includes an *a priori* distribution, and wave function collapse: but these are not purely quantum mechanical phenomenon, and occur also with measurements from a commutative subalgebra $C(X) \subset C(\mathbb{X})$. To illustrate, let $X = \{x_1, \ldots, x_N\}$ and consider the diagonal subalgebra $F(X)$ of $F(\mathbb{X}) := M_N(\mathbb{C})$. Subsets $Y \subset X$ yield subspaces $F(Y) \subset F(X)$ together with projections $p_Y \in F(\mathbb{X})$. The *a priori* distribution of p_Y is:

$$\mathbb{P}[p_Y = \theta \,|\, \varphi] = \varphi(p_Y^\theta),$$

and, conditional on $p_Y = \theta$, there is wave function collapse to $\widetilde{p_Y^\theta} \varphi \in \mathcal{S}(F(X))$.

A defining difference between classical and quantum measurement is the quantum phenomenon of projection observables that cannot be simultaneously measured in the sense that $(p \succ q) \ne (q \succ p)$. In classical measurement, all projection observables can be simultaneously measured, and this implies that while classical measurement *can* disturb a mixed state, the effects are purely probabilistic, capturing a decrease in uncertainty about the state. In the finite, classical case of $F(X)$, measurement with an appropriate sequence of classical projection measurements results in collapse to a pure state $\mathrm{ev}_{x_i} \in \mathcal{S}(F(X))$. Pure states are invariant for the diagonal subalgebra: further measurement does *not* disturb the state. In contrast, for any state $\varphi \in \mathcal{S}(F(\mathbb{X}))$ there is a projection $p \in F(\mathbb{X})$ that can disturb it, and so collapse to complete certainty is impossible.

Sequential measurement of finite spectrum observables $f \in C(\mathbb{X})$ can also be considered. Through the inverse of the isometric Gelfand–Naimark *-isomorphism $C(\mathbb{X}) \to \pi(C(\mathbb{X})) \subset B(\mathsf{H})$, a finite spectrum observable $f \in C(\mathbb{G})$ has a spectral decomposition $f = \sum_{k=1}^{|\sigma(f)|} f_i\, p^{f_i}$, that defines a partition of unity $\{p^{f_i}\}_{i=1,\ldots,|\sigma(f)|} \subset C(\mathbb{X})$, and

$$\mathbb{P}[f = f_i \,|\, \varphi] = \varphi(p^{f_i}). \tag{2.2}$$

Furthermore this gives expectations

$$\mathbb{E}[f \,|\, \varphi] := \varphi(f).$$

For continuous spectrum observables, by passing to the enveloping von Neumann algebra $C(\mathbb{X})^{**} \cong \pi_U(C(\mathbb{X}))''$, which will be denoted $\ell^\infty(\mathbb{X})$, and taking the normal extension of φ to a state ω_φ on $\ell^\infty(\mathbb{X})$, Borel functional calculus can be used to measure, for example, if f is in some Borel subset of its spectrum, via the projection $\mathbb{1}_S(f) \in \ell^\infty(\mathbb{X})$, so that

$$\mathbb{P}[f \in S \,|\, \varphi] := \mathbb{P}[\mathbb{1}_S(f) = 1 \,|\, \varphi] := \omega_\varphi(\mathbb{1}_S(f)).$$

If $f \in S$ is non-null, for wave function collapse, embed functions $f \in C(\mathbb{X})$ via $\imath : C(\mathbb{X}) \hookrightarrow \ell^\infty(\mathbb{X})$, and the state transitions to $\varphi \mapsto \widetilde{\mathbb{1}_S(f)}\varphi \in \mathcal{S}(C(\mathbb{X}))$ defined by:

$$\widetilde{\mathbb{1}_S(f)}\varphi(g) = \frac{\omega_\varphi(\mathbb{1}_S(f)\imath(g)\mathbb{1}_S(f))}{\omega_\varphi(\mathbb{1}_S(f))}.$$

Although not considered in this work, for not-necessarily finite spectrum observables $(f_i)_{i=1}^n$, the distribution of the sequential measurement $f_n \succ \cdots \succ f_1$ could be defined for Borel sets $S_i \subset \sigma(f_i)$:

$$\mathbb{P}[(f_n \succ \cdots \succ f_1) \in (S_n, \cdots, S_i) | \varphi] = \omega_\varphi(|\mathbb{1}_{S_n}(f_n) \cdots \mathbb{1}_{S_1}(f_1)|^2).$$

3. Quantum permutations

Fresh decks of playing cards produced by e.g. the US Playing Card Company always come in the same original order:

$$A\spadesuit, \ldots, K\spadesuit, A\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit,$$

Respectively enumerate using $c : \{1, 2, \ldots, 52\} \to \{A\spadesuit, \ldots, K\heartsuit\}$. The original order can be associated with the pure state $\mathrm{ev}_e \in M_p(S_{52})$. After a suitably randomised shuffle, an active permutation, the deck will be in some unknown order given by a mixed state, a *random permutation* $\varsigma \in M_p(S_{52})$, with the card in position j moved to position $\varsigma(j)$. Turn over the card in position i to reveal card $c(j)$. This observable, denoted $x^{-1}(i) \in F(S_{52})$, reveals that the random permutation sent j to i. This observable has spectrum $\sigma(x^{-1}(i)) = \{1, 2, \ldots, 52\}$, and thus spectral decomposition

$$x^{-1}(i) = \sum_{k=1}^{52} k\, u_{ik}^{S_{52}},$$

with $\{u_{ik}^{S_{52}}\}_{k=1,\ldots,52}$ a partition of unity. The distribution of $x^{-1}(i)$ given the state $\varsigma \in M_p(S_{52})$ can be denoted

$$\mathbb{P}[\varsigma^{-1}(i) = j] := \mathbb{P}[x^{-1}(i) = j \,|\, \varsigma] = \varsigma(u_{ij}^{S_{52}}). \tag{3.1}$$

Each card $c(j)$ must be mapped *somewhere* and so, for all $\varsigma \in M_p(S_{52})$

$$\sum_{k=1}^{52} \mathbb{P}[\varsigma^{-1}(k) = j] = \varsigma\left(\sum_{k=1}^{52} u_{kj}^{S_{52}}\right) = 1,$$

this implies that $\{u_{kj}^{S_{52}}\}_{k=1,\ldots,52}$ is also a partition of unity, giving another observable

$$x(j) := \sum_{k=1}^{52} k\, u_{kj}^{S_{52}},$$

and note that

$$\mathbb{P}[\varsigma(j) = i] := \mathbb{P}[x(j) = i \,|\, \varsigma] = \varsigma(u_{ij}^{S_{52}}) = \mathbb{P}[\varsigma^{-1}(i) = j].$$

The observable $x^{-1}(i)$ is measured by turning over the card in position i. How is $x(j)$ measured? Go back to the deck in the original order, turn card $c(j)$ face *up*, shuffle with ς. The position of card $c(j)$ is $x(j)$.

Following the sequential measurement

$$x^{-1}(51) \succ \cdots \succ x^{-1}(2) \succ x^{-1}(1),$$

the random permutation will collapse to a (deterministic) permutation $\mathrm{ev}_\sigma \in M_p(S_{52})$. If the sequential measurement is paused, say at $\ell < 51$ with

$$(x^{-1}(\ell) \succ \cdots \succ x^{-1}(2) \succ x^{-1}(1)) = (j_\ell, \ldots, j_2, j_1),$$

the state has collapsed to

$$\varsigma_\ell := \widetilde{u_{ij_\ell}^{S_{52}}} \cdots \widetilde{u_{ij_2}^{S_{52}}} \widetilde{u_{ij_1}^{S_{52}}} \varsigma,$$

then

$$\mathbb{P}[\varsigma_\ell(k) = j] := \mathbb{P}[x^{-1}(k) = j_k \,|\, \varsigma_\ell] = \delta_{j,j_k},$$

that is once a card $c(k)$ is observed in the position j_k once, that is determined once and for all.

Note that $(u_{ij}^{S_{52}})_{i,j=1}^{52} \in M_{52}(F(S_{52}))$ is a *magic unitary*.

There is no issue whatsoever talking about the set of random permutations, $M_p(S_N))$, nor an element of this set $\varsigma \in M_p(S_N)$. Inspired by the Gelfand–Birkhoff picture, imagine for a moment that the same can be done for *quantum permutations*: imagine that there is a C*-algebra $C(S_N^+)$ such that the set of quantum permutations on N symbols is given by $\mathcal{S}(C(S_N^+))$, and a quantum permutation is simply an element $\varsigma \in \mathcal{S}(C(S_N^+))$.

What would make a permutation *quantum*? In light of previous discussions perhaps what might make a permutation *quantum* is that quantum versions of observables $x^{-1}(i)$ and $x(j)$ be *not* simultaneously observable. This implies that, with a deck of cards shuffled with a quantum permutation, once the first card has been revealed, the observation of the second card might disturb the state in a such a way that non-classical events *can* occur. What would be a non-classical event? Turning over the first card to reveal the ace of hearts, then turning over the second card to reveal an ace of spaces, then turning over the first card again to find it is not longer the ace of hearts but the ace of diamonds:

$$(\varsigma^{-1}(1) = c^{-1}(A\diamondsuit)) \succ (\varsigma^{-1}(2) = c^{-1}(A\spadesuit)) \succ (\varsigma^{-1}(1) = c^{-1}(A\heartsuit)).$$

With the deck in the original order, $x(j)$ would be measured by turning card $c^{-1}(j)$ face *up*, shuffling with ς, and noting the position of $c^{-1}(j)$ after the shuffle. Similarly $x^{-1}(i)$ would be observable by revealing the card in position i. What would not be permitted would be shuffling with more than one card face up, or revealing more than one card at once.

Given a quantum permutation $\varsigma \in \mathcal{S}(C(S_N^+))$, similarly to before, the spectral decompositions of the observables $x^{-1}(i)$ and $x(j)$ should give a magic unitary $(u_{ij})_{i,j=1}^{52}$. Denote as before

$$\mathbb{P}[\varsigma(j) = i] := \mathbb{P}[x(j) = i \,|\, \varsigma] = \varsigma(u_{ij}).$$

The projective nature of wave function collapse, that conditional on $\varsigma(j) = i$, $\varsigma \mapsto \widetilde{u_{ij}\varsigma}$, implies that

$$\mathbb{P}[(\varsigma(j) = i) \succ (\varsigma(j) = i)] = \mathbb{P}[\varsigma(j) = i];$$

the probability of observing $\varsigma(j) = i$ after (just) observing $\varsigma(j) = i$ is one.

In the sequel this will all be made mathematically precise.

3.1. **Wang's Quantum Permutation Groups.** In a survey article, Banica, Bichon & Collins [6] attribute to Brown [15] the idea of taking a matrix group $G \subset U_N$, realising $C(G)$ as a universal commutative C*-algebra generated by the matrix coordinates $u_{ij} \in C(G)$ subject to some relations R, and then studying (if it exists), the noncommutative universal C*-algebra generated by abstract variables u_{ij} subject to the same relations R. This procedure, later called *liberation* in the context of compact quantum groups by Banica & Speicher [11], was carried out by Wang to create quantum versions of the orthogonal and unitary groups, and later quantum permutation groups.

Let $F(S_N)$ be the algebra of complex functions on S_N with basis $\{\delta_\sigma\}_{\sigma \in S_N}$. Define $\mathbb{1}_{j \to i} \in F(S_N)$ by:

$$\mathbb{1}_{j \to i}(\sigma) := \begin{cases} 1, & \text{if } \sigma(j) = i, \\ 0, & \text{otherwise.} \end{cases}$$

Where Δ is the transpose of the group law $m : S_N \times S_N \to S_N$, so that $\Delta(f) = f \circ m$, and employing $F(S_N \times S_N) \cong F(S_N) \otimes_{\text{alg.}} F(S_N)$, note that

$$\Delta(\mathbb{1}_{j \to i}) = \sum_{k=1}^N \mathbb{1}_{k \to i} \otimes_{\text{alg.}} \mathbb{1}_{j \to k}.$$

Furthermore

$$\delta_\sigma = \prod_{j=1}^N \mathbb{1}_{j \to \sigma(j)}, \tag{3.2}$$

and so the matrix $u^{S_N} = (\mathbb{1}_{j \to i})_{i,j=1}^N$ is a unitary with inverse the transpose of u^{S_N}, whose entries generate $F(S_N)$. Therefore $F(S_N)$ is a commutative algebra of functions on a compact matrix quantum group. Furthermore the $\mathbb{1}_{j \to i}$ are projections, and

$$\sum_{k=1}^N \mathbb{1}_{k \to i} = \mathbb{1}_{S_N} = \sum_{k=1}^N \mathbb{1}_{j \to k}.$$

Therefore the matrix $u^{S_N} = (\mathbb{1}_{j \to i})_{i,j=1}^N$ is a magic unitary. Indeed $F(S_N)$ has a presentation as a universal commutative C*-algebra:

$$F(S_N) \cong \mathrm{C}^*_{\mathrm{comm}}(u_{ij}^c \,|\, u^c \text{ an } N \times N \text{ magic unitary}).$$

Following Wang [46], *liberate* by considering the universal C*-algebra:

$$C(S_N^+) := \mathrm{C}^*(u_{ij} \,|\, u \text{ an } N \times N \text{ magic unitary}).$$

The universal property says that if $(v_{ij})_{i,j=1}^N$ is another $N \times N$ magic unitary, then $u_{ij} \mapsto v_{ij}$ is a $*$-homomorphism. It can be shown that

$$\left[\sum_{k=1}^N u_{ik} \otimes u_{kj}\right]_{i,j=1}^N \in M_N(C(S_N^+) \otimes C(S_N^+))$$

is a magic unitary, and thus $\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}$ is a $*$-homomorphism. It is straightforward to show that Δ is unital and coassociative, and so $C(S_N^+)$ is an algebra of continuous functions on a compact matrix quantum group, *the* quantum permutation group on N symbols.

3.2. **Quantum Permutation Groups.** If $\mathbb{G}$ is a compact matrix quantum group whose fundamental representation is an $N \times N$ magic unitary $u^{\mathbb{G}}$, then the universal property gives $\pi : C(S_N^+) \to C(\mathbb{G})$ a surjective $*$-homomorphism that intertwines the comultiplication:

$$\Delta_{C(\mathbb{G})} \circ \pi = (\pi \otimes \pi) \circ \Delta_{C(S_N^+)}, \tag{3.3}$$

which is to say that $\mathbb{G} < S_N^+$, $\mathbb{G}$ is a quantum subgroup of S_N^+. Furthermore, if $\mathbb{G} < S_N^+$ by a comultiplication-intertwining surjective $*$-homomorphism $\pi_1 : C(S_N^+) \to C(\mathbb{G})$, then $[\pi_1(u_{ij})]_{i,j=1}^N$ is a magic unitary that is a fundamental representation for $\mathbb{G}$.

DEFINITION 3.1. A *quantum permutation group* $\mathbb{G}$ is a compact matrix quantum group whose fundamental representation is a magic unitary.

On the algebra of regular functions $\mathcal{O}(\mathbb{G})$, also generated (as a $*$-algebra) by $u_{ij}^{\mathbb{G}} \in \mathcal{O}(\mathbb{G})$, the comultiplication, counit, and antipodal maps are given by:

$$\begin{aligned}
\Delta(u_{ij}^{\mathbb{G}}) &= \sum_{k=1}^N u_{ik}^{\mathbb{G}} \otimes_{\mathrm{alg.}} u_{kj}^{\mathbb{G}}, \\
\varepsilon(u_{ij}^{\mathbb{G}}) &= \delta_{i,j}, \\
S(u_{ij}^{\mathbb{G}}) &= u_{ji}^{\mathbb{G}}.
\end{aligned} \tag{3.4}$$

In general the counit does not extend to a character on a completion $C_\alpha(\mathbb{G})$ of $\mathcal{O}(\mathbb{G})$. The antipode satisfies $S^2 = I_{\mathcal{O}(\mathbb{G})}$ so that $\mathbb{G}$ is a *Kac algebra*.

The justification for calling S_N^+ *the* quantum permutation group on N symbols goes beyond the liberation of $F(S_N)$. Wang originally defines the (universal) quantum automorphism group of $\mathbb{C}^n$ (that leaves the counting measure invariant). This leads to the definition of S_N^+ given above. This work should further cement that S_N^+ is a quantum generalisation of S_N.

THEOREM 3.2. *For $N \leq 3$, $C(S_N^+) \cong F(S_N)$*   ●

See Section 4.1 for a new proof for $N = 3$.

THEOREM 3.3. *For $N \geq 4$, $C(S_N^+)$ is noncommutative and infinite dimensional.*

Proof. The standard argument for $N = 4$ uses the universal C*-algebra generated by two projections (see [6]). To be slightly more concrete, consider the discrete infinite dihedral group

$$D_\infty := \langle a, b \mid a^2 = b^2 = e \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2.$$

The infinite dihedral group is amenable, which implies that the reduced and universal group C*-algebras coincide. Denote the (noncommutative) group ring by $\mathbb{C}D_\infty$ and $C(\widehat{D_\infty})$ the C*-completion, which is in fact *-isomorphic to the universal C*-algebra generated by two projections [40]. Together with $\Delta(g) = g \otimes g$, the dual $\widehat{D_\infty}$ is a compact matrix quantum group, with unit $\mathbb{1}_{\widehat{D_\infty}} := e$, and fundamental representation

$$\tilde{u}^{\widehat{D_\infty}} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

In fact $\widehat{D_\infty}$ is quantum permutation group. Where $p := (e + a)/2$ and $q := (e + b)/2$, via $a = u_{11}^{\widehat{D_\infty}} - u_{21}^{\widehat{D_\infty}}$ (and similarly for b) the following is a fundamental representation for $\widehat{D_\infty}$ that is a magic unitary:

$$u^{\widehat{D_\infty}} := \begin{bmatrix} p & e - p & 0 & 0 \\ e - p & p & 0 & 0 \\ 0 & 0 & q & e - q \\ 0 & 0 & e - q & q \end{bmatrix} \tag{3.5}$$

Therefore $\widehat{D_\infty}$ is an infinite quantum subgroup of S_4^+, and it follows that $C(S_4^+)$ is infinite dimensional and noncommutative.

To extend to $N \geq 4$ use $u^{\widehat{D_\infty} < S_{4+\ell}} := \mathrm{diag}(u^{\widehat{D_\infty}}, \mathbb{1}_{\widehat{D_\infty}}, \cdots, \mathbb{1}_{\widehat{D_\infty}})$ •

Note also that replacing a and b with order two generators of D_N shows that $\widehat{D_N} < S_4^+$ (exhibiting Th. 1.1 (9), [5]). Showing that $\widehat{D_3} = \widehat{S_3} < S_4^+$ is the easiest way of showing that $C(S_4^+)$ is noncommutative.

THEOREM 3.4. *For $N \geq 5$, S_N^+ is not coamenable.*

Proof. The standard argument that S_N^+ is not coamenable for $N \geq 5$ uses fusion rules [1]. However, in similar spirit to the (standard) proof of Theorem 3.3, using the fact that a compact subgroup of a coamenable compact quantum group is coamenable [45], the exhibition of a non-coamenable subgroup of S_5^+ proves Theorem 3.4 for $N = 5$ (the extension to $N > 5$ follows in the same way as the extension of Theorem 3.3 to $N > 4$). Let a and b be the respective generators of $\mathbb{Z}_3 * \mathbb{Z}_2$. Let $C(\widehat{\mathbb{Z}_3 * \mathbb{Z}_2})$ be a completion of $\mathbb{C}(\mathbb{Z}_3 * \mathbb{Z}_2)$ to a compact quantum group. Where $\omega = e^{2\pi i/3}$, consider the following magic unitary:

$$u^{\widehat{\mathbb{Z}_3}} := \frac{1}{3} \begin{bmatrix} e + a + a^2 & e + \omega^2 a + \omega a^2 & e + \omega a + \omega^2 a^2 \\ e + \omega a + \omega^2 a^2 & e + a + a^2 & e + \omega^2 a + \omega a^2 \\ e + \omega^2 a + \omega a^2 & e + \omega a + \omega^2 a^2 & e + a + a^2 \end{bmatrix}$$

Note that $a = u_{11}^{\widehat{\mathbb{Z}_3}} + \omega^2 u_{21}^{\widehat{\mathbb{Z}_3}} + \omega u_{31}^{\widehat{\mathbb{Z}_3}}$. With the same notation $q = (e+b)/2$ as with the infinite dihedral group, except obviously with $e \in \mathbb{Z}_3 * \mathbb{Z}_2$, let

$$u^{\widehat{\mathbb{Z}_2}} := \begin{bmatrix} q & e-q \\ e-q & q \end{bmatrix}.$$

Consider the block magic unitary $u^{\widehat{\mathbb{Z}_3 * \mathbb{Z}_2}} \in M_5(\mathbb{C}(\mathbb{Z}_3 * \mathbb{Z}_2))$:

$$u^{\widehat{\mathbb{Z}_3 * \mathbb{Z}_2}} := \begin{bmatrix} u^{\widehat{\mathbb{Z}_3}} & 0 \\ 0 & u^{\widehat{\mathbb{Z}_2}} \end{bmatrix}.$$

This shows that $\widehat{\mathbb{Z}_3 * \mathbb{Z}_2} < S_5^+$. The dual of a discrete group is coamenable if and only if the group is amenable; $\mathbb{Z}_3 * \mathbb{Z}_2$ is not amenable [40], therefore its dual is not coamenable, and thus neither is S_5^+ ●

Banica [4] calls $\widehat{\mathbb{Z}_2 * \mathbb{Z}_3}$ by Bichon's group dual subgroup of S_5^+. More on duals in Section 5.2.

It is the case that $S_N \leq S_N^+$ is a quantum permutation group, known to be maximal for $N \leq 5$ [2], but conjectured to be maximal for all $N \in \mathbb{N}$. One motivation for the current work is to perhaps provide some intuition to attack such a problem. See Section 6 for more.

4. THE GELFAND-BIRKHOFF PICTURE FOR QUANTUM PERMUTATIONS

DEFINITION 4.1. Where $C(\mathbb{G})$ is an algebra of continuous functions on a quantum permutation group, *a quantum permutation* is an element of $S(C(\mathbb{G}))$

With the use of the *Birkhoff slice* (Section 4.1) this statement will be made cogent, and as will be seen in Section 5, there is a natural candidate for what should be considered a *quantum group law* $S(C(\mathbb{G})) \times S(C(\mathbb{G})) \to S(C(\mathbb{G}))$.

Returning to Gelfand's Theorem: as soon as an algebra of functions $C(\mathbb{G})$ is noncommutative, it is often remarked that it obviously cannot be the algebra of functions on a compact Hausdorff space (with the commutative pointwise multiplication). However the elements of $C(\mathbb{G})$ viewed through the lens of Kadison's function representation are affine functions on a compact Hausdorff space.

Note firstly that $S(C(\mathbb{G}))$ is a weak-* compact Hausdorff space [36], and recall the embedding $\imath : C(\mathbb{G}) \hookrightarrow \ell^\infty(\mathbb{G})$, $f \mapsto \imath(f)$:

$$\imath(f)(\rho) := \rho(f) \qquad (\rho \in C(\mathbb{G})^*).$$

Finally weak*-convergence of a net of states $\varphi_\lambda \to \varphi$, that for $f \in C(\mathbb{G})$

$$\varphi_\lambda(f) \to \varphi(f),$$

gives continuity to $\imath(f)$:

$$\imath(f)(\varphi_\lambda) \to \imath(f)(\varphi).$$

The multiplication $\imath(C(\mathbb{G})) \times \imath(C(\mathbb{G})) \to \imath(C(\mathbb{G}))$ is not the pointwise multiplication, but inherited from $C(\mathbb{G})$:

$$\imath(f)\imath(g) = \imath(fg) \neq \imath(gf) = \imath(g)\imath(f).$$

Through this lens the elements of $C(\mathbb{G})$ are affine functions $S(C(\mathbb{G})) \to \mathbb{C}$, that is for $\rho_1, \rho_2 \in S(C(\mathbb{G}))$, and $\lambda \in [0,1]$:

$$\imath(f)(\lambda \rho_1 + (1-\lambda)\rho_2) = \lambda\, \imath(f)(\rho_1) + (1-\lambda)\,\imath(f)(\rho_2).$$

An element of $\imath(C(\mathbb{G})) \subsetneq C(S(C(\mathbb{G})))$ is therefore completely determined by its values on the weak-$*$ closure of the set of pure states, the *pure state space* $\mathcal{P}(C(\mathbb{G})) := \overline{PS(C(\mathbb{G}))}^{w^*}$. This is precisely how a function on a finite group $f_0 : G \to \mathbb{C}$ extends to a function on the set of *random permutations* on G, $f_1 : M_p(G) \to \mathbb{C}$:

$$f_1(\lambda \operatorname{ev}_{g_1} + (1-\lambda)\operatorname{ev}_{g_2}) = \lambda\, f_0(g_1) + (1-\lambda)\, f_0(g_2).$$

While Gelfand's Theorem says, through the fact that the character space and pure state space coincide for unital commutative C*-algebras, that for a finite group G, the embedded $\imath(F(G))$ is the full algebra of continuous functions $F(\mathcal{P}(F(G)))$, in the noncommutative case, restricting even to $\mathcal{P}(C(\mathbb{G}))$, $\imath(C(\mathbb{G}))$ is a proper subset of $C(S(C(\mathbb{G})))$, so, while tempting, it is abuse of notation to define $\mathbb{G} := \mathcal{P}(C(\mathbb{G}))$ as the compact Hausdorff space, and continue to use the $C(\mathbb{G})$ notation. If in the classical case the algebra of functions, $F(G)$ is understood as the algebra of affine functions on the random permutations $M_p(G) \to \mathbb{C}$, and an algebra of continuous functions on a quantum permutation group $C(\mathbb{G})$ is understood as an algebra of affine functions $S(C(\mathbb{G})) \to \mathbb{C}$, then the relationship between $\mathcal{P}(C(\mathbb{G}))$ and $S(C(\mathbb{G}))$ reflects in the quantum case the relationship between G and $M_p(G)$ in the classical case.

These analogies are well captured by the following schematic:

$$\begin{array}{ccc} \text{Deterministic} & \longhookrightarrow & \text{Random} \\ \downarrow & & \downarrow \\ \text{Pure Quantum} & \longhookrightarrow & \text{Quantum} \end{array}$$

The objects on the left are pure states on C*-algebras; while the objects on the right are mixed states. The objects on top are states on commutative algebras; while the objects on the bottom are states on noncommutative algebras. The focus of this work is on the mixed states. This pair of dichotomies is discussed in [13].

Therefore with the focus on mixed rather than pure states, the set of quantum permutations will be denoted by $\mathbb{G} := S(C(\mathbb{G}))$, *a* quantum permutation written an element $\varsigma \in \mathbb{G}$, but the $C(\mathbb{G})$ notation will be kept, but with the implicit understanding that it is a proper subset of $C(S(C(\mathbb{G})))$ (not to mention the fact that for non-coamenable $\mathbb{G}$ there are different C*-completions of $\mathcal{O}(\mathbb{G})$, and thus different state spaces). Particularly if $\varsigma \in \mathbb{G}$ is *deterministic* (see Section 4.2), the notation $\varsigma = \operatorname{ev}_\sigma$ will be used. In this sense, if $C(\mathbb{G})$ has a counit, its *Birkhoff slice* (see Section 4.1, below) is $\jmath(e)$ by (3.4), and so can be denoted $\varepsilon := \operatorname{ev}_e$.

4.1. **The Birkhoff Slice.** Given a quantum permutation group $\mathbb{G} \leq S_N^+$ generated by a magic unitary $(u_{ij}^{\mathbb{G}})_{i,j=1}^N \in M_N(C(\mathbb{G}))$, via the Gelfand–Birkhoff picture, an element $\varsigma \in \mathbb{G}$ is a quantum permutation. In this picture, the projections $u_{ij}^{\mathbb{G}}$ are Bernoulli observables. Make the following interpretation:

$$\mathbb{P}[\varsigma(j) = i] := \mathbb{P}[u_{ij}^{\mathbb{G}} = 1 \,|\, \varsigma] := \varsigma(u_{ij}^{\mathbb{G}}). \tag{4.1}$$

These probabilities can be collected in a matrix:

$$\Phi(\varsigma)_{ij} := \varsigma(u_{ij}^{\mathbb{G}}).$$

That $(u_{ij}^{\mathbb{G}})_{i,j=1}^{N}$ is a magic unitary implies that $\Phi(\varsigma)$ is a doubly stochastic matrix, i.e. $\Phi(\varsigma)$ is in the Birkhoff polytope $\mathcal{B}_N$, and call the map $\Phi : \mathbb{G} \to \mathcal{B}_N$ the *Birkhoff slice*. It is called a slice as it only captures an ephemeral aspect of a quantum permutation; and is not injective.

In the case of compact matrix quantum groups, there is a natural generalisation of the Birkhoff slice, $\Phi : S(C(\mathbb{G})) \to M_N(\mathbb{C})$. The restriction of this map to characters, an injective map $\Phi : \Omega(C(\mathbb{G})) \to M_N(\mathbb{C})$, has been studied previously. Immediately Woronowicz uses this map to prove Theorem 1.3 [48]. Kalantar and Neufang [23], who associate to a (locally) compact quantum group $\mathbb{G}$, a (locally) compact classical group $\tilde{\mathbb{G}}$, use the map to show that in the case of a compact matrix quantum group, $\tilde{\mathbb{G}}$ is homeomorphic to $\Phi(\Omega(C(\mathbb{G})))$.

Assuming that $\mathbb{P}[\varsigma(k) = \ell] \neq 0$, the quantum permutation ς can be conditioned on $\varsigma(k) = \ell$, and conditional probabilities can be collected in a Birkhoff slice. Recall state conditioning:

$$\widetilde{u_{\ell k}^{\mathbb{G}}}(\varsigma) := \frac{\varsigma(u_{\ell k}^{\mathbb{G}} \cdot u_{\ell k}^{\mathbb{G}})}{\varsigma(u_{\ell k}^{\mathbb{G}})}$$

$$\Rightarrow \Phi(\widetilde{u_{\ell k}^{\mathbb{G}}}(\varsigma)) = \left[\frac{\varsigma(u_{\ell k}^{\mathbb{G}} u_{ij}^{\mathbb{G}} u_{\ell k}^{\mathbb{G}})}{\varsigma(u_{\ell k}^{\mathbb{G}})} \right]_{i,j=1}^{N}$$

$$=: [\mathbb{P}[\varsigma(j) = i \,|\, \varsigma(k) = \ell]_{i,j=1}^{N}$$

PROPOSITION 4.2. *Let $\mathbb{G}$ be a quantum permutation group on N symbols. For $\varsigma \in \mathbb{G}$, if $\varsigma(u_{ij}^{\mathbb{G}})$ is non-zero, the matrix $\Phi(\widetilde{u_{ij}^{\mathbb{G}}}\varsigma)$ has a one in the (i,j)-th entry. If ς is given by a vector state $\xi \in \mathsf{H}$, $\xi \in \mathrm{ran}(\pi(u_{ij}^{\mathbb{G}}))$ if and only if $\Phi(\varsigma)_{ij} = 1$* •

Proposition 4.2 implies that if e.g.

$$\Phi(\varsigma) = \begin{bmatrix} \Phi(\varsigma)_{11} & \Phi(\varsigma)_{12} & \cdots & \Phi(\varsigma)_{1N} \\ \Phi(\varsigma)_{21} & \Phi(\varsigma)_{22} & \cdots & \Phi(\varsigma)_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(\varsigma)_{N1} & \Phi(\varsigma)_{N2} & \cdots & \Phi(\varsigma)_{NN} \end{bmatrix}$$

$$\Rightarrow \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma) = \begin{bmatrix} \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma)_{11} & 0 & \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma)_{13} & \cdots & \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma)_{1N} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma)_{33} & \cdots & \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma)_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma)_{N1} & 0 & \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma)_{N3} & \cdots & \Phi(\widetilde{u_{22}^{\mathbb{G}}}\varsigma)_{NN} \end{bmatrix}$$

Inductively, assuming $\varsigma(u_{i_n j_n}^{\mathbb{G}} \cdots u_{i_1 j_1}^{\mathbb{G}}) \neq 0$

$$\Phi(\widetilde{u_{i_n j_n}^{\mathbb{G}}} \cdots \widetilde{u_{i_1 j_1}^{\mathbb{G}}}(\varsigma))_{ij} = \mathbb{P}[\varsigma(j) = i \,|\, (\varsigma(j_n) = i_n) \succ \cdots \succ (\varsigma(j_1) = i_1)].$$

Indeed

$$\mathbb{P}[(\varsigma(j) = i) \succ (\varsigma(j_n) = i_n) \succ \cdots \succ (\varsigma(j_1) = i_1)] = \varsigma(|u_{ij}^{\mathbb{G}} u_{i_n j_n}^{\mathbb{G}} \cdots u_{i_1 j_1}^{\mathbb{G}}|^2)$$

$$= \Phi(\widetilde{u_{i_n j_n}^{\mathbb{G}}} \cdots \widetilde{u_{i_1 j_1}^{\mathbb{G}}}(\varsigma))_{ij} \cdot \Phi(\widetilde{u_{i_{n-1} j_{n-1}}^{\mathbb{G}}} \cdots \widetilde{u_{i_1 j_1}^{\mathbb{G}}}(\varsigma))_{i_n j_n} \cdots \Phi(\varsigma)_{i_1 j_1}.$$

EXAMPLE 4.3. *No quantum permutations on three symbols* As a toy example of how the Gelfand-Birkhoff picture is a good intuition, consider the theorem that $S_3^+ = S_3$. This is just to say that $C(S_3^+)$, the universal C*-algebra generated by a 3×3 magic unitary $(u_{ij})_{i,j=1}^3$ is commutative. This was known by Wang [46], but Banica, Bichon, & Collins [6] describe the Fourier-type proof as "quite tricky". Lupini, Mančinska, & Roberson however give a more elementary proof [30].

By allowing talk of *a* quantum permutation the Gelfand–Birkhoff picture suggests *why* there are no quantum permutations on three symbols. Without assuming $C(S_3^+)$ commutative, consider the observable

$$x(1) = u_{11} + 2u_{21} + 3u_{31} \in C(S_3^+),$$

which asks of a quantum permutation $\varsigma \in S_3^+$ what it maps one to. Measure ς with $x(1)$ and the denote the result by $\varsigma(1)$. The intuition might be that as soon as $\varsigma(1)$ is known, $\varsigma(2)$ and $\varsigma(3)$ are entangled in the sense that measurement of $x(2)$ cannot be made without affecting $x(3)$ (but without affecting $x(1)$). This is only intuition: it might still be possible to exhibit e.g. the non-classical event:

$$(\varsigma(3) = 3) \succ (\varsigma(2) = 1) \succ (\varsigma(1) = 3), \tag{4.2}$$

but pausing before measuring $\varsigma(2)$ allows the noting of a relationship between the events $(\varsigma(2) = 1) \succ (\varsigma(1) = 3)$ and $(\varsigma(3) = 2) \succ (\varsigma(1) = 3)$ that implies (4.2) cannot happen.

Suppose that $\varsigma \in S_3^+$ and assume without loss of generality that measuring ς with $x(1)$ gives $\varsigma(1) = 3$ with non-zero probability $\varsigma(u_{31})$, and the quantum permutation transitions to $\widetilde{u_{31}}\varsigma \in S_3^+$.

Now consider, using the fact that $u_{21}u_{31} = 0 = u_{32}u_{31}$, and the rows and columns of u are partitions of unity:

$$u_{31} = (u_{12} + u_{22} + u_{32})u_{31} = (u_{21} + u_{22} + u_{23})u_{31}$$
$$\Rightarrow u_{12}u_{31} = u_{23}u_{31}$$
$$\Rightarrow u_{31}u_{12} = u_{31}u_{23},$$

by taking the adjoint of both sides. This implies that conditioning on $(\varsigma(2) = 1) \succ (\varsigma(1) = 3)$ is the same as conditioning on $(\varsigma(3) = 2) \succ (\varsigma(1) = 3)$:

$$\widetilde{u_{12}}\widetilde{u_{31}}\varsigma = \frac{\varsigma(u_{31}u_{12} \cdot u_{12}u_{31})}{\varsigma(u_{31}u_{12}u_{31})} = \frac{\varsigma(u_{31}u_{23} \cdot u_{23}u_{31})}{\varsigma(u_{31}u_{23}u_{31})} = \widetilde{u_{23}}\widetilde{u_{31}}\varsigma \tag{4.3}$$

Now

$$\Phi(\widetilde{u_{12}}\widetilde{u_{31}}(\varsigma_\xi)) = \begin{bmatrix} 0 & 1 & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}.$$

Using (4.3)

$$\Phi(\widetilde{u_{12}}\widetilde{u_{31}}\varsigma)_{23} = \widetilde{u_{12}}\widetilde{u_{31}}\varsigma(u_{23}) = \widetilde{u_{23}}\widetilde{u_{31}}\varsigma(u_{23}) = \frac{\varsigma(u_{31}u_{23}u_{23}u_{23}u_{31})}{\varsigma(u_{31}u_{23}u_{31})} = 1$$

$$\Rightarrow \Phi(\widetilde{u_{12}}\widetilde{u_{31}}(\varsigma)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \Phi(\widetilde{u_{12}}\widetilde{u_{31}}(\varsigma))_{31} & 0 & 0 \end{bmatrix},$$

and as Φ maps to doubly stochastic matrices, $\Phi(\widetilde{u_{12}}\widetilde{u_{31}}(\varsigma))$ is equal to the permutation matrix (132). This implies that $\widetilde{u_{12}}\widetilde{u_{31}}(\varsigma) \in S_3^+$ is deterministic, that is its Birkhoff slice is a permutation matrix.

Not convinced this implies that $C(S_3^+)$ is commutative? Here is a proof inspired by the above.

THEOREM 4.4. $C(S_3^+)$ is commutative.

Proof. It suffices to show that $u_{11}u_{22} = u_{22}u_{11}$ by showing:

$$u_{11}u_{22} = u_{11}u_{33} = u_{22}u_{33} = u_{22}u_{11}.$$

The first equality follows from:

$$u_{11}(u_{21} + u_{22} + u_{23}) = u_{11}(u_{13} + u_{23} + u_{33}),$$

the second from

$$(u_{11} + u_{21} + u_{31})u_{33} = (u_{21} + u_{22} + u_{23})u_{33},$$

and the third from

$$u_{22}(u_{31} + u_{32} + u_{33}) = u_{22}(u_{11} + u_{21} + u_{31}) \quad \bullet$$

4.2. **Deterministic Permutations.** Let $\jmath : S_N \hookrightarrow M_N(\mathbb{C})$ be the embedding that sends a permutation to its permutation matrix. A *deterministic permutation in* $\mathbb{G}$ is a quantum permutation $\varsigma \in \mathbb{G}$ such that $\Phi(\varsigma) = \jmath(\sigma)$ for some $\sigma \in S_N$. In this case write $\varsigma = \mathrm{ev}_\sigma$. Note

COROLLARY 4.5. *A quantum permutation is deterministic if and only if it is a character.*

Proof. Suppose that $\varsigma = \mathrm{ev}_\sigma$ is deterministic. Consider the GNS representation $(\mathsf{H}_\sigma, \pi_\sigma, \xi_\sigma)$ associated to ς. Note that by Proposition 4.2

$$\mathrm{ev}_\sigma(u_{ij}) = \langle \xi_\sigma, \pi_\sigma(u_{ij}^{\mathbb{G}})(\xi_\sigma) \rangle = 0 \text{ or } 1$$

implies that for all $f \in C(\mathbb{G})$, there exists $f_\sigma \in \mathbb{C}$ such that $\pi_\sigma(f)(\xi_\sigma) = f_\sigma \xi_\sigma$. Therefore

$$\mathrm{ev}_\sigma(gf) = \langle \xi_\sigma, \pi_\sigma(gf)\xi_\sigma \rangle = \langle \xi_\sigma, \pi_\sigma(g)\pi_\sigma(f)(\xi_\sigma) \rangle$$
$$= f_\sigma \langle \xi_\sigma, \pi_\sigma(g)\xi_\sigma \rangle = \mathrm{ev}_\sigma(g)\,\mathrm{ev}_\sigma(f) \quad \bullet$$

On the other hand, by the homomorphism property of a character $\varsigma \in \mathbb{G}$

$$\varsigma(u_{ij}) = \varsigma(u_{ij}^2) = \varsigma(u_{ij})^2 \Rightarrow \varsigma(u_{ij}) = 0 \text{ or } 1,$$

that is ς is deterministic $\bullet$

The following can be extracted from this proof:

COROLLARY 4.6. *A deterministic* $\mathrm{ev}_\sigma \in \mathbb{G}$ *is invariant under wave function collapse* $\quad \bullet$

The set of deterministic permutations therefore coincides with the set of characters $G_{\mathbb{G}} := \Omega(C(\mathbb{G})) \subset \mathbb{G}$, and is either empty or a finite group (Corollary 5.3). This implies that the set $G_{\mathbb{G}}$ coincides with the set $\widetilde{\mathbb{G}}$ of Kalantar and Neufang. A *random permutation in* $\mathbb{G}$ is a convex combination of deterministic permutations, and the convex hull of $G_{\mathbb{G}}$ is the set of random permutations in $\mathbb{G}$, which could also be denoted by $M_p(G_{\mathbb{G}})$.

The study of maximal classical subgroups such as $G_{\mathbb{G}} = \widetilde{\mathbb{G}} \leq \mathbb{G}$ has a long history. An equivalent approach, seen for example in Banica and Skalski [10], is to quotient $C(\mathbb{G})$ by

its commutator ideal. Formally these approaches can give empty sets: the character space is empty if and only if the commutator ideal is the whole algebra. Studying instead the universal version $C_u(\mathbb{G})$ at least guarantees a counit, so that $e \in G_\mathbb{G}$, and $G_\mathbb{G}$ is a group (Corollary 5.3).

With an algebra of functions on a finite quantum permutation group $F(\mathbb{G})$, a *truly quantum* permutation is any quantum permutation zero on all one dimensional factors. More generally, working with the enveloping von Neumann algebra $\ell^\infty(\mathbb{G})$, in which $\imath : C(\mathbb{G}) \hookrightarrow \ell^\infty(\mathbb{G})$ embeds, a deterministic permutation $\mathrm{ev}_\sigma \in \mathbb{G}$ extends to a normal state ω_σ on $\ell^\infty(\mathbb{G})$, and thus has a *support projection* $p_\sigma \in \ell^\infty(\mathbb{G})$ which is the smallest projection $p \in \ell^\infty(\mathbb{G})$ such that $\omega_\sigma(p) = 1$, so that for any projection $q \in \ell^\infty(\mathbb{G})$ such that $\omega_\sigma(q) = 1$, $p_\sigma \leq q$. Note that

$$\mathrm{ev}_\sigma(u^\mathbb{G}_{\sigma(j)j}) = 1 \Rightarrow \omega_\sigma(\imath(u^\mathbb{G}_{\sigma(j)j})) = 1 \Rightarrow p_\sigma \leq \imath(u^\mathbb{G}_{\sigma(j)j}) \Rightarrow p_\sigma = \imath(u^\mathbb{G}_{\sigma(j)j})p_\sigma = p_\sigma\imath(u^\mathbb{G}_{\sigma(j)j}).$$

Any pair $\sigma_1 \neq \sigma_2 \in G_\mathbb{G}$ are distinguished by some $\sigma_1(j) \neq \sigma_2(j)$,

$$p_{\sigma_1}p_{\sigma_2} = p_{\sigma_1}\imath(u^\mathbb{G}_{\sigma_1(j)j})\imath(u^\mathbb{G}_{\sigma_2(j)j})p_{\sigma_2} = p_{\sigma_1}\imath(u^\mathbb{G}_{\sigma_1(j)j}u^\mathbb{G}_{\sigma_2(j)j})p_{\sigma_2} = 0.$$

Define:

$$p_C = \sum_{\sigma \in G_\mathbb{G}} p_\sigma, \tag{4.4}$$

and define a quantum permutation $\varsigma \in \mathbb{G}$ as *truly quantum* if its normal extension $\omega_\varsigma \in S(\ell^\infty(\mathbb{G}))$ has the property that $\omega_\varsigma(p_C) = 0$. If $G_\mathbb{G}$ is empty, $p_C = 0$ and all quantum permutations in $\mathbb{G}$ are truly quantum.

A quick consideration shows that if an algebra of functions on a quantum permutation is a direct sum with a one-dimensional factor $\mathbb{C}f_i$, then the state $f^i : f_i \mapsto 1$ is deterministic.

EXAMPLE 4.7. The *Kac–Paljutkin quantum group of order eight* [22], $\mathfrak{G}_0$, has algebra of functions structure:

$$F(\mathfrak{G}_0) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3 \oplus \mathbb{C}f_4 \oplus M_2(\mathbb{C}). \tag{4.5}$$

Where $I_2 \in M_2(\mathbb{C})$ the identity, and the projection

$$p := \left(0, 0, 0, 0, \begin{pmatrix} \frac{1}{2} & \frac{1}{2}e^{-i\pi/4} \\ \frac{1}{2}e^{+i\pi/4} & \frac{1}{2} \end{pmatrix}\right),$$

a concrete exhibition of $\mathfrak{G}_0 < S_4^+$ (Th. 1.1 (8), [5]) comes by the magic unitary:

$$u^{\mathfrak{G}_0} := \begin{bmatrix} f_1 + f_2 & f_3 + f_4 & p & I_2 - p \\ f_3 + f_4 & f_1 + f_2 & I_2 - p & p \\ p^T & I_2 - p^T & f_1 + f_3 & f_2 + f_4 \\ I_2 - p^T & p^T & f_2 + f_4 & f_1 + f_3 \end{bmatrix}. \tag{4.6}$$

The one dimensional factors give deterministic permutations, $f^1 = \mathrm{ev}_e$, $f^2 = \mathrm{ev}_{(34)}$, $f^3 = \mathrm{ev}_{(12)}$ and $f^4 = \mathrm{ev}_{(12)(34)}$, so that $G_{\mathfrak{G}_0} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 < \mathfrak{G}_0$. Given a quantum permutation $\varsigma \in \mathfrak{G}_0$, measurement with an $x(j)$ will see collapse to either a random permutation or a state on the $M_2(\mathbb{C})$ factor: a *truly quantum* permutation.

Given a truly quantum permutation $\varsigma \in \mathfrak{G}_0$, the random variables $\varsigma(1)$ and $\varsigma(2)$ are entangled in the following sense: if measurement of $x(1)$ gives $\varsigma(1) = 3$, then subsequent measurement of $x(2)$ will yield $\varsigma(2) = 4$ with probability one (and similarly for $\varsigma(1) = 4$ and $\varsigma(2) = 3$ and also $x(1) \succ x(2)$). Similarly $\varsigma(3)$ and $\varsigma(4)$ are entangled in this way: this reminds of

S_3^+ where given the measurement of $\varsigma(1)$, the random variables $\varsigma(2)$ and $\varsigma(3)$ are entangled. However as soon as e.g. $\varsigma(1)$ is known, there is complete uncertainty about $\varsigma(3)$ and $\varsigma(4)$, for example, for every truly quantum permutation $\varsigma \in \mathfrak{G}_0$, and J_2 the matrix of all ones:

$$\Phi(\widetilde{u_{31}^{\mathfrak{G}_0}\varsigma}) = \left[\begin{array}{cc} 0_2 & \frac{1}{2}J_2 \\ I_2 & 0_2 \end{array} \right].$$

With $F(\mathfrak{G}_0) \subset B(\mathbb{C}^6)$, nonclassical behaviour can be exhibited with e.g. the vector state $\varsigma_{e_5} \in \mathfrak{G}_0$:

$$\mathbb{P}[(\varsigma_{e_5}(1) = 4) \succ (\varsigma_{e_5}(3) = 1) \succ (\varsigma_{e_5}(1) = 3)] = \frac{1}{8}.$$

For a truly quantum permutation, certain sequential measurements cannot reveal quantum behaviour. Consider a sequential measurement

$$x(j_n) \succ \cdots \succ x(j_2) \succ x(j_1);$$

if the constituent measurements are all $x(1)$ and $x(2)$ measurements; or all $x(3)$ and $x(4)$ measurements; or a number of $x(1)$ and $x(2)$ measurements followed by $x(3)$ and $x(4)$ measurements (or vice versa), then quantum behaviour will not be observed. Instead these sequential measurements will incorrectly suggest that a quantum permutation $\varsigma \in \mathfrak{G}_0$ is a random permutation (deterministic if there is a mix of $x(1)/x(2)$ and $x(3)/x(4)$ measurements) in the complement of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the dihedral group of order eight.

5. QUANTUM GROUP LAW AND IDENTITY

5.1. **Quantum Group Law.** In the classical case of a finite group $G \leq S_N$, for $g_2, g_1 \in G$, the group law is encoded within the convolution of the pure states ev_{g_1} and ev_{g_2}:

$$(\mathrm{ev}_{g_2} \star \mathrm{ev}_{g_1})\mathbb{1}_{j \to i} = (\mathrm{ev}_{g_2} \otimes \mathrm{ev}_{g_1})\Delta(\mathbb{1}_{j \to i}) = (\mathrm{ev}_{g_2} \otimes \mathrm{ev}_{g_1}) \sum_k (\mathbb{1}_{k \to i} \otimes \mathbb{1}_{j \to k}) = \mathbb{1}_{j \to i}(g_1 g_2).$$

The same game can be played with quantum permutations:

DEFINITION 5.1. The *quantum group law* $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$ is the convolution, $\varsigma_2 \star \varsigma_1 := (\varsigma_2 \otimes \varsigma_1)\Delta$.

PROPOSITION 5.2. *Let $\varsigma_2, \varsigma_1 \in \mathbb{G}$ be quantum permutations. The Birkhoff slice is multiplicative:*

$$\Phi(\varsigma_2 \star \varsigma_1) = \Phi(\varsigma_2)\Phi(\varsigma_1).$$

Proof. Calculate

$$\Phi(\varsigma_2 \star \varsigma_1)_{ij} = (\varsigma_2 \star \varsigma_1)\Delta(u_{ij}^{\mathbb{G}}) = (\varsigma_2 \otimes \varsigma_1) \sum_k (u_{ik}^{\mathbb{G}} \otimes u_{kj}^{\mathbb{G}}) = \sum_k \varsigma_2(u_{ik}^{\mathbb{G}})\varsigma_1(u_{kj}^{\mathbb{G}}) = [\Phi(\varsigma_2)\Phi(\varsigma_1)]_{ij} \bullet$$

COROLLARY 5.3. *The set of deterministic permutations $G_{\mathbb{G}}$ is either empty or a group. It is a group if and only if $\varepsilon \in \mathbb{G}$. Therefore if a quantum permutation group $\mathbb{G}$ is coamenable, or the algebra of continuous functions $C(\mathbb{G}) \cong C_u(\mathbb{G})$, then $G_{\mathbb{G}}$ is a group. In particular, if $\mathbb{G} \leq S_4^+$, then $G_{\mathbb{G}}$ is a group $\bullet$*

If $C(\mathbb{G})$ admits a counit, it plays precisely the role of the identity:

$$\varsigma \star \varepsilon = \varsigma = \varepsilon \star \varsigma \qquad (\varsigma \in \mathbb{G}).$$

Restricted to $G_{\mathbb{G}}$, precomposing ev_σ with the antipode $S : C(\mathbb{G}) \to C(\mathbb{G})$ is an inverse:

$$\mathrm{ev}_\sigma \circ S = \mathrm{ev}_{\sigma^{-1}} \Rightarrow \mathrm{ev}_{\sigma^{-1}} \star \mathrm{ev}_\sigma = \mathrm{ev}_e = \varepsilon.$$

For a general quantum permutation, it might be more accurate to call $\varsigma^{-1} := \varsigma \circ S$ the *reverse* of ς in the sense that

$$\varsigma^{-1}(|u_{i_n j_n} \cdots u_{i_1 j_1}|^2) = \varsigma(|u_{j_n i_n} \cdots u_{j_1 i_1}|^2) \tag{5.1}$$

$$\Rightarrow \mathbb{P}[(\varsigma^{-1}(i_n) = j_n) \succ \cdots \succ (\varsigma^{-1}(i_1) = j_1)] = \mathbb{P}[(\varsigma(j_n) = i_n) \succ \cdots \succ (\varsigma(j_n) = i_n)]$$

Remarkably for a piece about compact quantum groups, the *Haar state* has not yet been introduced. The following is equivalent to more conventional definitions.

DEFINITION 5.4. A quantum permutation group $\mathbb{G}$ has a quantum permutation $h_{\mathbb{G}}$ called the *Haar state* that is the unique annihilator for the quantum group law, that is for all $\varsigma \in \mathbb{G}$

$$h_{\mathbb{G}} \star \varsigma = h_{\mathbb{G}} = \varsigma \star h_{\mathbb{G}}.$$

The non-zero elements of the Birkhoff slice $\Phi(h_{\mathbb{G}})$ are equal along rows and columns. The Haar state can be thought of as the "maximally random" quantum permutation: in the classical case of S_N it corresponds to uniform measure on S_N.

5.2. Abelian Quantum Permutation Groups.

Given a compact group G, the algebra of continuous functions analogue of "*G is abelian*" is that "*C(G) is cocommutative*". In this sense an abelian compact quantum group is given by a cocommutative algebra of continuous functions $C(\widehat{\Gamma})$, that is an algebra of continuous functions on the dual of a discrete group Γ. As the quantum permutations in $\widehat{\Gamma}$ are in the state space of $C(\widehat{\Gamma})$, which is some class of positive definite functions on Γ with pointwise, commutative multiplication, this idea that duals of discrete groups are abelien is trivial through Definition 4.1.

Consider a cyclic group of order N, $\langle \gamma \rangle$. For $\omega := \exp(2\pi i/N)$, consider the following vector in $F(\widehat{\langle \gamma \rangle})^N$:

$$(u^{\widehat{\langle \gamma \rangle}})_{,1} := \frac{1}{N} \begin{bmatrix} e + \gamma + \gamma^2 + \cdots + \gamma^{N-1} \\ e + \omega\gamma + \omega^2\gamma^2 + \cdots + \omega^{N-1}\gamma^{N-1} \\ e + \omega^2\gamma + (\omega^2)^2\gamma^2 + \cdots + (\omega^2)^{N-1}\gamma^{N-1} \\ \cdots \\ e + \omega^{N-1}\gamma + (\omega^{N-1})^2\gamma^2 + \cdots + (\omega^{N-1})^{N-1}\gamma^{N-1} \end{bmatrix}. \tag{5.2}$$

A magic unitary for $\widehat{\langle \gamma \rangle} \cong \widehat{\mathbb{Z}_N} \cong \mathbb{Z}_N < S_N^+$ is the circulant matrix defined by this vector:

$$[u^{\widehat{\langle \gamma \rangle}}]_{i,j} := \frac{1}{N} \sum_{\ell=1}^{N} \omega^{(i-j)\ell}\gamma^{\ell}.$$

Note that

$$\gamma = u_{11}^{\widehat{\langle \gamma \rangle}} + \omega^{N-1}u_{21}^{\widehat{\langle \gamma \rangle}} + \omega^{N-2}u_{31}^{\widehat{\langle \gamma \rangle}} + \cdots + \omega u_{N,1}^{\widehat{\langle \gamma \rangle}}.$$

Let $\Gamma = \langle \gamma_1, \ldots \gamma_k \rangle$ be a finitely generated discrete group, with generators of finite order $N_1, N_2, \ldots, N_k$. Then the dual $\widehat{\Gamma}$ is a quantum permutation group on $N := \sum_p N_p$ symbols via the block magic unitary:

$$u^{\widehat{\Gamma}} = \begin{pmatrix} u^{\widehat{\gamma_1}} & 0 & \cdots & 0 \\ 0 & u^{\widehat{\gamma_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & u^{\widehat{\gamma_k}} \end{pmatrix}.$$

Note that due to the fact that the $u^{\widehat{\gamma_p}}$ are circulant matrices, the entries of each $u^{\widehat{\gamma_p}}$ commute. Such a construction was used to prove Theorem 3.4. It is long known that duals of finite groups G are quantum permutation groups, but earlier references such as [7] placed $\widehat{G} < S^+_{|G|^2}$. That result can be shown by considering each group element an order $|G|$ generator, so making for each g_p a $|G| \times |G|$ magic unitary $u^{\widehat{\langle g_p \rangle}}$, and forming the block matrix $\mathrm{diag}(u^{\widehat{\langle g_1 \rangle}}, \ldots, u^{\widehat{\langle g_{|G|} \rangle}})$. Smaller embeddings of duals of finite groups abound. An induction on $|G|$ shows that $\widehat{G} \leq S^+_{|G|}$. A much smaller embedding $\widehat{S_N} < S^+_{N+2}$ is provided by $S_N = \langle (12), (12 \ldots N) \rangle$. Slightly better, for $N \geq 3$, is $\widehat{S_N} < S^+_{N+1}$ via $S_N = \langle (12), (23 \ldots N) \rangle$. In fact, except for $N = 5, 6, 8$, S_N is generated by an element of order two and an element of order three [35], and so S^+_5 contains all the duals $\widehat{S_N}$ for $N \geq 9$. The dual of any dihedral group, including the infinite dihedral group, is a quantum subgroup of S^+_4. On the other hand, the smallest embedding of the dual of the quaternion group is $\widehat{Q} \leq S^+_8$ via $Q = \langle j, k \rangle$.

Let G be a finite group. Where $\mathrm{Irr}(G)$ is an index set for a maximal set of pairwise inequivalent *unitary* irreducible representations $\rho_\alpha : G \to M_{d_\alpha}(\mathbb{C})$, the algebra of functions on the dual has algebra
$$F(\widehat{G}) = \bigoplus_{\alpha \in \mathrm{Irr}(G)} M_{d_\alpha}(\mathbb{C}).$$
When looking at concrete examples, sometimes it is easy to look at the regular representation:
$$\pi(F(\widehat{G})) \subset B(\mathbb{C}^{|G|}), \quad g : e_h \mapsto e_{gh}.$$
Each one dimensional representation gives a deterministic permutation. The quantum permutations are positive definite functions on G. The function $\mathbb{1}_G \in \widehat{G}$ is the counit $F(\widehat{G}) \to \mathbb{C}$, and has Birkhoff slice $\jmath(e)$. That $\widehat{G}$ is abelian implies that the group of deterministic permutations is abelian. If G is a simple group, either there are no truly quantum permutations, and $G \cong \mathbb{Z}_p$ for a prime p, and $\widehat{G} = G$, or $G_{\widehat{G}}$ is the trivial group. The dual of the symmetric group for $N \geq 2$ has only two deterministic permutations: one is the counit $\varepsilon = \mathbb{1}_G$, and the other is the sign representation, an order two deterministic permutation: $\sum_{\sigma \in S_N} \mathrm{sgn}(\sigma)\delta_\sigma$. The dual of the quaternion group has four deterministic permutations $G_{\widehat{Q}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the dual of a dihedral group has either two or four depending on the degree.

For every odd prime p, the semi-direct product of $\mathbb{Z}_p$ and the multiplicative group $(\mathbb{Z}_p)^\times$, acting by multiplication on $\mathbb{Z}_p$, is a group of order $p(p-1)$ with p characters and thus the dual has p deterministic permutations. In this picture, Pontryagin duality for a finite abelian group G is nothing but $\widehat{G}$ having no truly quantum permutations: all the representations are one dimensional, and hence deterministic.

Suppose that $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle$ is a discrete group such that $\widehat{\Gamma} < S^+_N$. Partition the symbols $1, \ldots, N$ into blocks $B_1, \ldots, B_k$ of length $|\gamma_p|$. The fact that the blocks of $u^{\widehat{\Gamma}}$ are circulant matrices implies that if for $j \in B_p$, the measurement of a quantum permutation $\varsigma \in \widehat{\Gamma}$ with $x(j)$ will see wave function collapse such that the restriction of the state to B_p is now deterministic in the sense that if, for $r, s \in B_p$, $\varsigma(u^{\widehat{\Gamma}}_{rs}) \neq 0$, then the matrix $[\Phi(\widetilde{u^{\widehat{\Gamma}}_{rs}}(\varsigma))]_{i,j \in B_p}$ is a permutation matrix. For example, for $N \geq 9$, and generators σ_1 of order two, and σ_2 of

order three, consider $\widehat{S_N} < S_5^+$ with blocks $B_1 = \{1, 2\}$ from $u^{\widehat{\langle \sigma_1 \rangle}}$ and $B_2 = \{3, 4, 5\}$ from $u^{\widehat{\langle \sigma_2 \rangle}}$. Let $\varsigma \in \widehat{S_N}$ be a quantum permutation. Measure with

$$x(4) = 3u_{34}^{\widehat{S_N}} + 4u_{44}^{\widehat{S_N}} + 5u_{54}^{\widehat{S_N}}.$$

Suppose that the measurement yields $x(4) = 5$. Then the quantum permutation collapses to:

$$\widetilde{u_{54}^{\widehat{S_N}}}(\varsigma) = \frac{\varsigma(u_{54}^{\widehat{S_N}} \cdot u_{54}^{\widehat{S_N}})}{\varsigma(u_{54}^{\widehat{S_N}})},$$

and the circulant nature of $u^{\widehat{\langle \sigma_2 \rangle}}$ implies that the Birkhoff slice

$$\Phi(\widetilde{u_{54}^{\widehat{S_N}}}\varsigma) = \left(\begin{array}{cc|ccc} \alpha & 1-\alpha & 0 & 0 & 0 \\ 1-\alpha & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right),$$

that is the measurement of $x(4) = 5$ collapses the quantum permutation in such a way that it is deterministic on B_2. Unlike in the case of S_3^+, while $x(1)$ and $x(2)$ are now entangled, the measurement of $x(1)$ can disturb the state in such a way that the complete certainty about B_2 is disturbed. The same phenomenon necessarily occurs for the dual of any non-abelian finite groups. The circulant nature of the blocks implies that for all $j_1, j_2 \in B_p$, the measurement of $x(j_1)$ determines $x(j_2)$, and all such measurements could be denoted $x(B_p)$. It could be speculated that the dual of a discrete group $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle$ could model a k particle "entangled" quantum system, where the p-th particle, corresponding to the block B_p, has $|\gamma_p|$ states, labelled $1, \ldots, |\gamma_p|$. Full information about the state of all particles is in general impossible, but measurement with $x(B_p)$ will see collapse of the pth particle to a definite state. Only the deterministic permutations in $\widehat{\Gamma}$ would correspond to classical states.

6. Phenomena

6.1. **Quasi-subgroups.** In the Gelfand-Birkhoff picture a random walk on a quantum permutation group is a sequence $(\varsigma^{\star k})_{k \geq 0}$ in $\mathbb{G}$ (see [33] for more). Of particular interest are *ergodic* random walks, those random walks such that the sequence converges in the weak-* topology to the *Haar state*, $h_{\mathbb{G}}$. A necessary (but dramatically far from sufficient) condition for the convolution powers $(\varsigma^{\star k})_{k \geq 0}$ to converge to the Haar state are that:

$$\Phi(\varsigma^k) = \Phi(\varsigma)^k \to \Phi(h_{\mathbb{G}}).$$

It seems like this observation could help achieve some first partial results in the extension of the finite quantum group random walk ergodic theorem [34] to the case of quantum permutation groups, alas there are many quantum permutations $\varsigma \neq h_{\mathbb{G}}$ such that $\Phi(\varsigma) = \Phi(h_{\mathbb{G}})$. There are even examples of random permutations whose associated random walk is not ergodic, for example $\nu \in M_p(S_3)$ given by:

$$\nu = \frac{1}{3}(\mathrm{ev}_{(12)} + \mathrm{ev}_{(13)} + \mathrm{ev}_{(23)}) \Rightarrow \Phi(\nu) = \frac{1}{3}J_3 = \Phi(h_{S_3}).$$

Very crude results are available though: for example, if $\mathbb{G} < S_N^+$ and any non-trivial N-th root of unity is an eigenvalue of $\Phi(\varsigma)$, then the associated random walk is not ergodic. A qualitative aspect would be that if $\Phi(\varsigma)$ is such that the convergence of $\Phi(\varsigma)^k \to \Phi(h_{\mathbb{G}})$ is

slow, then the associated random walk on $\mathbb{G}$ should take a long time to converge to the Haar state.

If G is a classical finite group, subsets $S \subset G$ that are closed under the group law are subgroups. More generally, for the algebra of continuous functions $C(G)$ on a classical compact group, states $C(G) \to \mathbb{C}$ correspond via integration to Borel probability measures in $M_p(G)$, and in the context of this work could be called *random permutations* (of an infinite set if G is infinite). In this context the quantum group law might be called the random group law, and the Kawada–Itô theorem says that random permutations idempotent with respect to the random group law are Haar measures/states on compact subgroups of G [25].

Suppose that $\mathbb{H} \leq \mathbb{G}$ by $\pi : C(\mathbb{G}) \to C(\mathbb{H})$. Quantum permutations $\varsigma_2', \varsigma_1' \in \mathbb{H}$ are quantum permutations $\varsigma_2, \varsigma_1 \in \mathbb{G}$ by $\varsigma_i := \varsigma_i' \circ \pi$. Note that

$$\varsigma_i(\ker \pi) = \varsigma_i' \pi(\ker \pi) = 0,$$

similarly

$$(\varsigma_2 \star \varsigma_1) \ker \pi = (\varsigma_2 \otimes \varsigma_1)\Delta(\ker \pi) = (\varsigma_2' \otimes \varsigma_1')(\pi \otimes \pi)\Delta(\ker \pi) = (\varsigma_2' \otimes \varsigma_1')\Delta\pi(\ker \pi) = 0,$$

so that $\mathbb{H}$ is closed under the quantum group law of $\mathbb{G}$, and the Haar state of $\mathbb{H}$ in $\mathbb{G}$, $h_{\mathbb{H}} \circ \pi$, is an idempotent state in $\mathbb{G}$, called a *Haar idempotent*.

Compact quantum groups, however, can have non-Haar idempotents. These are quantum permutations $\varsigma \in \mathbb{G}$ idempotent with respect to the quantum group law, $\varsigma \star \varsigma = \varsigma$, that are not the Haar state on any compact subgroup. In the Gelfand picture, idempotent states correspond to measures uniform on virtual objects called *quasi-subgroups*[2]. Pal's idempotents in the Kac–Paljutkin quantum group provide examples of non-Haar idempotents [39]. As the current work allows us to talk of a *set* of quantum permutations $\mathbb{S} \subset \mathbb{G}$, where f^1, f^4 are dual to f_1, $f_2 \in F(\mathfrak{G}_\circ)$, and E^{11}, E^{22} dual to E_{11}, E_{22} in the $M_2(\mathbb{C})$ factor of $F(\mathfrak{G}_0)$, consider the convex hulls $\mathbb{S}_i := \mathrm{co}(\{f^1, f^4, E^{ii}\}) \subset \mathfrak{G}_0$, with associated idempotent states $\frac{1}{4}f^1 + \frac{1}{4}f^4 + \frac{1}{2}E^{ii} \in \mathfrak{G}_0$. Consider $\varsigma \in \mathbb{S}_i$. The Birkhoff slice is given by, for some $\alpha, \beta \in [0, 1]$:

$$\Phi(\varsigma) = \begin{pmatrix} \beta M_\alpha & \frac{(1-\beta)}{2}J_2 \\ \frac{(1-\beta)}{2}J_2 & \beta M_\alpha \end{pmatrix}; \qquad \text{where } M_\alpha = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}.$$

Quasi-subgroups of finite quantum groups behave very much like quantum subgroups: they are closed under the quantum group law (Prop. 3.12, [34]), they contain the identity ε, and they are closed under precomposition with the antipode ([29]).

Let $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle$ be a discrete group with generators of finite order. The quasi-subgroups $S_\Lambda \subset \widehat{\Gamma}$ are given by non-trivial subgroups $\Lambda \leq \Gamma$:

$$S_\Lambda := \{\varsigma \in \widehat{\Gamma} : \varsigma(\lambda) = 1 \text{ for all } \lambda \in \Lambda\}.$$

The associated idempotent state is $\mathbb{1}_\Lambda$, and gives a quantum subgroup only when $\Lambda \triangleleft \Gamma$. This is an illustration of the fact that quotients $\Gamma \to \Gamma/\Lambda$ linearly extend to quotients $C(\widehat{\Gamma}) \to C(\widehat{\Gamma/\Lambda})$, which give rise to quantum subgroups $\widehat{\Gamma/\Lambda} < \widehat{\Gamma}$

$$\pi : C(\widehat{\Gamma}) \to C(\widehat{\Gamma/\Lambda}), \quad \sum_{\gamma \in \Gamma} \alpha_\gamma \gamma \mapsto \sum_{\gamma \in \Gamma} \alpha_\gamma [\gamma].$$

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[2]this terminology is from [24] and has nothing to do with cancellative magmas

Suppose that $\gamma_p \in \Lambda$. Then, where $\Phi(\cdot)_{B_p}$ refers to the $u^{\widehat{\langle \gamma_p \rangle}}$ block of $u^{\widehat{\Gamma}}$:

$$\Phi(\mathbb{1}_\Lambda)_{B_p} = I_{|\gamma_p|},$$

but as Φ is multiplicative, for $k \in \mathbb{N}$:

$$\Phi(\mathbb{1}_\Lambda^{\star k})_{B_p} = I_{|\gamma_p|} \not\to \frac{1}{|\gamma_p|} J_{|\gamma_p|} = \Phi(h_{\widehat{\Gamma}})_{B_p},$$

so that the random walk given by $\mathbb{1}_\Lambda \in \widehat{\Gamma}$ is not ergodic.

Restricting now to finite Γ, the algebra $F(\widehat{\Gamma})$ is also a von Neumann algebra and so contains the support projections of all quantum permutations $\varsigma \in \widehat{\Gamma}$. The support projection of the idempotent quantum permutation $\mathbb{1}_\Lambda \in \widehat{\Gamma}$ is $\chi_\Lambda = \sum_{\lambda \in \Lambda} \lambda/|\Lambda|$, and the random walk given by $\mathbb{1}_\Lambda$ remains concentrated on S_Λ in the sense that $\mathbb{1}_\Lambda^{\star k}(\chi_\Lambda) = 1$ for all $k \in \mathbb{N}$. If no $\gamma_p \in \Lambda$, and $\Gamma \leq S_N^+$, then consider $\lambda \in \Lambda$ and $\Gamma < S_{N+|\lambda|}^+$ via:

$$u^{\widehat{\Gamma} < S_{N+|\lambda|}^+} = \begin{pmatrix} u^{\widehat{\Gamma}} & 0 \\ 0 & u^{\widehat{\langle \lambda \rangle}} \end{pmatrix}.$$

Then the non-ergodicity of the random walk associated with $\mathbb{1}_\Lambda$ can be seen in the λ block: $\Phi(\mathbb{1}_\Lambda)_{B_\lambda} = I_{|\lambda|}$. Note that it is not required for $\varsigma \in S_\Lambda$ that $\varsigma(\gamma) = 0$ for γ in the complement of Λ in Γ. Instead like in the classical, commutative case of a random permutation $\nu \in M_p(\Gamma)$ being supported on $\Lambda < \Gamma$ exhibited by $\nu^{\star k}(\mathbb{1}_{\Gamma - \Lambda}) = 0$ for all $k \in \mathbb{N}$, if $\varsigma(\gamma) = 1$ for all $\lambda \in \Lambda$, so supported on the quasi-subgroup $S_\Lambda \subset \widehat{\Gamma}$, then the role of $\mathbb{1}_{\Gamma - \Lambda}$ is played by the projection $\overline{\chi_\Lambda} := e - \chi_\Lambda \in F(\widehat{\Gamma})$, and it is the case that $\varsigma^{\star k}(\overline{\chi_\Lambda}) = 0$ for all $k \in \mathbb{N}$.

In the case of finite quantum groups, an idempotent state $\phi_\mathbb{S}$ is associated to a *group-like projection* $\mathbb{1}_\mathbb{S}$, which is also its support projection (see [34] for more including original references), and therefore it is tenable to define:

DEFINITION 6.1. Let $\mathbb{G}$ be a finite quantum permutation group with an idempotent state $\phi_\mathbb{S} \in \mathbb{G}$ and associated group-like projection $\mathbb{1}_\mathbb{S} \in F(\mathbb{G})$. The associated *quasi-subgroup* $\mathbb{S} \subset \mathbb{G}$ is given by:

$$\mathbb{S} := \{\varsigma \in \mathbb{G} : \varsigma(\mathbb{1}_\mathbb{S}) = 1\}. \tag{6.1}$$

A very good question is: why are quasi-subgroups given by non-Haar idempotents not considered quantum subgroups? The conventional analysis for an idempotent state $\phi \in \mathbb{G}$ on a finite quantum group $\mathbb{G}$ is to consider the ideals of the associated idempotent states:

$$N_\phi := \{g \in F(\mathbb{G}) : \phi(g^* g) = 0\}.$$

Franz & Skalski (Th. 4.5, [18]) show that ϕ is a Haar idempotent precisely when N_ϕ is two-sided, equivalently self-adjoint, equivalently S-invariant.

That not all quasi-subgroups are subgroups can be explained in the language of measurement. For example for Pal's quasi-subgroup $\mathbb{S}_1 := \mathrm{co}(\{f^1, f^4, E^{11}\}) \subset \mathfrak{G}_0$, the support of the idempotent state

$$\phi_1 = \frac{1}{4} f^1 + \frac{1}{4} f^4 + \frac{1}{2} E^{11},$$

is $\mathbb{1}_{\mathbb{S}_1} := f_1 + f_2 + E_{11}$. Consider $\varsigma := E^{11} \in \mathbb{S}_1$:

$$\mathbb{P}[\varsigma(3) = 2] = \frac{1}{2},$$

and so the quantum permutation conditioned on $\varsigma(3) = 2$ is:

$$\widetilde{u_{13}^{\mathfrak{G}_0}}\varsigma = \frac{\varsigma(u_{13}^{\mathfrak{G}_0} \cdot u_{13}^{\mathfrak{G}_0})}{\varsigma(u_{13}^{\mathfrak{G}_0})} = 2E^{11}(u_{13}^{\mathfrak{G}_0} \cdot u_{13}^{\mathfrak{G}_0}).$$

But:

$$\widetilde{u_{13}^{\mathfrak{G}_0}}\varsigma(\mathbb{1}_{\mathbb{S}_1}) = 2E^{11}(u_{13}^{\mathfrak{G}_0}\mathbb{1}_{\mathbb{S}_1}u_{13}^{\mathfrak{G}_0}) = \frac{1}{2},$$

and this implies with (6.1) that measurement with $u_{13}^{\mathfrak{G}_0}$ has conditioned the quantum permutation outside the quasi-subgroup.

As another example, consider $\widehat{S_3} \leq S_N^+$ given by $u^{\widehat{S_N}} = \operatorname{diag}(u^{\widehat{\langle(12)\rangle}}, \dots)$. The quasi-subgroup $S_{\langle(23)\rangle} \subset \widehat{S_3}$ is the set of quantum permutations associated with the non-Haar idempotent $\mathbb{1}_{\langle(23)\rangle}$. The associated group-like projection is $\chi_{\langle(23)\rangle} = (e + (23))/2$, so that:

$$S_{\langle(23)\rangle} := \left\{ \varsigma \in \widehat{S_3} : \varsigma(e) = \varsigma((23)) = 1 \right\}.$$

Take $\mathbb{1}_{\langle(23)\rangle} \in S_{\langle(23)\rangle}$ so that $\mathbb{P}[\mathbb{1}_{\langle(23)\rangle}(1) = 1] = 1/2$. Then

$$\widetilde{u_{11}^{\widehat{S_3}}}\mathbb{1}_{\langle(23)\rangle}(\delta_{(23)}) = \frac{\mathbb{1}_{\langle(23)\rangle}(u_{11}^{\widehat{S_3}}\delta_{23}u_{11}^{\widehat{S_3}})}{\mathbb{1}_{\langle(23)\rangle}(u_{11}^{\widehat{S_3}})} = 2\,\mathbb{1}_{\langle(23)\rangle}\left(\frac{1}{4}((13) + (23) + (123) + (132))\right) = \frac{1}{2},$$

so that $\widetilde{u_{11}^{\widehat{S_3}}}\mathbb{1}_{\langle(23)\rangle} \notin S_{\langle(23)\rangle}$.

So quasi-subgroups that are not quantum subgroups are just like quantum subgroups: until you start measuring them and it is seen that they are not stable under wave function collapse. It can be shown that for C*-algebras generated by projections such as algebras of functions $F(\mathbb{G})$ on finite quantum permutation groups, if $p \in F(\mathbb{G})$ is a projection and $J := F(\mathbb{G})p$ is a left but not both-sided ideal, then there exists a projection $q \in F(\mathbb{G})$ such that $qpq \notin J$. That implies that for every quasi-subgroup $\mathbb{S} \subset \mathbb{G}$ that is not a quantum subgroup, there is a quantum permutation $\varsigma \in \mathbb{S}$ and a projection $q \in F(\mathbb{G})$ such that $\widetilde{q}\varsigma \notin \mathbb{S}$.

This cannot happen with a genuine quantum subgroup $\mathbb{H} \leq \mathbb{G}$ given by $\pi : F(\mathbb{G}) \to F(\mathbb{H})$. Let $\mathbb{1}_{\mathbb{H}} \in F(\mathbb{G})$ be the support projection of the Haar idempotent $h_{\mathbb{H}} \circ \pi$, and define $\mathbb{H} \leq \mathbb{G}$ by

$$\mathbb{H} := \{\varsigma \in \mathbb{G} : \varsigma(\mathbb{1}_{\mathbb{H}}) = 1\}.$$

Let $\varsigma \in \mathbb{H}$ be measured with a projection $p \in F(\mathbb{G})$. Suppose $\varsigma(p) > 0$, and given $\varsigma \in \mathbb{H}$, $p = 1$ has been observed with wave function collapse to $\widetilde{p}\varsigma$. Then, where $q_{\mathbb{H}} := \mathbb{1}_{\mathbb{G}} - \mathbb{1}_{\mathbb{H}}$, and $\mathbb{1}_{\mathbb{H}} \geq p_\varsigma\mathbb{1}_{\mathbb{H}} = p_\varsigma$ the support projection of ς,

$$\widetilde{p}\varsigma(q_{\mathbb{H}}) = \frac{\varsigma(pq_{\mathbb{H}}p)}{\varsigma(p)} = \frac{\varsigma(p_\varsigma pq_{\mathbb{H}}p)}{\varsigma(p)} = \frac{\varsigma(p_\varsigma\mathbb{1}_{\mathbb{H}}pq_{\mathbb{H}}p)}{\varsigma(p)} = 0,$$

as $\mathbb{1}_{\mathbb{H}}$ is central [18]. Therefore $\widetilde{p}\varsigma(\mathbb{1}_{\mathbb{H}}) = 1$ so that $\widetilde{p}\varsigma \in \mathbb{H}$.

6.2. **Cyclic Cosets.** Being concentrated on a quasi-subgroup is one barrier to a random walk $(\varsigma^{\star k})_{k \geq 0}$ converging to the Haar state. Another is periodicity. In the finite case, assuming that a random walk is not concentrated on a quasi-subgroup, the support of ς for a non-ergodic random walk is concentrated on a cyclic coset of quasi-subgroup:

DEFINITION 6.2. [34] Let $\mathbb{G}$ be a finite quantum permutation group. A quantum permutation $\varsigma \in \mathbb{G}$ is *supported on a cyclic coset of a proper quasi-subgroup* if there exists a pair of projections p_0, p_1, such that $p_0 p_1 = 0$, $p_0 + p_1 \leq \mathbb{1}_{\mathbb{G}}$, $\varsigma(p_1) = 1$, p_0 is a group-like projection, $(\varsigma \otimes I_{F(\mathbb{G})})\Delta(p_1) = p_0$, and there exists $d > 1$ such that $((\varsigma \otimes I_{F(\mathbb{G})})\Delta)^d(p_1) = p_1$.

In the classical case, p_0, p_1 are indicator functions $\mathbb{1}_N$, $\mathbb{1}_{Ng}$ for some $N \triangleleft H$, $H \leq G$, such that, for some $d > 1$, $H/N \cong \mathbb{Z}_d$. In the *irreducible* case, where ς is not concentrated on a subgroup, $N \triangleleft G$, and the p_0, p_1 are elements of a full partition of $\mathbb{1}_G$.

In the 'abelian', dual group case a result of Freslon [19] can be illustrated using the Birkhoff slice. Freslon considers the case where $\varsigma \in \widehat{\Gamma}$ coincides with a non-trivial character on $\Lambda < \Gamma$. In the fashion of the previous section, let $\lambda \in \Lambda$ give a function $\lambda \in F(\widehat{\Gamma})$ and consider $\widehat{\Gamma} \leq S_N^+$ by including λ in the set of generators forming the magic unitary $u^{\widehat{\Gamma}}$. If $\varsigma(\lambda) = 1$ and ς is a character so that $\varsigma(\lambda^\ell) = 1$ then ς is concentrated on the quasi-subgroup S_Λ. Therefore let $\varsigma(\lambda) = e^{i\theta}$, say $e^{2\pi i m/|\lambda|}$. Considering (5.2), and recalling ς restricted to Λ is a character, it is the case that:
$$\Phi(\varsigma)_{B_\lambda} = \jmath((12\dots|\lambda|)^{|\lambda|-m+1}),$$
and the multiplicative nature of the Birkhoff slice implies that $\Phi(\varsigma^{\star k})_{B_\lambda}$ is periodic, not converging to $\Phi(h_{\widehat{\Gamma}})_{B_\lambda}$.

Note that after measurement with $x(B_p)$, $\varsigma \in \widehat{\Gamma}$ collapses quantum permutation whose restriction to $\langle \gamma_p \rangle$ is a character. This follows from the fact that $|\varsigma(\gamma)| \leq 1$ and the form of (5.2). In the classical case, a random walk given by $\nu \in M_p(G)$ which is irreducible but periodic is concentrated on a coset Ng of a proper normal subgroup, and the subsequence $\nu^{\star(k|g|+1)}$ converges to the uniform measure on Ng, that is the convolution of an idempotent h_G and a character/deterministic permutation ev_g:
$$\nu^{\star(k|g|+1)} \to h_N \star \mathrm{ev}_g, \text{ and } \nu^{\star(k|g|+s)} \to h_N \star \mathrm{ev}_{g^s}.$$
In this case, ev_g commutes with the idempotent h_H and so:
$$(h_N \star \mathrm{ev}_g)^{\star k} = h_N \star \mathrm{ev}_{g^k}.$$
In the case of irreducible random walks on quantum permutation groups it is not always the case that periodicity comes from a character/deterministic permutation commuting with an idempotent. It can be shown that the random walk on $\widehat{Q}$ given by:
$$\varsigma = \delta_1 - \delta_{-1} - i\delta_j + i\delta_{-j}$$
is irreducible and periodic, but is not equal to any character/deterministic permutation on $\widehat{Q}$ times an idempotent state. Instead passing to $\langle j \rangle < Q$, the character $\langle j \rangle \to \mathbb{C}$, $\chi : j^s \mapsto \exp(3\pi i s/2)$, is such that $\varsigma = \mathbb{1}_{\langle j \rangle}\chi = \chi\mathbb{1}_{\langle j \rangle}$, but χ is not equal to a restriction of a character on Q. Therefore, a character $\mathrm{ev}_g \in \mathbb{G}$ commuting with an idempotent is sufficient but not necessary for periodic behaviour. One final remark, in the classical case, it is *not* the case that periodic implies concentrated on a coset of a normal subgroup. What is the case is that periodic implies concentrated on a coset of a proper normal subgroup $N \triangleleft H$ of a subgroup

$H \leq G$. In this case, there is the convolution of a deterministic permutation $\mathrm{ev}_g \in H$ and an idempotent h_N, but every deterministic permutation $\mathrm{ev}_g \in H$ is a deterministic permutation $\mathrm{ev}_g \in G$. This is not generally true in the quantum case. The final possibility might be a quantum or even random permutation $\varsigma \in \mathbb{G}$ commuting with an idempotent $\phi_{\mathbb{S}}$ such that $\varsigma^{\star k} = \varepsilon$, however this is impossible because such a quantum permutation satisfies:

$$\Phi(\varsigma^{\star k}) = \Phi(\varsigma)^k = I_N,$$

but the only invertible doubly stochastic matrices come from deterministic permutations.

6.3. **Fixed Points Phenomena and Quantum Transpositions.** Given a quantum permutation group $\mathbb{G} \leq S_N^+$ with algebra of functions $C(\mathbb{G})$, and fundamental representation $u^{\mathbb{G}} \in M_N(C(\mathbb{G}))$, define the number of fixed points observable:

$$\mathrm{fix}^{\mathbb{G}} := \sum_{j=1}^{N} u_{jj}^{\mathbb{G}}.$$

In general, the spectrum of $\mathrm{fix}^{\mathbb{G}}$ contains non-integers: indeed by looking at a faithful representation $\pi(C(\widehat{D_\infty})) \subset B(L^2([0,1], M_2(\mathbb{C})))$ [40], it can be seen that $\sigma(\mathrm{fix}^{\widehat{D_\infty}}) = [0,4]$. For fix in universal $C(S_N^+)$, $\sigma(\mathrm{fix}) = [0,N]$. When the spectrum of $\mathrm{fix}^{\mathbb{G}}$ is finite, such as in the case of a finite quantum permutation group, there is a spectral decomposition in $C(\mathbb{G})$:

$$\mathrm{fix}^{\mathbb{G}} = \sum_{\lambda \in \sigma(\mathrm{fix}^{\mathbb{G}})} \lambda \, p_\lambda, \tag{6.2}$$

and if $\varsigma \in \mathbb{G}$ is such that

$$\mathbb{P}[\mathrm{fix}^{\mathbb{G}}(\varsigma) = \lambda] = \varsigma(p_\lambda) > 0,$$

then

$$\widetilde{p_\lambda \varsigma} := \frac{\varsigma(p_\lambda \cdot p_\lambda)}{\varsigma(p_\lambda)} \in \mathbb{G},$$

is a quantum permutation with λ fixed points, as is any quantum permutation with $\varsigma(p_\lambda) = 1$. Note that if a quantum permutation has λ fixed points, the trace of $\Phi(\varsigma)$ is λ.

DEFINITION 6.3. Where $\mathrm{fix}^{\mathbb{G}} = \sum_{i=1}^{N} u_{ii}^{\mathbb{G}}$ has spectral decomposition (6.2), a quantum permutation in a finite quantum permutation group $\mathbb{G} < S_N^+$ *has λ fixed points* if $\varsigma(p_\lambda) = 1$. A quantum permutation in $\mathbb{G} < S_N^+$ with $N - 2$ fixed points is a quantum *transposition in* $\mathbb{G} < S_N^+$.

Define a magic unitary for $\widehat{S_3} < S_4^+$ by

$$u^{\widehat{S_3}} = \begin{bmatrix} u^{\widehat{\langle(12)\rangle}} & 0 \\ 0 & u^{\widehat{\langle(13)\rangle}} \end{bmatrix}.$$

The spectrum $\sigma(\mathrm{fix}^{\widehat{S_3}}) = \{0, 1, 3, 4\}$. The deterministic permutations ev_e (given by the trivial representation) and $\mathrm{ev}_{(12)(34)}$ (given by the sign representation) have four and zero fixed points. A quantum permutation with three fixed points is:

$$\varsigma = \delta_e + \frac{1}{2}\delta_{(12)} + \frac{1}{2}\delta_{(13)} - \delta_{(23)} - \frac{1}{2}\delta_{(123)} - \frac{1}{2}\delta_{(132)}.$$

It has Birkhoff slice:
$$\Phi(\varsigma) = \begin{pmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 1/4 & 3/4 \end{pmatrix}.$$

Placing $\widehat{S_3} < S_8^+$ via $u^{\widehat{S_3} < S_8^+} = \mathrm{diag}(u^{\widehat{S_3}}, u^{\widehat{S_3}})$ is one way to get a transposition, however note, reminding of $A_8 \lhd S_8$, there is a periodicity to $\varsigma \in S_8^+$:

$$\lim_{k \to \infty} \varsigma^{\star 2k} = \phi_0 := \delta_e + \delta_{(23)} \text{ and } \lim_{k \to \infty} \varsigma^{\star(2k+1)} = \phi_1 := \delta_e - \delta_{(23)}.$$

Note $\phi_0 = \delta_e + \delta_{(23)} \in \widehat{S_3}$ is an idempotent state with support projection $p_0 := (e + (23))/2 \in F(\widehat{G})$, and ϕ_1 has support projection $p_1 := (e - (23))/2$. It is the case that:

$$\mathrm{ev}_{(12)(34)} = \mathbb{1}_{\langle(123)\rangle} - \mathbb{1}_{\{(12),(13),(23)\}}$$

and so

$$\phi_1 = \mathrm{ev}_{(12)(34)} \phi_0 = \phi_0 \, \mathrm{ev}_{(12)(34)}.$$

This implies that:

$$\phi_1^{\star k} = (\phi_0 \, \mathrm{ev}_{(12)(34)})^{\star k} = \phi_0 \, \mathrm{ev}_{(12)(34)^k} = \begin{cases} \phi_0, & \text{if } k \text{ even} \\ \phi_1, & \text{if } k \text{ odd} \end{cases}.$$

The quantum permutation ς is concentrated on p_1, a cyclic coset of the quasi-subgroup $S_{\langle(23)\rangle} \subset \widehat{S_3}$ encountered in the previous section. This implies that the support of the irreducible random walk associated with ς alternates between p_0 and p_1. On the technical level, this is unlike the periodicity of the state uniform on permutations of odd parity because ϕ_0 is not the Haar state on a quantum subgroup of $\widehat{S_3}$, so it doesn't make sense to say that $\langle(23)\rangle$ is normal in S_3. See [34] for more.

Another quantum phenomenon is that there are quantum permutations with $N - 1$ fixed points which are *not* the identity. Consider the finite quantum group $\widehat{S_4} < S_5^+$ given by the magic unitary:

$$u^{\widehat{S_4^+}} = \begin{pmatrix} u^{\langle(12)\rangle} & 0 \\ 0 & u^{\langle(234)\rangle} \end{pmatrix} \in M_5(F(\widehat{S_4})).$$

Representing $\pi(F(\widehat{S_4})) \subset B(\mathbb{C}^{24})$ with the regular representation, and employing a CAS, it can be found that

$$\sigma(\mathrm{fix}^{\widehat{S_4}}) = \left\{ 0, \frac{5 - \sqrt{17}}{2}, 1, 2, 3, 4, \frac{5 + \sqrt{17}}{2}, 5 \right\},$$

so that the phenomenon of quantum permutations with a non-integer number of fixed points occurs for $\widehat{S_4}$. Define subsets of S_4:

$$X_1 := \langle(34)\rangle, \ X_2 := (12)\langle(34)\rangle, \ X_3 := \{\sigma : \sigma(1) = 1\} \backslash X_1, \ X_4 := \{\sigma : \sigma(2) = 2\} \backslash X_1,$$

$$X_5 := \{(13)(24), (14)(23), (1423), (1324)\}, \text{ and } X_6 := S_4 \backslash \left(\bigcup_{\ell=1}^{5} X_\ell \right).$$

Then the following quantum permutation has four fixed points and is not the identity/counit:

$$\varsigma := \mathbb{1}_{X_1} + \frac{1}{3}\mathbb{1}_{X_2} + \frac{5}{6}\mathbb{1}_{X_3} - \frac{1}{2}\mathbb{1}_{X_4} - \frac{2}{3}\mathbb{1}_{X_5} - \frac{1}{6}\mathbb{1}_{X_6}, \tag{6.3}$$

and has Birkhoff slice:

$$\Phi(\varsigma) = \begin{pmatrix} 2/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 8/9 & 1/18 & 1/18 \\ 0 & 0 & 1/18 & 8/9 & 1/18 \\ 0 & 0 & 1/18 & 1/18 & 8/9 \end{pmatrix}$$

As the convolution power in $\widehat{S_4}$ is pointwise multiplication, $\varsigma^{\star k} \to \mathbb{1}_{X_1}$, and there is convergence to a non-Haar idempotent.

For any integer $\ell \geq 2$, quantum permutation groups $\mathbb{G} \leq S_N^+$ are also quantum permutation groups $\mathbb{G} < S_{\ell N}^+$ via:

$$u^{\mathbb{G} < S_{\ell N}^+} := \mathrm{diag}(u^{\mathbb{G}}, u^{\mathbb{G}}, \ldots, u^{\mathbb{G}}) \in M_{\ell N}(C(\mathbb{G})),$$

and so if $\varsigma \in \mathbb{G}$ has $N - \frac{2}{\ell}$ fixed points, $\varsigma \in \mathbb{G} \leq S_{\ell N}$ is a transposition, that is it has $\ell N - 2$ fixed points. Therefore, $\varsigma \in \widehat{S_4}$ given by (6.3) is a quantum transposition $\varsigma \in \widehat{S_4} < S_{10}^+$ whose convolution powers do not exhibit periodicity.

When $\sigma(\mathrm{fix}^{\mathbb{G}})$ is no longer finite pass to the universal enveloping von Neumann algebra $\ell^\infty(\mathbb{G})$ of $C(\mathbb{G})$, which contains $\imath : C(\mathbb{G}) \hookrightarrow \ell^\infty(\mathbb{G})$ and the spectral projections of elements of $C(\mathbb{G})$. Consider $\imath(\mathrm{fix}^{\mathbb{G}}) \in \ell^\infty(\mathbb{G})$, with spectral projections $\mathbb{1}_S(\mathrm{fix}^{\mathbb{G}})$, in particular $p_\lambda := \mathbb{1}_{\{\lambda\}}(\mathrm{fix}^{\mathbb{G}})$. Where ω_ς is the normal extension of $\varsigma \in \mathbb{G}$, define:

$$\mathbb{P}[\varsigma \text{ has } \lambda \text{ fixed points}] := \omega_\varsigma(p_\lambda),$$

and say that ς has λ fixed points if $\omega_\varsigma(p_\lambda) = 1$.

Let $\pi : C(S_N^+) \to C(\mathbb{G})$ give a quantum subgroup $\mathbb{G} < S_N^+$, with $\mathrm{fix}^{\mathbb{G}}$ of finite spectrum. Where $\varsigma_0 \in \mathbb{G}$ has λ_0 fixed points, the quantum permutation $\varsigma := \varsigma_0 \circ \pi \in S_N^+$ has λ_0 fixed points in S_N^+. Consider the C*-algebras generated by fix and $\mathrm{fix}^{\mathbb{G}}$, $\mathrm{C}^*(\mathrm{fix}) \cong C([0,N])$, and, $\mathrm{C}^*(\mathrm{fix}^{\mathbb{G}}) \cong C(\sigma(\mathrm{fix}^{\mathbb{G}}))$. For $f \in \mathrm{C}^*(\mathrm{fix})$,

$$\pi(f) = \sum_{\lambda \in \sigma(\mathrm{fix}^{\mathbb{G}})} f(\lambda)\delta_\lambda.$$

By assumption, for $\lambda_0 \in \sigma(\mathrm{fix}^{\mathbb{G}})$, $\varsigma_0(\delta_{\lambda_0}) = 1$ and so

$$\varsigma(f) = \varsigma_0\left(\sum_{\lambda \in \sigma(\mathrm{fix}^{\mathbb{G}})} f(\lambda)\delta_\lambda\right) = f(\lambda_0) \Rightarrow \varsigma = \mathrm{ev}_{\lambda_0}.$$

The enveloping von Neumann algebra $\mathrm{C}^*(\mathrm{fix})^{**} \cong \ell^\infty([0,N]) \subset B(\ell^2([0,N]))$ and ς extends to $\omega_\varsigma = \mathrm{ev}_{\lambda_0} \in \ell^\infty([0,N])^*$. The spectral projection $\mathbf{1}_{\{\lambda_0\}}(\mathrm{fix})$ is:

$$\ell^2([0,N]) \ni f \mapsto f(\lambda_0)\delta_{\lambda_0} \in \ell^2([0,N]), \tag{6.4}$$

so that $\mathbf{1}_{\{\lambda_0\}}(\mathrm{fix})$ may be identified with $\delta_{\lambda_0} \in \ell^\infty([0,N])$ and indeed $\omega_\varsigma(\mathbf{1}_{\{\lambda_0\}}(\mathrm{fix})) = \mathrm{ev}_{\lambda_0}(\delta_{\lambda_0}) = 1$, so that ς also has λ_0 fixed points.

The quantum transposition φ_{tr} studied by Freslon, Teyssier and Wang [20] is a *central state*, and it is the only central quantum transposition in S_N^+. *Central states* such as φ_{tr} have some nice properties: that for any irreducible representation $n \in \mathbb{N}_{\geq 0}$, $\varphi_{\text{tr}}(\rho_{ij}^{(n)}) = \varphi_{\text{tr}}(n)\delta_{i,j}$, and as the matrix elements of the irreducible representations form a basis of $C(S_N^+)$, they are completely determined by their restriction to the central algebra $C(S_N^+)_0$ generated by the characters as:

$$\varphi_{\text{tr}}(\chi_n) = \sum_{i=1}^{d_n} \varphi_{\text{tr}}(\rho_{ii}^{(n)}) = d_n \varphi_{\text{tr}}(n).$$

The central algebra is commutative, and it follows from spectral theory that:

$$C(S_N^+)_0 \cong C([0, N]).$$

The isomorphism from the characters to $C([0, N])$ is given by $\chi_n \mapsto (t \mapsto U_{2n}(\sqrt{t}/2))$, where U_n are the Chebyshev polynomials of the second kind, and therefore, restricted to the central algebra, $\chi_0 + \chi_1 = \text{fix}$. The state φ_{tr} is given by $\text{ev}_{N-2} \in C([0, N])^*$. The normal extension of φ_{tr} is also ev_{N-2}, and indeed $\text{ev}_{N-2}(p_{N-2}) = 1$, and because of (6.4), ev_{N-2} is the unique central quantum transposition in S_N^+.

6.4. Maximality of $S_N < S_N^+$.

As mentioned in Section 3.2, there is the following maximality result:

THEOREM 6.4. *For $N \leq 5$, there is no intermediate $S_N < \mathbb{G} < S_N^+$* •

The result is conjectured to be true for $N > 5$ also. A strong way to interpret the conjecture is to say that all that has to be added to S_N to get the whole of S_N^+ is an arbitrary quantum permutation. In this section the $u_{ij} \in C(S_N^+)$, and, without making it notationally explicit, all quantum permutations are assumed elements of S_N^+ via, for $\varsigma_0' \in \mathbb{G}$ and $\text{ev}_\sigma \in S_N$:

$$\varsigma_0(u_{ij}) = \varsigma_0' \circ \pi_{\mathbb{G}}(u_{ij}) = \varsigma_0'(u_{ij}^{\mathbb{G}}), \text{ and } \text{ev}_\sigma(u_{ij}) = \text{ev}_\sigma' \circ \pi_{S_N}(u_{ij}) = \mathbb{1}_{j \to i}(\sigma)$$

It is thus possible to convolve quantum permutations in $\mathbb{G} < S_N^+$ with deterministic permutations in S_N but not in $G_{\mathbb{G}}$.

Let $\mathbb{G}_0 < S_N^+$ and $\varsigma' \in \mathbb{G}_0$ any (non-classical) quantum permutation. Working with $C_u(\mathbb{G}_0)$, and h_{S_N} the Haar state on $F(S_N)$ define:

$$\varsigma'' := \frac{1}{2}\varepsilon + \frac{1}{2}\varsigma' \in \mathbb{G}_0, \text{ and } \varsigma := h_{S_N} \star \varsigma'' \in S_N^+.$$

The Cesáro averages

$$\varsigma_n := \frac{1}{n} \sum_{k=1}^{n} \varsigma^{\star k} \xrightarrow{w^*} \phi_\varsigma,$$

an idempotent state in S_N^+. There are three possibilities here.

(1) $\phi_\varsigma = h_{S_N^+}$, the Haar state on S_N^+;
(2) $\phi_\varsigma = h_{\mathbb{G}}$, for an intermediate quantum group $S_N < \mathbb{G} < S_N^+$;
(3) ϕ_ς is a non-Haar idempotent, giving an intermediate quasi-subgroup $S_N \subsetneq \mathbb{S} \subsetneq S_N^+$.

The conjecture of maximality $S_N < S_N^+$ believes that (2) cannot happen but does not preclude (3). If neither (2) nor (3) can happen, then *any* non-classical quantum permutation in *any* quantum permutation group together with h_{S_N} generates the whole of S_N^+.

This section illustrates how convolving deterministic permutations with non-classical quantum permutations can 'move' the quantumness around. This first result shows that deterministic permutations can be used to permute labels in the following sense:

PROPOSITION 6.5. *Suppose that $\varsigma_0 \in \mathbb{G} < S_N^+$ is such that:*

$$\alpha := \mathbb{P}[(\varsigma_0(j_n) = i_n) \succ \cdots \succ (\varsigma_0(j_1) = i_1)] = \varsigma_0(|u_{i_n j_n} \cdots u_{i_1 j_1}|^2) > 0,$$

then $\varsigma := \mathrm{ev}_{\sigma_2} \star \varsigma_0 \star \mathrm{ev}_{\sigma_1^{-1}}$ is such that:

$$\mathbb{P}[(\varsigma(\sigma_1(j_n)) = \sigma_2(i_n)) \succ \cdots \succ (\varsigma(\sigma_1(j_1)) = \sigma_2(i_1))] = \varsigma(|u_{\sigma_2(i_n)\sigma_1(j_n)} \cdots u_{\sigma_2(i_1)\sigma_1(j_1)}|^2) = \alpha.$$

Proof. Calculate $\varsigma(|u_{\sigma_2(i_n)\sigma_1(j_n)} \cdots u_{\sigma_2(i_1)\sigma_1(j_1)}|^2)$

$$= (\mathrm{ev}_{\sigma_2} \otimes \varsigma_0 \otimes \mathrm{ev}_{\sigma_1^{-1}}) \Delta^{(2)}(u_{\sigma_2(i_1)\sigma_1(j_1)} \cdots u_{\sigma_2(i_n)\sigma_1(j_n)} \cdots u_{\sigma_2(i_1)\sigma_1(j_1)})$$

$$= (\mathrm{ev}_{\sigma_2} \otimes \varsigma_0 \otimes \mathrm{ev}_{\sigma_1^{-1}}) \Delta^{(2)}(u_{\sigma_2(i_1)\sigma_1(j_1)}) \cdots \Delta^{(2)}(u_{\sigma_2(i_n)\sigma_1(j_n)}) \cdots \Delta^{(2)}(u_{\sigma_2(i_1)\sigma_1(j_1)})$$

$$= (\mathrm{ev}_{\sigma_2} \otimes \varsigma_0 \otimes \mathrm{ev}_{\sigma_1^{-1}}) \left(\sum_{k_1,k_2=1}^{N} u_{\sigma_2(i_1)k_1} \otimes u_{k_1 k_2} \otimes u_{k_2 \sigma_1(j_1)} \right) \times$$

$$\cdots \left(\sum_{k_{2n-1},k_{2n}=1}^{N} u_{\sigma_2(i_n)k_{2n-1}} \otimes u_{k_{2n-1}k_{2n}} \otimes u_{k_{2n}\sigma_1(j_n)} \right) \times$$

$$\cdots \left(\sum_{k_{4n-3},k_{4n-2}=1}^{N} u_{\sigma_2(i_1)k_{4n-3}} \otimes u_{k_{4n-3}k_{4n-2}} \otimes u_{k_{4n-2}\sigma_1(j_1)} \right)$$

$$= \sum_{k_1,\ldots,k_{4n-2}=1}^{N} \mathrm{ev}_{\sigma_2}(u_{\sigma_2(i_1)k_1} \cdots u_{\sigma_2(i_n)k_{2n-1}} \cdots u_{\sigma_2(i_1)k_{4n-3}}) \times$$

$$\varsigma_0(u_{k_1 k_2} \cdots u_{k_{2n-1}k_{2n}} \cdots u_{k_{4n-3}k_{4n-2}}) \, \mathrm{ev}_{\sigma_1^{-1}}(u_{k_2\sigma_1(j_1)} \cdots u_{k_{2n}\sigma_1(j_n)} \cdots u_{k_{4n-2}\sigma_1(j_1)})$$

The deterministic permutations are characters and moreover $\mathrm{ev}_\sigma(u_{ij}) = 1$ if and only if $\sigma(j) = i$, and this implies $k_1 = i_1, \ldots, k_{2n-1} = i_n, \ldots, k_{4n-3} = i_1$ and also $k_2 = j_1, \ldots, k_{2n} = j_n, \ldots, k_{4n-2} = j_1$ and so

$$\varsigma(|u_{\sigma_2(i_n)\sigma_1(j_n)} \cdots u_{\sigma_2(i_1)\sigma_1(j_1)}|^2) = \varsigma_0(u_{i_1 j_1} \cdots u_{i_n j_n} \cdots u_{i_1 j_1}) = \alpha > 0 \qquad \bullet$$

Consider, for example, $S_N < S_N^+$ and $\mathfrak{G}_0 < S_N^+$ given by $u^{\mathfrak{G}_0 < S_N^+} = \mathrm{diag}(u^{\mathfrak{G}_0}, \mathbb{1}_{\mathfrak{G}_0}, \cdots, \mathbb{1}_{\mathfrak{G}_0})$. The quantum permutations in $\mathfrak{G}_0$ can be combined with deterministic permutations to get new quantum permutations in S_N^+, for example, where ς_{e_5} is the vector state on $\mathfrak{G}_0$, for $\varsigma := \mathrm{ev}_{(14)} \star \varsigma_{e_5} \star \mathrm{ev}_{(16)}$

$$\mathbb{P}[(\varsigma(6) = 3) \succ (\varsigma(4) = 2) \succ (\varsigma(6) = 1)] = \varsigma(|u_{36}u_{24}u_{16}|^2) = \varsigma_{e_5}(u_{41}u_{24}u_{31}u_{24}u_{41}) = \frac{1}{8} > 0.$$

PROPOSITION 6.6. *For any quantum permutation* $\varsigma \in \mathbb{G} < S_N^+$ *and* $\sigma \in S_N$:

$$(\varsigma \star \mathrm{ev}_\sigma \star \varsigma)(|u_{i_4 j_4} u_{i_3 j_3} u_{i_2 j_2} u_{i_1 j_1}|^2)$$

$$= \sum_{k_1, \cdots, k_7 = 1}^{N} \varsigma(u_{i_1 \sigma(k_1)} u_{i_2 \sigma(k_2)} u_{i_3 \sigma(k_3)} u_{i_4 \sigma(k_4)} u_{i_3 \sigma(k_5)} u_{i_2 \sigma(k_6)} u_{i_1 \sigma(k_7)}) \times$$

$$\varsigma(u_{k_1 j_1} u_{k_2 j_2} u_{k_3 j_3} u_{k_4 j_4} u_{k_5 j_3} u_{k_6 j_2} u_{k_7 j_1}) \qquad \bullet$$

This can be used to show that, for example, where $\varsigma = \varsigma_{e_5} \star \mathrm{ev}_{(23)} \star \varsigma_{e_5}$

$$\varsigma(|u_{11} u_{32} u_{13} u_{31}|^2) = \mathbb{P}[(\varsigma(1) = 1) \succ (\varsigma(2) = 3) \succ (\varsigma(3) = 1) => (\varsigma(1) = 3)] > 0.$$

As a final, isolated, example, note that measurement of $\varsigma_{e_5} \in \mathfrak{G}_0$ can see $3 \mapsto 1$ and subsequently $3 \mapsto 2$, however the probability that it ever subsequently maps three to anything other than one or two is zero. However the quantum permutation $\varsigma := \mathrm{ev}_{(23)} \star \varsigma_{e_5} \star \mathrm{ev}_{(24)} \star \varsigma_{e_5} \star \mathrm{ev}_{(23)} \in S_N^+$ is such that:

$$\mathbb{P}[(\varsigma(1) = 4) \succ (\varsigma(2) = 2) \succ (\varsigma(1) = 3) \succ (\varsigma(2) = 2) \succ (\varsigma(1) = 1)] > 0.$$

To show this is to show that $\varsigma(|u_{41} u_{22} u_{31} u_{22} u_{11}|^2) > 0$. This quantity is equal to

$$= (\mathrm{ev}_{(23)} \otimes \varsigma_{e_5} \otimes \mathrm{ev}_{(24)} \otimes \varsigma_{e_5} \otimes \mathrm{ev}_{(23)}) \Delta^{(4)}(u_{11} u_{22} u_{31} u_{22} u_{41} u_{22} u_{31} u_{22} u_{11})$$

$$= \sum_{k_1, \ldots, k_{36} = 1}^{4} \mathrm{ev}_{(23)}(u_{1 k_1} u_{2 k_2} u_{3 k_3} u_{2 k_4} u_{4 k_5} u_{2 k_6} u_{3 k_7} u_{2 k_8} u_{1 k_9})$$

$$\times \varsigma_{e_5}(u_{k_1 k_{10}} \cdots u_{k_9 k_{18}}) \, \mathrm{ev}_{(24)}(u_{k_{10} k_{19}} \cdots u_{k_{18} k_{27}}) \varsigma_{e_5}(u_{k_{19} k_{28}} \cdots u_{k_{27} k_{36}})$$

$$\times \mathrm{ev}_{(23)}(u_{k_{28} 1} u_{k_{29} 2} u_{k_{30} 1} u_{k_{31} 2} u_{k_{32} 1} u_{k_{33} 2} u_{k_{34} 1} u_{k_{35} 2} u_{k_{36} 1})$$

$$= \sum_{k_{10}, \ldots, k_{27}}^{4} \varsigma_{e_5}(u_{1 k_{10}} u_{3 k_{11}} u_{2 k_{12}} u_{3 k_{13}} u_{4 k_{14}} u_{3 k_{15}} u_{2 k_{16}} u_{3 k_{17}} u_{1 k_{18}})$$

$$\times \mathrm{ev}_{(24)}(u_{k_{10} k_{19}} \cdots u_{k_{18} k_{27}}) \varsigma_{e_5}(u_{k_{19} 1} u_{k_{20} 3} u_{k_{21} 1} u_{k_{22} 3} u_{k_{23} 1} u_{k_{24} 3} u_{k_{25} 1} u_{k_{26} 3} u_{k_{27} 1})$$

Now consider that the ordered pairs

$$(k_{10}, k_{19}), \ldots, (k_{18}, k_{27}) \overset{!}{\in} \{(1,1), (2,4), (3,3), (4,2)\}.$$

Recall that ς_{e_5} is pre-composed with $\pi_{\mathfrak{G}_0} : C(S_N^+) \to F(\mathfrak{G}_0)$. Considering the algebra structure (4.5) on $F(\mathfrak{G}_0)$, the magic unitary $u^{\mathfrak{G}_0 < S_N^+} = \mathrm{diag}(u^{\mathfrak{G}_0}, \mathbb{1}_{\mathfrak{G}_0}, \ldots, \mathbb{1}_{\mathfrak{G}_0})$ (4.6), and that ς_{e_5} is zero on the one-dimensional factors, there is only one choice for the ordered pairs $(k_{10}, k_{19}), \ldots, (k_{18}, k_{27})$, namely alternating $(3,3)$ and $(1,1)$, except for $(k_{14}, k_{23}) = (2,4)$. This leaves:

$$\varsigma(|u_{41} u_{22} u_{31} u_{22} u_{11}|^2) = \varsigma_{e_5}(u_{13} u_{31} u_{23} u_{31} u_{42} u_{31} u_{23} u_{31} u_{13}) \varsigma_{e_5}(u_{31} u_{13} u_{31} u_{13} u_{41} u_{13} u_{31} u_{13} u_{31}),$$

which is strictly positive.

6.5. **Orbits and Orbitals.** One barrier to attacks on the maximality conjecture via methods inspired by the previous section is the non-transitivity of *orbitals*. The study of orbits and orbitals was initiated by Lupini, Mančinska, and Roberson [30]. Banica and Freslon [9] independently studied orbits. In this section one-orbitals (or orbits), two-orbitals, and three-orbitals are studied, in the language of quantum permutations, and a new (conjectured) counter-example to the transitivity of the three-orbital relation is given.

In the case of $\mathbb{G} < S_N^+$, for any non-zero $u_{ij}^{\mathbb{G}} \in C(\mathbb{G})$, there is a (pure) quantum permutation $\varsigma \in \mathbb{G}$ such that $\varsigma(j) = i$ with probability one. Working in the universal, $C_u(\mathbb{G})$, setting, define the *orbit relation*, or *one-orbital relation*, on N by $i \sim_1 j$ if $u_{ij}^{\mathbb{G}} \neq 0$. The identity $\varepsilon \in \mathbb{G}$ is a quantum permutation that fixes all points, so $u_{ii}^{\mathbb{G}}(\varepsilon) = 1$, therefore $u_{ii}^{\mathbb{G}} \neq 0$, and thus $\sim_1$ is reflexive. Suppose that $i \sim_1 j$ so $u_{ij}^{\mathbb{G}} \neq 0$, and $\varsigma \in \mathbb{G}$ a (pure) quantum permutation such that $\varsigma(u_{ij}^{\mathbb{G}}) = \|u_{ij}^{\mathbb{G}}\| = 1$, then $j \sim_1 i$ by the reverse of ς, $\varsigma^{-1} := \varsigma \circ S$, as

$$\mathbb{P}[\varsigma^{-1}(i) = j] = \varsigma^{-1}(u_{ji}^{\mathbb{G}}) = \varsigma(S(u_{ji}^{\mathbb{G}})) = \varsigma(u_{ij}^{\mathbb{G}}) = 1.$$

Let ς_2 and ς_1 be such that $\varsigma_2(u_{i\ell}^{\mathbb{G}}) = \|u_{i\ell}^{\mathbb{G}}\| = 1$ and $\varsigma_1(u_{\ell j}^{\mathbb{G}}) = \|u_{\ell j}^{\mathbb{G}}\| = 1$. Consider the quantum permutation $\varsigma_2 \star \varsigma_1 \in \mathbb{G}$:

$$\varsigma_2 \star \varsigma_1(u_{ij}^{\mathbb{G}}) = (\varsigma_2 \otimes \varsigma_1)\Delta(u_{ij}^{\mathbb{G}}) = \sum_{k=1}^{N} \varsigma_2(u_{ik}^{\mathbb{G}})\varsigma_2(u_{kj}^{\mathbb{G}}).$$

However if $\varsigma_2(u_{i\ell}^{\mathbb{G}}) = 1$ and $\varsigma_1(u_{\ell j}^{\mathbb{G}}) = 1$, all of the terms with $k \neq \ell$ are zero, so

$$\varsigma_2 \star \varsigma_1(u_{ij}^{\mathbb{G}}) = \varsigma_2(u_{i\ell}^{\mathbb{G}})\varsigma_1(u_{\ell j}^{\mathbb{G}}) = 1,$$

and so $u_{ij}^{\mathbb{G}} \neq 0$, so that $\sim_1$ is an equivalence relation on N. This can also be seen by considering $\Phi(\varsigma_2 \star \varsigma_1) = \Phi(\varsigma_2)\Phi(\varsigma_1)$.

Higher order orbitals may also be defined as relations on N^k. Say that:

$$(i_m, \cdots, i_1) \sim_m (j_m, \ldots j_1) \Leftrightarrow u_{i_m j_m}^{\mathbb{G}} \cdots u_{i_1 j_1}^{\mathbb{G}} \neq 0.$$

Similarly to the above, with $\varepsilon \in \mathbb{G}$, and $\mathbb{G}$ closed under reversal, $\sim_m$ is reflexive and symmetric. The following is Lemma 3.4 from [30], but with the proof rewritten in the language of the current work:

PROPOSITION 6.7. *The two-orbital, $\sim_2$, is transitive.*

Proof. Note that for projections $p, q \in C(\mathbb{G})$, the C*-identity gives:

$$\|pq\|^2 = \|(pq)^* pq\| = \|qpq\|,$$

and so $pq = 0$ exactly when $qpq = 0$. Suppose that $u_{i_2\ell_2}^{\mathbb{G}} u_{i_1\ell_1}^{\mathbb{G}} \neq 0$ and $u_{\ell_2 j_2}^{\mathbb{G}} u_{\ell_1 j_1}^{\mathbb{G}} \neq 0$ so that $u_{i_1\ell_1}^{\mathbb{G}} u_{i_2\ell_2}^{\mathbb{G}} u_{i_1\ell_1}^{\mathbb{G}} \neq 0$ and $u_{\ell_1 j_1}^{\mathbb{G}} u_{\ell_2 j_2}^{\mathbb{G}} u_{\ell_1 j_1}^{\mathbb{G}} \neq 0$, so that $(i_2, i_1) \sim_2 (\ell_2, \ell_1)$ and $(\ell_2, \ell_1) \sim_2 (j_2, j_1)$. Let ς_2', ς_1' be such that $\varsigma_2'(u_{i_1\ell_1}^{\mathbb{G}} u_{i_2\ell_2}^{\mathbb{G}} u_{i_1\ell_1}^{\mathbb{G}}) = \|u_{i_1\ell_1}^{\mathbb{G}} u_{i_2\ell_2}^{\mathbb{G}} u_{i_1\ell_1}^{\mathbb{G}}\| \neq 0$, and $\varsigma_1'(u_{\ell_1 j_1}^{\mathbb{G}} u_{\ell_2 j_2}^{\mathbb{G}} u_{\ell_1 j_1}^{\mathbb{G}}) = \|u_{\ell_1 j_1}^{\mathbb{G}} u_{\ell_2 j_2}^{\mathbb{G}} u_{\ell_1 j_1}^{\mathbb{G}}\| \neq 0$. This means that ς_2' maps ℓ_2 to i_2 after mapping ℓ_1 to i_1 with non-zero probability (and similar for ς_1'). Note in particular, by e.g. (2.1), that $\varsigma_2'(u_{i_1\ell_1}^{\mathbb{G}})$, $\varsigma_1'(u_{\ell_1 j_1}^{\mathbb{G}}) \neq 0$.

Conditioning ς_2' and ς_1' so that they map $\ell_1 \to i_1$ and $j_1 \to \ell_1$ with probability one gives quantum permutations $\varsigma_2 := \widetilde{u^{\mathbb{G}}_{i_1\ell_1}}\varsigma_2'$ and $\varsigma_1 := \widetilde{u^{\mathbb{G}}_{\ell_1 j_1}}\varsigma_1'$ that can still map $\ell_2 \to i_2$ and $j_2 \to \ell_2$;

$$\varsigma_2(u^{\mathbb{G}}_{i_2\ell_2}) = \frac{\varsigma_2'(u^{\mathbb{G}}_{\ell_1 j_1} u^{\mathbb{G}}_{i_2\ell_2} u^{\mathbb{G}}_{\ell_1 j_1})}{\varsigma_2'(u^{\mathbb{G}}_{\ell_1 j_1})} = \frac{\|u^{\mathbb{G}}_{\ell_1 j_1} u^{\mathbb{G}}_{i_2\ell_2} u^{\mathbb{G}}_{\ell_1 j_1}\|}{\varsigma_2'(u^{\mathbb{G}}_{\ell_1 j_1})} > 0,$$

with similar for ς_1. From here

$$(\varsigma_2 \star \varsigma_1)(u^{\mathbb{G}}_{i_1 j_1} u^{\mathbb{G}}_{i_2 j_2} u^{\mathbb{G}}_{i_1 j_1}) = (\varsigma_2 \otimes \varsigma_1)\Delta(u^{\mathbb{G}}_{i_1 j_1} u^{\mathbb{G}}_{i_2 j_2} u^{\mathbb{G}}_{i_1 j_1})$$

$$= \sum_{k,k_1,k_2=1}^{N} \varsigma_2(u^{\mathbb{G}}_{i_1 k_1} u^{\mathbb{G}}_{i_2 k} u^{\mathbb{G}}_{i_1 k_2})\varsigma_1(u^{\mathbb{G}}_{k_1 j_1} u^{\mathbb{G}}_{k j_2} u^{\mathbb{G}}_{k_2 j_1})$$

$$= \sum_{k,k_1,k_2=1}^{N} \frac{\varsigma_2'(u^{\mathbb{G}}_{i_1 \ell_1} u^{\mathbb{G}}_{i_1 k_1} u^{\mathbb{G}}_{i_2 k} u^{\mathbb{G}}_{i_1 k_2} u^{\mathbb{G}}_{i_1 \ell_1})}{\varsigma_2'(u^{\mathbb{G}}_{i_1 \ell_1})} \frac{\varsigma_1'(u^{\mathbb{G}}_{\ell_1 j_1} u^{\mathbb{G}}_{k_1 j_1} u^{\mathbb{G}}_{k j_2} u^{\mathbb{G}}_{k_2 j_1} u^{\mathbb{G}}_{\ell_1 j_1})}{\varsigma_1'(u^{\mathbb{G}}_{\ell_1 j_1})}$$

All of the terms with $k_1 \neq \ell_1$ or $k_2 \neq \ell_1$ are zero:

$$\Rightarrow (\varsigma_2 \star \varsigma_1)(u^{\mathbb{G}}_{i_1 j_1} u^{\mathbb{G}}_{i_2 j_2} u^{\mathbb{G}}_{i_1 j_1}) = \sum_{k=1}^{N} \varsigma_2(u^{\mathbb{G}}_{i_2 k})\varsigma_1(u^{\mathbb{G}}_{k j_2})$$

$$= \underbrace{\varsigma_2(u^{\mathbb{G}}_{i_2 \ell_2})\varsigma_1(u^{\mathbb{G}}_{\ell_2 j_2})}_{>0} + \underbrace{\sum_{k \neq \ell_2} \varsigma_2(u^{\mathbb{G}}_{i_2 k})\varsigma_1(u^{\mathbb{G}}_{k j_2})}_{\geq 0} > 0.$$

Therefore $u^{\mathbb{G}}_{i_1 j_1} u^{\mathbb{G}}_{i_2 j_2} u^{\mathbb{G}}_{i_1 j_1} \neq 0$, so that $u^{\mathbb{G}}_{i_2 j_2} u^{\mathbb{G}}_{i_1 j_1} \neq 0$ so that $(i_2, i_1) \sim_2 (j_2, j_1)$, that is $\sim_2$ is transitive (and thus an equivalence relation on $N \times N$ •

Lupini, Mančinska, and Roberson [30] (as well as Banica [3]) expressed the belief that $\sim_3$ is not transitive in general. The algebra of functions on a finite quantum group, as a finite dimensional C*-algebra, is a direct sum of $F(G_{\mathbb{G}})$, the direct sum of the one dimensional factors, and B, the direct sum of the higher dimensional factors. Counterexamples to $\sim_3$ transitive can occur in the finite case when for the elements along the diagonal $u^{\mathbb{G}}_{ii}F(\mathbb{G}) \subset F(G_{\mathbb{G}})$; that is if a quantum permutation $\varsigma \in \mathbb{G}$ is such that $\mathbb{P}[\varsigma(i) = i] = \varsigma(u^{\mathbb{G}}_{ii}) > 0$, $\widetilde{u^{\mathbb{G}}_{ii}}\varsigma$ is a random permutation that cannot exhibit non-classical behaviour. For example, if $\mathbb{P}[\varsigma(i_1) = i_1] > 0$, $\widetilde{u^{G}_{i_1 i_1}}\varsigma$ is a random permutation, and so for all $\varsigma \in \mathbb{G}$

$$\mathbb{P}[(\varsigma(i_1) \neq i_1) \succ (\varsigma(i_2) = i_2) \succ (\varsigma(i_1) = i_1)] = 0,$$

which implies that for any $j_3 \neq i_1$, $u^{\mathbb{G}}_{j_3 i_1} u^{\mathbb{G}}_{i_2 i_2} u^{\mathbb{G}}_{i_1 i_1} = 0$. Therefore to find:

$$u^{\mathbb{G}}_{i_1 j_3} u^{\mathbb{G}}_{i_2 j_2} u^{\mathbb{G}}_{i_1 j_1} \neq 0, \text{ and } u^{\mathbb{G}}_{j_3 i_1} u^{\mathbb{G}}_{j_2 i_2} u^{\mathbb{G}}_{j_1 i_1} \neq 0,$$

yields the non-transitivity of $\sim_3$.

This phenomenon occurs in both the Kac-Paljutkin quantum group $\mathfrak{G}_0$ and also the dual $\widehat{Q}$ of the quaternions. In the case of $\mathfrak{G}_0$, the uncertainty phenomenon implies that the quantum permutation given by $\varsigma_2 := \widetilde{u^{\mathfrak{G}_0}_{41}}\varsigma_{e_5}$ satisfies:

$$\mathbb{P}[(\varsigma(1) = 3) \succ (\varsigma(3) = 1) \succ (\varsigma(1) = 4)] = \frac{1}{4} \Rightarrow u^{\mathfrak{G}_0}_{31} u^{\mathfrak{G}_0}_{13} u^{\mathfrak{G}_0}_{41} \neq 0 \Rightarrow (3, 1, 4) \sim_3 (1, 3, 1).$$

Similarly $\varsigma_1 := \widetilde{u_{14}^{\mathfrak{G}_0}} \varsigma_{e_5}$ shows that $u_{14}^{\mathfrak{G}_0} u_{31}^{\mathfrak{G}_0} u_{14}^{\mathfrak{G}_0} \neq 0$, and so $(1,3,1) \sim_3 (4,1,4)$. For transitivity, it would have to be the case that $(3,1,4) \sim_3 (4,1,4)$, that is $u_{34}^{\mathfrak{G}_0} u_{11}^{\mathfrak{G}_0} u_{44}^{\mathfrak{G}_0} \neq 0$, but as for all $\varsigma' \in \mathfrak{G}_0$ such that $\mathbb{P}[\varsigma'(4) = 4] > 0$, $\varsigma := \widetilde{u_{44}^{\mathfrak{G}_0}} \varsigma'$ is a random permutation:

$$\mathbb{P}[(\varsigma(4) = 3) \succ (\varsigma(1) = 1) \succ (\varsigma(4) = 4)] = 0 \Rightarrow u_{34}^{\mathfrak{G}_0} u_{11}^{\mathfrak{G}_0} u_{44}^{\mathfrak{G}_0} = 0,$$

and so $\sim_3$ is not transitive for $\mathfrak{G}_0 < S_4^+$.

Let $\widehat{Q} < S_8^+$:

$$u^{\widehat{Q}} = \begin{pmatrix} u^{\widehat{\langle j \rangle}} & 0 \\ 0 & u^{\widehat{\langle k \rangle}} \end{pmatrix}.$$

Both $u_{11}^{\widehat{Q}}$ and $u_{55}^{\widehat{Q}}$ are orthogonal projection onto direct sums of two one dimensional spaces (respectively the one dimensional factors associated with $\{e, (57)(68)\} \subset G_{\widehat{Q}}$, and $\{e, (13)(24)\} \subset G_{\widehat{Q}}$). Direct calculation shows that $u_{78}^{\widehat{Q}} u_{12}^{\widehat{Q}} u_{58}^{\widehat{Q}} \neq 0$ so $(7,1,2) \sim_3 (8,2,8)$, and $u_{85}^{\widehat{Q}} u_{21}^{\widehat{Q}} u_{85}^{\widehat{Q}} \neq 0$ so $(8,2,8) \sim_3 (5,1,5)$, but $u_{75}^{\widehat{Q}} u_{11}^{\widehat{Q}} u_{55}^{\widehat{Q}} = 0$ as conditioning by $u_{55}^{\widehat{Q}}$ gives a random permutation.

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