QUANTUM PERMUTATIONS: INTUITIONS, INTERPRETATIONS, EXAMPLES, PHENOMENA (LONG DRAFT)

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Abstract. In this exploratory essay an alternative to the 'Gelfand picture' of quantum groups is proposed for quantum permutation groups.

Contents

Note for The Reader. *If you have come here to learn about quantum permutation groups, or to read about some research breakthrough, I am afraid you have come to the wrong place. If your specific interest is quantum permutation groups, and are looking for good references, might I suggest the Banica 'tome' on quantum permutations* [6]. *It has 400 pages and 200 references and could therefore be describes as a one stop shop for quantum permutations. Other references I like may be found at* `https://math.stackexchange.com/a/3667937/19352`

This piece could be for you if you are curious about compact quantum groups.

This is a long draft version of what will be an expository piece which will hopefully find itself on the arXiv in the near future. I am fully au fait with the weaknesses of the piece in its current guise. Nonetheless, I feel there is merit to posting this long draft on my website in its current iteration. The speculative and conjectural aspects probably belong rather in a blog post. There is a lot of mathematical physics going on, and too much about Hilbert space representations. The piece originated in a talk that exclusively used vector states, and a lot of the current iteration is explaining the move from vector to algebraic states, but this is my journey and doesn't belong in a piece of writing. So while I want to preserve the big long journey, and

2000 *Mathematics Subject Classification.* 46L53,81R50.
Key words and phrases. Compact quantum group, quantum permutation group.

the wild speculation and conjecture here, I am now going to go about polishing this work. In the revisions, the mathematical physics and projective Hilbert spaces will go, I will cut all the reference to quantum permutation vectors, and instead only algebraic quantum mechanics aka quantum probability will be used. This will have implications for example the start of Section 4. I will slim down Section 5.2, and consolidate Definition 5.2 with the quantum group law section. I am going to try and be more thoughtful about notation, particular what I am doing with $f(x) = x(f)$. Switching between the two mid formula is not tenable. I will also write in a more conventional and formal style, in particular the formatting of examples as subsections rather than the wild flowers we see in this long piece. I must be a bit cleverer and slicker about the new elementary proof of $S_3^+ = S_3$, and tighten and better reference Section 6.2. I will tighten Section 7.2 and I don't need to talk about the infinite dihedral group here as the focus will be off vector states. From the final section I will keep the section on quasi subgroups – focussing on the new definition and see can it look to the compact case more generally. Maybe a little on periodicity. I definitely want to talk about fixed points and transpositions. That goes to the heart of the work. We will lose all speculation, forget about pure states, and consolidate Sections 8.6 and 8.7. Hopefully that will leave me with a far superior piece of expository academic writing.

Oh, and I will completely rewrite this introduction below. I just want to get the meat of these ideas on my website ASAP before polishing this off. This intro was a rush job so I can get onto the important job of polishing this off to a nice piece of expository writing.

1. Introduction

Compact quantum groups as defined by Woronowicz [56] are beautiful and intriguing objects that have been around now for about 35 years. For a passing reader however their abstract nature provides a steep learning curve: for the conventional view is that compact quantum groups do not exist even as mathematical objects, the passing reader is in trouble for they cannot get their hands on a concrete element of a quantum group. More mature readers are naturally more comfortable with abstraction, and can handle working in analogy, etc., and general readers with an appropriate amount of patience can learn enough to get some kind of handle on these abstract beasts, however, in this work an attempt at making quantum *permutation* groups more accessible for even an undergraduate student.

With some very simple examples, such as the dual of a finite group, and some appropriate interpretation, one can get some kind of a handle on compact quantum groups with only a little bit more than a bit of linear algebra. Furthermore, in the appropriate interpretation that will be presented in this work, it is perfectly tenable to think of a compact quantum permutation group as a *set* $\mathbb{G}$, and thus talk about *a* quantum permutation, an element of this set.

This interpretation simply says that for an algebra of functions on a quantum permutation group $C(\mathbb{G})$ with generators $\{u_{ij}^{\mathbb{G}}\}_{i,j=1}^N$, the set $\mathbb{G}$ which defines the quantum permutation group is the set of states, norm one positive functionals $C(\mathbb{G}) \to \mathbb{C}$, and the generators $u_{ij}^{\mathbb{G}}$ give the probability that a quantum permutation $\varsigma \in \mathbb{G}$ maps j to i. Measurement of a quantum permutation $\varsigma \in \mathbb{G}$ with an self-adjoint $f \in C(\mathbb{G})$ will see wave function collapse of the quantum permutation. The quantum permutation may now be measured again and in a sequence of three measurements quantum behaviour can be observed.

The big question here is: is this mathematics at all? There is very little in the way of genuine original research: I think the counterexample to the transitivity of orbitals might be about it. Otherwise the mathematical content could be described merely as a new interpretation (whose content is ostensibly just notational), some equivalent definitions (e.g. the Haar state and quasi-subgroups), and some baby examples. Oh and some new proofs.

In reality this piece is about interpretation: how to think about quantum permutations. Quoting William Thurston is never a bad idea:

> *This question brings to the fore something that is fundamental and pervasive: that what we* [mathematicians] *are doing is finding ways for people to understand and think about mathematics.*

My personal view is that mathematicians could commit more often to paper their intuitions and interpretations of more abstract mathematics. I think by doing this, more abstract mathematics can become accessible to a wider mathematical audience. I think even a second year undergraduate student could use some CAS to play around with duals of finite groups. I don't think it is beyond the bounds of reason for an undergraduate student to do this using the Pauli representation of $C(S_4^+)$ [11]. I think there is a lot to be said to playing with the concrete examples of algebras of functions on quantum permutation groups. It is probably important at this point to say there are finite quantum groups which are not quantum permutation groups [9].

Speaking personally again, I don't think I have ever enjoyed thinking about mathematics so much as I have in the course of producing this work. With this new interpretation of quantum permutations, I have found it fun, cool and engaging to think about questions.

The real hope might be that such interpretations might help prove new results in the field of quantum permutation groups. I think the jury is very much still out on that.

It would be remiss not to give a few references that experts have said came to mind when I discussed some of these ideas with them: [20, 38, 46, 50].

So without further ado, let us leave this terribly and hastily written introduction and let us do some mathematics:

2. Compact quantum groups

In 1995, Alain Connes posed the question: *What is the quantum automorphism group of a space*? For the case of finite spaces, this question was answered in 1998 by Shuzou Wang [53]. There are two main ways of defining this quantum automorphism group but first some noncommutative terminology/philosophy is required.

2.1. **The Gelfand picture.** The prevailing point of view in the study of (locally) compact quantum groups is to employ what could be called the *Gelfand picture*. This starts with a categorical equivalence given by Gelfand's Theorem:

$$\text{compact Hausdorff spaces} \simeq (\text{unital commutative C}^*\text{-algebras})^{\text{op}}.$$

Starting with a compact Hausdorff space X, the algebra of continuous functions on X, $C(X)$, is a unital commutative C*-algebra; and starting with a unital commutative C*-algebra A, the spectrum, $\Omega(A)$, the set of *characters*, non-zero homomorphisms $A \to \mathbb{C}$, is a compact

Hausdorff space such that $A \cong C(\Omega(A))$. Therefore a general unital commutative C*-algebra can be denoted $A = C(X)$. Inspired by this, one can define the category of 'compact quantum spaces' as

$$\text{compact quantum spaces} :\simeq (\text{unital C}^*\text{-algebras})^{\text{op}}.$$

In analogy with the commutative case, a general not-necessarily commutative unital C*-algebra can be denoted $A = C(\mathbb{X})$, and $\mathbb{X}$ referred to as the spectrum of A. Of course $\mathbb{X}$ is not a set any more but a so-called *virtual* object.

2.2. **Compact Quantum Groups.** In the well-known setting of C*-algebraic compact quantum groups as defined by Woronowicz [56], it is through the Gelfand picture that one speaks of a quantum group $\mathbb{G}$, through a unital noncommutative C*-algebra A that is considered an algebra of continuous functions on it, $A = C(\mathbb{G})$. If X and Y are compact topological spaces, then, where $\otimes$ is the minimal tensor product:

$$C(X \times Y) \cong C(X) \otimes C(Y).$$

Let S be a compact semigroup. The transpose of the continuous multiplication $m : S \times S \to S$ is a $*$-homomorphism, the *compultiplication*:

$$\Delta : C(S) \to C(S \times S) \cong C(S) \otimes C(S).$$

The associativity of the multiplication gives *coassociativity* to the comultiplication:

$$(\Delta \otimes I_{C(S)}) \circ \Delta = (I_{C(S)} \otimes \Delta) \circ \Delta.$$

If $C(S)$ satisfies *Woronowicz cancellation*

$$\overline{\Delta(C(S))(\mathbb{1}_S \otimes C(S))} = \overline{\Delta(C(S))(C(S) \otimes \mathbb{1}_S)} = C(S) \otimes C(S);$$

then it has cancellation, and a compact semigroup with cancellation is a group. In this sense Woronowicz cancellation is a $C(G)$-analogue of cancellation. This inspires:

Definition 2.1. **An** (Woronowicz C*-) *algebra of continuous functions on a compact quantum group* $\mathbb{G}$ is a unital C*-algebra $C(\mathbb{G})$ together with a unital $*$-morphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ that satisfies coassociativity and *Woronowicz cancellation*:

$$\overline{\Delta(C(\mathbb{G}))(\mathbb{1}_{\mathbb{G}} \otimes C(\mathbb{G}))} = \overline{\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes \mathbb{1}_{\mathbb{G}})} = C(\mathbb{G}) \otimes C(\mathbb{G}).$$

Note the stress on *an*: a compact quantum group $\mathbb{G}$ comes with a dense Hopf $*$-algebra of *regular functions*, $\mathcal{O}(\mathbb{G})$, but this can have more than one completion. There is a maximal, *universal* completion $C_u(\mathbb{G})$, and a minimal, *reduced* completion $C_r(\mathbb{G})$. However all of these completions give the same dense Hopf $*$-algebra of regular functions in the following sense: given any algebra of continuous functions on a compact quantum group, surject onto the dense subalgebra $\mathcal{O}(\mathbb{G})$, complete to $C_\alpha(\mathbb{G})$, and then surject onto the dense subalgebra $\mathcal{O}_\alpha(\mathbb{G})$ associated to $C_\alpha(\mathbb{G})$:

$$C(\mathbb{G}) \twoheadrightarrow \mathcal{O}(\mathbb{G}) \hookrightarrow C_\alpha(\mathbb{G}) \twoheadrightarrow \mathcal{O}_\alpha(\mathbb{G}),$$

it turns out that $\mathcal{O}(\mathbb{G}) \cong \mathcal{O}_\alpha(\mathbb{G})$ as algebras of regular functions on *algebraic compact quantum groups* [51]. In this sense, a compact quantum group can be identified with non-isomorphic algebras of continuous functions $C_\alpha(\mathbb{G})$ and $C_\beta(\mathbb{G})$ if their dense algebras of regular functions are isomorphic. Non-isomorphic algebra of continuous functions can arise for duals for discrete groups. The dual $\widehat{\Gamma}$ is a compact quantum group, with algebra of regular functions given by the group ring, $\mathcal{O}(\widehat{\Gamma}) = \mathbb{C}\Gamma$. All the completions of $\mathcal{O}(\widehat{\Gamma})$ are canonically isomorphic exactly when Γ is amenable: when all the completions of the algebra of regular functions on a compact quantum group $\mathcal{O}(\mathbb{G})$ are canonically isomorphic, in particular $C_u(\mathbb{G}) \cong C_r(\mathbb{G})$, the compact quantum group is said to be *coamenable.*

What makes a compact quantum group a generalisation of a compact group? Consider a commutative Woronowicz C*-algebra A. Gelfand's Theorem states $A \cong C(\Omega(A))$, the unital $*$-homomorphism $\Delta : C(\Omega(A)) \to C(\Omega(A) \times \Omega(A))$ gives a continuous map $m : \Omega(A) \times \Omega(A) \to \Omega(A)$, the coassociativity of Δ gives associativity to m, and so $(\Omega(A), m)$ is a compact semigroup with Woronowicz cancellation, and so a group. That compact quantum groups with commutative algebras of continuous functions are in fact (classical) compact groups is Gelfand duality: that the virtual quantum objects are still studied through their algebra of functions is the essence of the Gelfand picture.

A particular type of compact quantum group, earlier defined by Woronowicz [55], is a *compact matrix quantum group.*

Definition 2.2. If a compact quantum group $\mathbb{G}$ is such that

- $C(\mathbb{G})$ generated (as a C*-algebra) by the entries of a unitary matrix $u = (u_{ij})_{i,j=1}^N \in M_N(C(\mathbb{G}))$, and
- u and u^t are invertible, and
- $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$, $u_{ij} \mapsto \sum_{k=1}^N u_{ik} \otimes u_{kj}$ is a $*$-homomorphism,

then $\mathbb{G}$ is a *compact matrix quantum group* with *fundamental representation* $u \in M_N(C(\mathbb{G}))$.

Theorem 2.3. *(Woronowicz) If a compact matrix quantum group G has commutative algebra of functions $C(G)$, then G is homemorphic to a closed compact subgroup of the unitary group, $U_N(\mathbb{C})$* •

The quantum groups studied in this work are all compact matrix quantum groups such that $u \in M_N(C(\mathbb{G}))$, the *fundamental representation*, is a *magic unitary.* That is the entries of u are projections in a C*-algebra:

$$u_{ij} = u_{ij}^2 = u_{ij}^*,$$

that are partitions of unity on rows and columns:

$$\sum_{k=1}^N u_{ik} = \mathbb{1}_{\mathbb{G}} = \sum_{k=1}^N u_{kj}.$$

Such compact quantum groups are called *quantum permutation groups.*

3. Some quantum mechanics

The preceding sections outlines the conventional view of compact quantum groups. The Gelfand picture allows nominal talk about a compact quantum group as a whole, but in general does not permit the consideration of an *element* of compact quantum group. Aspects of quantum mechanics can be used to inspire a way of doing this for quantum permutation groups.

3.1. **The Weaver Picture for Quantum Mechanics.** In his book Weaver [54] states and argues the point that:

> "The fundamental idea of mathematical quantisation is that sets are replaced by Hilbert spaces... [and] the quantum version of a [real]-valued function on a set is a [self-adjoint] operator on a Hilbert space."

Weaver attributes the Hilbert space as set point of view to Birkhoff and von Neumann [17], and the operator as function point of view to Mackey [40]. In this picture, the elements of the projective version of a Hilbert space $P(\mathsf{H})$ form a quantum space, and the self-adjoint operators are random variables $P(\mathsf{H})\to\mathbb{R}$, with superposition together with the Born rule providing probability, and spectral projections providing wave function collapse. Later on, with a quantum permutation group $\mathbb{G}$ faithfully represented by $\pi(C(\mathbb{G}))\subset B(\mathsf{H})$, it will be possible to interpret elements of $P(\mathsf{H})$ as quantum permutations.

To illustrate the Weaver picture, consider a quantum vector state ψ in the projective version, $P(\mathsf{H})$, of a Hilbert space H, and the C*-algebra of bounded operators $B(\mathsf{H})$, of which the self-adjoint operators are *observables.* Consider a projection, a *Bernoulli observable* $p\in B(\mathsf{H})$. It has an orthonormal eigenbasis $\Omega_p:=\{e_\alpha\}_\alpha$, and, where $p^1:=p$ and $p^0:=I_H-p$, decomposes the Hilbert space into eigenspaces:

$$\mathsf{H}=\operatorname{ran}p^1\oplus\operatorname{ran}p^0.$$

The observable cannot discriminate between elements of $\operatorname{ran}p^1$ (and respectively $\operatorname{ran}p^0$). It can be interpreted that when p 'looks' at H it 'sees' the set Ω_p, and resembles a random variable:

$$p:\Omega_p\to\{0,1\}.$$

Superposition, via a vector state $\psi\in P(\mathsf{H})$ provides probabilistic data, and where $\widehat{\psi}\in\mathsf{H}$ is a norm one representative of $\psi\in P(\mathsf{H})$:

$$\mathbb{P}[p=\theta\,|\,\psi]:=\langle\widehat{\psi},p^\theta(\widehat{\psi})\rangle=\|p^\theta(\widehat{\psi})\|^2.$$

In the sequel any element of $P(\mathsf{H})$ that appears in a Hilbert space calculation that should be normalised will be assumed to be normalised. In the event of $p=\theta$ the state transitions $\psi\mapsto p^\theta(\psi)\in P(\mathsf{H})$. This is *wave function collapse.* Note this is assumed possible only if $p=\theta$ is non-null in the sense that

$$\mathbb{P}[p=\theta\,|\,\psi]=\langle\psi,p^\theta(\psi)\rangle\neq 0.$$

Take another projection $q\in B(\mathsf{H})$. Similarly to p, when q 'looks' at H it 'sees' a set Ω_q, and q resembles a random variable $q:\Omega_q\to\{0,1\}$. The issue is that the random variables p and q may see different sets and cannot simultaneously agree on what they observe.

Suppose that the 'event' $p=\theta_1$ has been observed so that the state is now $p^{\theta_1}(\psi)\in P(\mathsf{H})$. The probability that measurement *now* produces $q=\theta_2$, and $p^{\theta_1}(\psi)\mapsto q^{\theta_2}p^{\theta_1}(\psi)$, is:

$$\mathbb{P}[q=\theta_2\,|\,p^{\theta_1}(\psi)]:=\left\langle\frac{p^{\theta_1}(\psi)}{\|p^{\theta_1}(\psi)\|},q^{\theta_2}\left(\frac{p^{\theta_1}(\psi)}{\|p^{\theta_1}(\psi)\|}\right)\right\rangle=\frac{\langle p^{\theta_1}(\psi),q^{\theta_2}(p^{\theta_1}(\psi))\rangle}{\|p^{\theta_1}(\psi)\|^2}.$$

Define now the event $((q=\theta_2)\succ(p=\theta_1)\,|\,\psi)$, said 'given the state ψ, q is measured to be θ_2 *after* p is measured to be θ_1'. Using the expression above a probability can be ascribed to this event:

$$\begin{aligned}\mathbb{P}\left[(q=\theta_2)\succ(p=\theta_1)\,|\,\psi\right] &:= \mathbb{P}[p=\theta_1\,|\,\psi]\cdot\mathbb{P}[q=\theta_2\,|\,p^{\theta_1}(\psi)] \\ &= \langle\psi,p^{\theta_1}(\psi)\rangle\cdot\frac{\langle p^{\theta_1}(\psi),q^{\theta_2}(p^{\theta_1}(\psi))\rangle}{\|p^{\theta^1}(\psi)\|^2} \\ &= \|q^{\theta_2}p^{\theta_1}(\psi)\|^2.\end{aligned}$$

Inductively, for a finite number of projections $(p_i)_{i=1}^n$, and $\theta_i\in\{0,1\}$:

$$\mathbb{P}\left[(p_n=\theta_n)\succ\cdots\succ(p_1=\theta_1)\,|\,\psi\right]=\|p_n^{\theta_n}\cdots p_1^{\theta_1}(\psi)\|^2.$$

In general, $pq\neq qp$ and so

$$\mathbb{P}\left[(q=\theta_2)\succ(p=\theta_1)\,|\,\psi\right]\neq\mathbb{P}\left[(p=\theta_1)\succ(q=\theta_1)\,|\,\psi\right],$$

and this helps interpret that q and p are not simultaneously observable. However the *sequential projection measurement* $q\succ p$ is 'observable' in the sense that it resembles a random variable with values in $\{0,1\}^2$. Inductively the sequential projection measurement $p_n\succ\cdots\succ p_1$ resembles an $\{0,1\}^n$-valued random variable.

If p and q *do* commute, they share an orthonormal eigenbasis, and it can be interpreted that they can 'agree' on what set they 'see' when they 'look' at H, and can thus be determined simultaneously. Alternatively, if they commute then the distributions of $q\succ p$ and $p\succ q$ are equal in the sense that

$$\mathbb{P}\left[(q=\theta_2)\succ(p=\theta_1)\,|\,\psi\right]=\|q^{\theta_2}p^{\theta_1}(\psi)\|^2=\mathbb{P}\left[(p=\theta_1)\succ(q=\theta_2)\,|\,\psi\right],$$

it doesn't matter what order they are measured in, the outputs of the measurements can be multiplied together, and this observable can be called $pq=qp$.

Through superposition, measurement with a projection includes an *a priori* distribution, and also wave function collapse: but these are not purely quantum mechanical phenomenon, and occur also with measurements from a commutative subalgebra $A\subset B(\mathsf{H})$. To illustrate, let $X=\{x_1,\dots,x_N\}$ and consider the diagonal subalgebra of $B(\ell^2(X))$. Subsets $Y\subset X$ yield subspaces $\ell^2(Y)\subset\ell^2(X)$ together with projections $p_Y\in B(\ell^2(X))$. Superposition in $P(\ell^2(X))$ defines an *a priori* distribution on p_Y via:

$$\mathbb{P}[p_Y=\theta\,|\,\psi]=\langle\psi,p_Y^\theta(\psi)\rangle,$$

and, conditional on $p_Y=\theta$, there is wave function collapse to $p_Y^\theta(\psi)\in P(\ell^2(X))$. A state of the form $e_{x_i}\in\ell^2(X)$ is a *pure state.* A state that is the superposition of two or more pure states is a *mixed state*: classical measurement can *disturb* a mixed state.

A defining difference between classical and quantum measurement is the quantum phenomenon of projection observables that cannot be simultaneously measured in the sense that $(p\succ q)\neq(q\succ p)$. In classical measurement, all projection observables can be simultaneously measured, and this implies that while classical measurement *can* disturb a mixed state, the effects are purely probabilistic, capturing a decrease in uncertainty about the state. In the finite case of $\ell^2(X)$, measurement with an appropriate finite collection of classical projection measurements results in collapse to a pure state $e_{x_i}\in\ell^2(X)$. Pure states are invariant for the diagonal subalgebra: further measurement does *not* disturb the state. In contrast, for any state $\psi \in P(\ell^2(X))$ there is a quantum measurement $p \in B(\ell^2(X))$ that can disturb it, and so collapse to complete certainty is impossible.

Sequential measurement of finite spectrum observables $f \in B(\mathsf{H})$ can also be considered: where the spectral decomposition $f = \sum_{k=1}^{|\sigma(f)|} f_i\, p^{f_i}$ defines a partition of unity $(p^{f_i})_{i=1,\dots,|\sigma(f)|}$, and

$$\mathbb{P}[f = f_i \,|\, \psi] = \langle \psi, p^{f_i}(\psi)\rangle = \|p^{f_i}(\psi)\|^2. \tag{3.1}$$

Furthermore this gives expectations

$$\mathbb{E}[f \,|\, \psi] := \langle \psi, f(\psi)\rangle.$$

Borel functional calculus can be used to measure, for example, if f is in e.g. $[a,b) \subset \mathbb{R}$, via the projection $\mathbb{1}_{[a,b)}(f) \in B(\mathsf{H})$, so that

$$\mathbb{P}[a \leq f < b \,|\, \psi] := \mathbb{P}[\mathbb{1}_{[a,b)}(f) = 1 \,|\, \psi] = \langle \psi, \mathbb{1}_{[a,b)}(f)(\psi)\rangle,$$

and the state transitions $\psi \mapsto \mathbb{1}_{[a,b)}(f)(\psi))$. Note that $\mathbb{1}_{\{f_i\}}(f) \neq 0$ exactly when $f_i \in \sigma(f)$ is an eigenvalue, and so it is possible to measure if f is equal to an eigenvalue.

4. About quantum permutations

Fresh decks of playing cards produced by e.g. the US Playing Card Company always come in the same original order:

$$A\spadesuit, \dots, K\spadesuit, A\clubsuit, \dots, K\clubsuit, A\diamondsuit, \dots, K\diamondsuit, A\heartsuit, \dots, K\heartsuit,$$

Respectively enumerate using $c : \{1, 2, \dots, 52\} \to \{A\spadesuit, \dots, K\heartsuit\}$. The original order can be associated with the pure state $e_e \in \ell^2(S_{52})$. After a suitably randomised shuffle, an active permutation, the deck will be in some unknown order given by a mixed state, a *random* permutation $\varsigma \in P(\ell^2(S_{52}))$, with the card in position j moved to position $\varsigma(j)$. Turn over the card in position i to reveal card $c(j)$. This observable, denoted $x^{-1}(i) \in B(\ell^2(S_{52}))$, reveals that the random permutation sent j to i. This observable has spectrum $\sigma(x^{-1}(i)) = \{1, 2, \dots, 52\}$, and thus spectral decomposition

$$x^{-1}(i) = \sum_{k=1}^{52} k\, u_{ik}^{S_{52}},$$

with $(u_{ik}^{S_{52}})_{k=1,\dots,52}$ a partition of unity. The distribution of $x^{-1}(i)$ given the state $\varsigma \in P(\ell^2(S_{52}))$ can be denoted

$$\mathbb{P}[\varsigma^{-1}(i) = j] := \mathbb{P}[x^{-1}(i) = j \,|\, \varsigma] = \langle \varsigma, u_{ij}^{S_{52}}(\varsigma)\rangle. \tag{4.1}$$

Each card $c(j)$ must be mapped *somewhere* and so, for all $\varsigma \in P(\ell^2(S_{52}))$

$$\sum_{k=1}^{52} \mathbb{P}[\varsigma^{-1}(k) = j] = \left\langle \varsigma, \sum_{k=1}^{52} u_{kj}^{S_{52}}(\varsigma) \right\rangle = 1,$$

this implies that $(u_{kj}^{S_{52}})_{k=1,\dots,52}$ is also a partition of unity, giving another observable

$$x(j) := \sum_{k=1}^{52} k\, u_{kj}^{S_{52}},$$

and note that

$$\mathbb{P}[\varsigma(j)=i]:=\mathbb{P}[x(j)=i\,|\,\varsigma]=\langle\varsigma,u_{ij}^{S_{52}}(\varsigma)\rangle=\mathbb{P}[\varsigma^{-1}(i)=j].$$

The observable $x^{-1}(i)$ is measured by turning over the card in position i. How is $x(j)$ measured? Go back to the deck in the original order, turn card $c(j)$ face *up*, shuffle with ς. The position of card $c(j)$ is $x(j)$.

Following the sequential measurement

$$x^{-1}(51)\succ\cdots\succ x^{-1}(2)\succ x^{-1}(1),$$

the random permutation will collapse to a (deterministic) permutation $e_\sigma\in\ell^2(S_{52})$. If the sequential measurement is paused, say at $\ell<51$ with

$$(x^{-1}(\ell)\succ\cdots\succ x^{-1}(2)\succ x^{-1}(1))=(j_\ell,\dots,j_2,j_1),$$

the state has collapsed to

$$\varsigma_\ell:=u_{ij_\ell}^{S_{52}}\cdots u_{ij_2}^{S_{52}}u_{ij_1}^{S_{52}}(\varsigma),$$

then

$$\mathbb{P}[\varsigma_\ell(k)=j]:=\mathbb{P}[x^{-1}(k)=j_k\,|\,\varsigma_\ell]=\delta_{j,j_k},$$

that is once a card $c(k)$ is observed in the position j_k once, that is determined once and for all.

Note that $(u_{ij}^{S_{52}})_{i,j=1}^{52}\in M_{52}(B(\ell^2(S_{52})))$ is a *magic unitary.*

There is no issue whatsoever talking about the set of random permutations, $P(\ell^2(S_N))$, nor an element of this set $\varsigma\in P(\ell^2(S_N))$. Inspired by the Weaver picture, imagine for a moment that the same can be done for *quantum permutations*: imagine that there is a Hilbert space H such that the set of quantum permutations on N symbols is given by $P(\mathsf{H})$, and a quantum permutation is simply an element $\varsigma\in P(\mathsf{H})$.

What would make a permutation *quantum*? In light of previous discussions perhaps what might make a permutation *quantum* is that quantum versions of observables $x^{-1}(i)$ and $x(j)$ be *not* simultaneously observable. This implies that, with a deck of cards shuffled with a quantum permutation, once the first card has been revealed, the observation of the second card might disturb the state in a such a way that non-classical events like,

$$(\varsigma(1)\neq c^{-1}(A\heartsuit))\succ(\varsigma(2)=c^{-1}(A\spadesuit))\succ(\varsigma(1)=c^{-1}(A\heartsuit))$$

can occur.

With the deck in the original order, $x(j)$ would be measured by turning card $c^{-1}(j)$ face *up*, shuffling with ς, and noting the position of $c^{-1}(j)$ after the shuffle. Similarly $x^{-1}(i)$ would be observable by revealing the card in position i. What would not be permitted would be shuffling with more than one card face up, or revealing more than one card at once.

Given a quantum permutation $\varsigma\in P(\mathsf{H})$, similarly to before, the spectral decompositions of the observables $x^{-1}(i)$ and $x(j)$ should give a magic unitary $(u_{ij})_{i,j=1}^{52}$. Denote as before

$$\mathbb{P}[\varsigma(j)=i]:=\mathbb{P}[x(j)=i\,|\,\varsigma]=\langle\varsigma,u_{ij}\varsigma\rangle.$$

The projective nature of wave function collapse, that conditional on $\varsigma(j) = i$, $\varsigma \mapsto u_{ij}(\varsigma)$, implies that

$$\mathbb{P}[(\varsigma(j) = i) \succ (\varsigma(j) = i)] = \mathbb{P}[\varsigma(j) = i];$$

the probability of observing $\varsigma(j) = i$ after (just) observing $\varsigma(j) = i$ is one.

In the sequel this will all be made mathematically precise.

4.1. **Wang's Quantum Permutation Groups.** In a survey article, Banica, Bichon & Collins [8] attribute to Brown [19] the idea of taking a matrix group $G \subset U_N$, realising $C(G)$ as a universal commutative C*-algebra generated by the matrix coordinates $u_{ij} \in C(G)$ subject to some relations R, and then studying (if it exists), the noncommutative universal C*-algebra generated by abstract variables u_{ij} subject to the same relations R. This procedure, called *liberation* in the context of compact quantum groups by Banica & Speicher [14], was carried out by Wang to create quantum versions of the orthogonal and unitary groups, and later quantum permutation groups.

Let $F(S_N)$ be the algebra of complex functions on S_N with basis $\{\delta_\sigma\}_{\sigma \in S_N}$. In the sequel, finite-dimensional algebras of functions will be denoted with $F(\mathbb{G})$ rather than $C(\mathbb{G})$. Define $\mathbb{1}_{j\to i} \in F(S_N)$ by:

$$\mathbb{1}_{j\to i}(\sigma) := \begin{cases} 1, & \text{if } \sigma(j) = i, \\ 0, & \text{otherwise.} \end{cases}$$

Where Δ is the transpose of the group law $m : S_N \times S_N \to S_N$, so that $\Delta(f) = f \circ m$, and employing $F(S_N \times S_N) \cong F(S_N) \otimes F(S_N)$, note that

$$\Delta(\mathbb{1}_{j\to i}) = \sum_{k=1}^{N} \mathbb{1}_{k\to i} \otimes \mathbb{1}_{j\to k}.$$

Furthermore

$$\delta_\sigma = \prod_{j=1}^{N} \mathbb{1}_{j\to\sigma(j)}, \tag{4.2}$$

and so the matrix $u^{S_N} = (\mathbb{1}_{j\to i})_{i,j=1}^N$ is a unitary with inverse the transpose of u^{S_N}, whose entries generate $F(S_N)$. Therefore $F(S_N)$ is a commutative algebra of functions on a compact matrix quantum group. Furthermore the $\mathbb{1}_{j\to i}$ are projections, and

$$\sum_{k=1}^{N} \mathbb{1}_{k\to i} = \mathbb{1}_{S_N} = \sum_{k=1}^{N} \mathbb{1}_{j\to k}.$$

Therefore the matrix $u^{S_N} = (\mathbb{1}_{j\to i})_{i,j=1}^N$ is a magic unitary. Indeed $F(S_N)$ has a presentation as a universal commutative C*-algebra:

$$F(S_N) \cong \mathrm{C}^*_{\mathrm{comm}}(u^c_{ij} \,|\, u^c \text{ an } N \times N \text{ magic unitary}).$$

Liberation is to consider the same abstract generators and relations, but not include commutativity. Following Wang [53] consider the universal C*-algebra:

$$C(S_N^+) := \mathrm{C}^*(u_{ij} \,|\, u \text{ an } N \times N \text{ magic unitary}).$$

The universal property says that if $(v_{ij})_{i,j=1}^N$ is another $N\times N$ magic unitary, then $u_{ij}\mapsto v_{ij}$ is a $*$-homomorphism. It can be shown that

$$\left[\sum_{k=1}^N u_{ik}\otimes u_{kj}\right]_{i,j=1}^N \in M_N(C(S_N^+)\otimes C(S_N^+))$$

is a magic unitary, and thus $\Delta(u_{ij})=\sum_{k=1}^N u_{ik}\otimes u_{kj}$ is a $*$-homomorphism. It is straightforward to show that Δ is unital and coassociative, and so $C(S_N^+)$ is an algebra of continuous functions on a compact matrix quantum group, *the* quantum permutation group on N symbols.

4.2. **Quantum Permutation Groups.** If $\mathbb{G}$ is a compact matrix quantum group whose fundamental representation is an $N\times N$ magic unitary $u^{\mathbb{G}}$, then the universal property gives $\pi: C(S_N^+)\to C(\mathbb{G})$ a surjective $*$-homomorphism that intertwines the comultiplication:

$$\Delta_{C(\mathbb{G})}\circ\pi=(\pi\otimes\pi)\circ\Delta_{C(S_N^+)},$$

which is to say that $\mathbb{G}<S_N^+$, $\mathbb{G}$ is a quantum subgroup of S_N^+. Furthermore, if $\mathbb{G}<S_N^+$ by a comultiplication-intertwining surjective $*$-homomorphism $\pi_1: C(S_N^+)\to C(\mathbb{G})$, then $[\pi_1(u_{ij})]_{i,j=1}^N$ is a magic unitary that is a fundamental representation for $\mathbb{G}$.

Definition 4.1. A *quantum permutation group* is a compact matrix quantum group whose fundamental representation is a magic unitary.

The justification for calling S_N^+ *the* quantum permutation group on N symbols goes beyond the liberation of $F(S_N)$. Wang originally defines the (universal) quantum automorphism group of $\mathbb{C}^n$ (that leaves the trace invariant). This leads to the definition of S_N^+ given above. This essay should further cement that S_N^+ is a quantum generalisation of S_N.

Theorem 4.2. *For $N\leq 3$, $C(S_N^+)\cong F(S_N)$* $\bullet$

See Section 6.1 for a new proof for $N=3$.

Theorem 4.3. *For $N\geq 4$, $C(S_N^+)$ is noncommutative and infinite dimensional.*

Proof. The standard argument for $N=4$ uses the universal C*-algebra generated by two projections (see [8]). To be more concrete, consider the discrete infinite dihedral group

$$D_\infty := \langle a,b\,|\,a^2=b^2=e\rangle\cong\mathbb{Z}_2*\mathbb{Z}_2.$$

The infinite dihedral group is amenable, which implies that the reduced and universal group C*-algebras coincide. Denote the (noncommutative) group ring by $\mathbb{C}D_\infty$ and $C(\widehat{D_\infty})$ *the* C*-completion, which is *-isomorphic to the universal C*-algebra generated by two projections [49]. Together with $\Delta(g)=g\otimes g$, the dual $\widehat{D_\infty}$ is a compact matrix quantum group with fundamental representation

$$\tilde{u}^{\widehat{D_\infty}} := \begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}.$$

In fact $\widehat{D_\infty}$ is quantum permutation group. Where $p:=(e+a)/2$ and $q:=(e+b)/2$, via $a=u_{11}^{\widehat{D_\infty}}-u_{21}^{\widehat{D_\infty}}$ (and similarly for b) the following is a fundamental representation for $\widehat{D_\infty}$

that is a magic unitary:

$$u^{\widehat{D_\infty}} := \begin{bmatrix} p & e-p & 0 & 0 \\ e-p & p & 0 & 0 \\ 0 & 0 & q & e-q \\ 0 & 0 & e-q & q \end{bmatrix} \tag{4.3}$$

Therefore $\widehat{D_\infty}$ is an infinite quantum subgroup of S_4^+, and it follows that $C(S_4^+)$ is infinite dimensional and noncommutative •

Note also that replacing a and b with order two generators of D_N shows that $\widehat{D_N} < S_4^+$) (exhibiting Th. 1.1 (9), [7]). Showing that $\widehat{S_3} < S_4^+$ is the easiest way of showing that $C(S_4^+)$ is noncommutative. See Section 7.2.1 for more on $C(\widehat{D_\infty})$.

THEOREM 4.4. *For $N \geq 5$, S_N^+ is not coamenable.*

Proof. The standard argument that S_N^+ is not coamenable for $N \geq 5$ uses fusion rules [3]. However, in similar spirit to the (standard) proof of Theorem 4.3, using the fact that a compact subgroup of a coamenable compact quantum group is coamenable [52], the exhibition of a non-coamenable subgroup of S_5^+ proves Theorem 4.4 for $N = 5$ (the extension to $N > 5$ is straightforward). Let a and b be the respective generators of $\mathbb{Z}_3 * \mathbb{Z}_2$, which is not amenable so there is not a unique C*-completion of $\mathbb{C}(\mathbb{Z}_3 * \mathbb{Z}_2)$. Let $C(\widehat{\mathbb{Z}_3 * \mathbb{Z}_2})$ be one such completion to a compact quantum group. Where $\omega = e^{2\pi i/3}$, consider the following magic unitary:

$$u^{\widehat{\mathbb{Z}_3}} := \frac{1}{3}\begin{bmatrix} e+a+a^2 & e+\omega^2 a+\omega a^2 & e+\omega a+\omega^2 a^2 \\ e+\omega a+\omega^2 a^2 & e+a+a^2 & e+\omega^2 a+\omega a^2 \\ e+\omega^2 a+\omega a^2 & e+\omega a+\omega^2 a^2 & e+a+a^2 \end{bmatrix}$$

Note that $a = u_{11}^{\widehat{\mathbb{Z}_3}} + \omega^2 u_{21}^{\widehat{\mathbb{Z}_3}} + \omega u_{31}^{\widehat{\mathbb{Z}_3}}$. With the same notation $q = (e+b)/2$ as with the infinite dihedral group, except obviously with $e \in \mathbb{Z}_3 * \mathbb{Z}_2$, let

$$u^{\widehat{\mathbb{Z}_2}} := \frac{1}{2}\begin{bmatrix} q & e-q \\ e-q & q \end{bmatrix}.$$

Consider the block magic unitary $u^{\widehat{\mathbb{Z}_3 * \mathbb{Z}_2}} \in M_5(\mathbb{C}(\mathbb{Z}_3 * \mathbb{Z}_2))$:

$$u^{\widehat{\mathbb{Z}_3 * \mathbb{Z}_2}} := \begin{bmatrix} u^{\widehat{\mathbb{Z}_3}} & 0 \\ 0 & u^{\widehat{\mathbb{Z}_2}} \end{bmatrix}.$$

This shows that $\widehat{\mathbb{Z}_3 * \mathbb{Z}_2} < S_5^+$. The dual of a discrete group is coamenable if and only if the group is amenable; $\mathbb{Z}_3 * \mathbb{Z}_2$ is not amenable [49], therefore its dual is not coamenable, and thus neither is S_5^+ •

Banica [6] calls $\widehat{\mathbb{Z}_2 * \mathbb{Z}_3}$ by Bichon's group dual subgroup of S_5^+. More on duals in Section 7.2.

It is the case that $S_N \leq S_N^+$ is a quantum permutation group, known to be maximal for $N \leq 5$ [4], but conjectured to be maximal for all $N \in \mathbb{N}$. One motivation for the current work is to perhaps provide some intuition to attack such a problem. See Section 8 for more.

5. Some more quantum mechanics

5.1. **Algebraic quantum mechanics.** Careful readers might note a possible future tension in this work between the fact that Section 3 concerns the full algebra of bounded operators on a Hilbert space, while an algebra of continuous functions on a quantum permutation group is a C*-algebra. In quantum mechanics and quantum field theory one learns that not all of the bounded operators on a Hilbert space should necessarily be considered in a theory, perhaps only a norm closed or strongly closed self-adjoint subalgebra [18, 48]. Consider a given closed self-adjoint subalgebra $A \subset B(\mathsf{H}_0)$: in the context of infinite degrees of freedom, there can be many unitarily inequivalent faithful Hilbert space representations to consider:

$$\pi_\alpha : A \to B(\mathsf{H}_\alpha).$$

With unitarily inequivalent representations understood to yield different physical systems, the question arises: which of the representations $\pi_\alpha(A) \subset B(\mathsf{H}_\alpha)$ corresponds to the actual physical system that the quantum physicist wants to study? To explain how different representations can give different systems, if working with a von Neumann algebra $\mathcal{M}$, while the spectrum of an observable $f \in \mathcal{M}$ is the same in every faithful representation, the eigenvalues are not. Suppose $\lambda \in \sigma(f)$ is an eigenvalue of $\pi_\alpha(f) \in B(\mathsf{H}_\alpha)$ but not of $\pi_\beta(f) \in B(\mathsf{H}_\beta)$: this implies, through the spectral projection $\mathbb{1}_{\{\lambda_i\}}(f)$, the event $f = \lambda_i$ is non-null in $\pi_\alpha(\mathcal{M})$ but null in $\pi_\beta(\mathcal{M})$.

In the context of this essay, removed from real world physics, unitary inequivalence of Hilbert space representations of algebras of continuous functions on quantum permutation groups will play a role. In Section 6.2 a Hilbert space representation $\pi(C(\mathbb{G})) \subset B(\mathsf{H})$ will be considered, and it will be explained how particular intersections of ranges of $\pi(u_{ij}^{\mathbb{G}})$ give *deterministic permutation vectors* in $P(\mathsf{H})$. In fact, it will be seen in Section 7.2.1 that one representation $(\pi_1 \oplus \pi_V)(C(\widehat{D_\infty})) \subset B(\mathsf{H}_1)$ contains as a set of deterministic permutation vectors the Klein four group, $\mathbb{Z}_2 \times \mathbb{Z}_2$, while in $\pi_1(C(\widehat{D_\infty})) \subset B(\mathsf{H}_2)$ there are *no* deterministic permutations vectors. If the choice of a Hilbert space representation is necessary to talk about measurement, and the state space restricted to the projective version of the Hilbert space, then classical behaviour, with respect to all observables $\pi(C(\mathbb{G})) \subset B(\mathsf{H})$, cannot be observed in a representation with no deterministic permutation vectors.

A solution to the quandary of what representation to pick is to not pick any representation at all and instead work at the C*-algebraic level — this is *algebraic quantum mechanics*, a (very) rough outline of which will be outlined below (see [35] for a comprehensive treatment). Frequently however it shall be easier to work in Hilbert space mechanics: as will be seen Hilbert space *vector states* $\psi \in P(\mathsf{H})$ provide easy to describe *algebraic states*: and wave function collapse is easier in the Hilbert space picture too. Denote for a C*-algebra A the set of *algebraic states*, the positive linear functionals of norm one, by $S(A)$. A state $\varphi \in S(A)$ is *pure* if it has the property that whenever ρ is a positive linear functional such that $\rho \leq \varphi$, necessarily there is a number $t \in [0, 1]$ such that $\rho = t\varphi$. Otherwise ρ is a *mixed state.* In contrast to the full $B(\mathsf{H})$ case, vector states on $\pi(A) \subset B(\mathsf{H})$ need not be pure states.

To describe this toy version algebraic quantum mechanics is to rewrite Section 3, but in terms of C*-algebras. Considering that algebras of continuous functions on quantum permutations groups have been introduced in Section 4.1, let us use such objects. First however some interpretation.

5.2. **The Weaver picture for quantum permutations.** The *Weaver picture*, that the projective version of a Hilbert space *is* a quantum space, has not yet made its way into the world of compact quantum groups. One can speculate as to why this has not happened yet. One reason might be that making interpretations in the world of physics is a brave endeavour, full of many pitfalls, paradoxes etc., and perhaps quantum group theorists are loathe to engage in such 'philosophy'. However in pure mathematics, there is nothing to say don't interpret objects that are difficult to understand, and surely everything is gained and nothing lost by having good interpretation/intuition. The author recalls giving a talk about finite quantum groups to a room full of group theorists and giving a wry smile when a subsequent speaker, Ted Hurley, stated that sometimes group theorists lose sight of the fact that *elements* of groups are objects of interest. Unlike the Gelfand picture, for the case of quantum permutation groups, the Weaver picture does allow meaningful talk of the *elements* of a quantum group: the first attempt at this is to declare that the elements of a quantum permutation group are the elements of the projective Hilbert space:

Definition 5.1. *A quantum permutation vector* is an element of $P(\mathsf{H})$, where $\pi(C(\mathbb{G}))\subset B(\mathsf{H})$ is a faithful representation of an algebra of continuous functions on a quantum permutation group.

With the use of the *Birkhoff slice* (Section 6.1) this statement can be made meaningful.

As discussed in Section 2.2, given an algebra of a regular functions $\mathcal{O}(\mathbb{G})$ on a compact quantum group $\mathbb{G}$, there exist different algebras of continuous functions, different C*-completions, say $C_\alpha(\mathbb{G})$ and $C_\beta(\mathbb{G})$. Questions of an interpretative nature then might be asked such as:

> *Can it be said that $C_\alpha(\mathbb{G})$ and $C_\alpha(\mathbb{G})$ give* different *quantum spaces but the* same *compact quantum group*?

If it is declared that for a given faithful representation $\pi(C(\mathbb{G}))\subset B(\mathsf{H})$ that elements of $P(\mathsf{H})$ *are* quantum permutations, then things are muddied further because even with coamenable compact quantum groups, such as the dual of the infinite dihedral group (see Section 7.2.1), where there is only one C*-completion of $\mathcal{O}(\mathbb{G})$, and so the Gelfand picture says there is only one quantum space, there can be unitarily inequivalent faithful representations. The Weaver picture might say that these inequivalent representations give different quantum groups: certainly for the dual of the infinite dihedral group different behaviour will be seen in two inequivalent representations. Then the spectre of different compact quantum groups having the same algebra of continuous functions raises its head. Together with motivations from quantum physics, this gives another reason to make the choice not to distinguish any particular Hilbert space representation, and instead work at the C*-algebraic level.

The conclusion to this will be an extension of Definition 5.1.

5.3. **Algebraic quantum mechanics for quantum permutations.** Let $C(\mathbb{G})$ be an algebra of functions on a quantum permutation group with unit $\mathbb{1}_{\mathbb{G}}:=I_{C(\mathbb{G})}$. A projection $p\in C(\mathbb{G})$ is a *Bernoulli* observable (note that $C(\mathbb{G})$ is generated by projections). Denote as before $p^1:=p$ and $p^0:=\mathbb{1}_{\mathbb{G}}-p$. Denote by $S(C(\mathbb{G}))$ the states on $C(\mathbb{G})$, the positive linear functionals of norm one, and choose $\varphi\in S(C(\mathbb{G}))$. Measurement of the state φ with p can give two outcomes, $p=1$ or $p=0$ with distribution:

$$\mathbb{P}[p=\theta\,|\,\varphi]:=\varphi(p^\theta).$$

In the event of $p=\theta$ the state transition, the wave function collapse, is

$$\varphi \mapsto \frac{\varphi(p\cdot p)}{\varphi(p)} \in S(C(\mathbb{G})).$$

Denote this collapsed state by $\widetilde{p^{\theta}}(\varphi)$, the state φ conditioned by $p=\theta$.

Like in Section 3, conditional probabilities may be defined. For example if $q\in C(\mathbb{G})$ is another projection:

$$\mathbb{P}[q=\theta_2\,|\,\widetilde{p^{\theta_1}}(\varphi)] = \frac{\varphi(p^{\theta_1}q^{\theta_2}p^{\theta_1})}{\varphi(p^{\theta_1})}.$$

Sequential measurements may also be considered, via:

$$\mathbb{P}[(q=\theta_2)\succ(p=\theta_1)\,|\,\varphi] = \varphi(p^{\theta_1}q^{\theta_2}p^{\theta_1}),$$

and inductively, for a finite number of projections $(p_i)_{i=1}^n$, and $\theta_i\in\{0,1\}$:

$$\mathbb{P}\left[(p_n=\theta_n)\succ\cdots\succ(p_1=\theta_1)\,|\,\varphi\right] = \varphi(p_1^{\theta_1}\cdots p_{n-1}^{\theta_{n-1}}p_n^{\theta_n}p_{n-1}^{\theta_{n-1}}\cdots p_1^{\theta_1}). \quad (5.1)$$

This is standard quantum probability theory [39], and can be done at the algebraic, $\mathcal{O}(\mathbb{G})$, level. It is worth noting that, through e.g (Th. 3.3.7, [45]) that

$$\mathbb{P}[(p_2=\theta_2)\succ(p_1=\theta_1)|\varphi] \leq \mathbb{P}[p_1=\theta_1|\varphi], \quad (5.2)$$

so that in particular if $\mathbb{P}[(p_2=\theta_2)\succ(p_1=\theta_1)|\varphi]>0$ then $\mathbb{P}[p_1=\theta_1|\varphi]>0$.

Going beyond projections to finite spectrum observables, through the inverse of the isometric Gelfand–Naimark *-isomorphism $C(\mathbb{G})\to\pi(C(\mathbb{G}))\subset B(\mathsf{H})$, finite spectrum observables $f\in C(\mathbb{G})$ have spectral decompositions, and therefore quantities like $\mathbb{P}[f=f_i\,|\,\varphi]=\varphi(p^{f_i})$ and $\mathbb{E}[f\,|\,\varphi]=\varphi(f)$ may be defined. Extending to continuous spectrum observables is more involved. On the one hand there is nothing stopping us defining the expectation of f under φ in the same way, but in general spectral projections like $\mathbb{1}_{[a,b)}(f)$ and $\mathbb{1}_{\{\lambda\}}(f)$ are not elements of $C(\mathbb{G})$. To get around this issue, work with the universal enveloping von Neumann version of $C(\mathbb{G})$, $C(\mathbb{G})^{**}\cong\pi_U(C(\mathbb{G}))''$. This contains a copy of $C(\mathbb{G})$, say $\imath: C(\mathbb{G})\hookrightarrow C(\mathbb{G})^{**}$, and the spectral projections of an observable $f\in C(\mathbb{G})$ are given by $\mathbb{1}_S(\imath(f))$. A state φ on $C(\mathbb{G})$ extends to normal state ω_φ on $C(\mathbb{G})^{**}$. Define

$$\mathbb{P}[f\in S\,|\,\varphi] := \omega_\varphi(\mathbb{1}_S(\imath(f))).$$

If this is non-zero, the state φ on $C(\mathbb{G})$ can be conditioned to:

$$\widetilde{\mathbb{1}_S(f)}\varphi : g\mapsto \frac{\omega_\varphi(\mathbb{1}_S(f)\imath(g)\mathbb{1}_S(\imath(f)))}{\omega_\varphi(\mathbb{1}_S(\imath(f)))},$$

which is a state on $C(\mathbb{G})$.

By noting that a faithful representation $\pi(C(\mathbb{G}))\subset B(\mathsf{H})$ together with a vector state $\psi\in P(\mathsf{H})$ gives an (algebraic) state $\varphi_\psi\in S(C(\mathbb{G}))$,

$$\varphi_\psi(f) := \langle\psi,\pi(f)(\psi)\rangle,$$

Hilbert space vector states are associated with algebraic states. The GNS construction shows that the converse holds. In the sequel, which contains a lot of finite dimensional algebras in any case, it will generally be easier to talk through the language of vector states, particularly as they readily provide examples of states. Furthermore the Hilbert space formalism is more familiar. In any case, the extension of Definition 5.1 to the algebraic case is immediate:

DEFINITION 5.2. Where $C(\mathbb{G})$ is an algebra of continuous functions on a quantum permutation group, *a quantum permutation* is an element of $S(C(\mathbb{G}))$

As will be seen in Section 7, there is a natural candidate for what should be considered a quantum group law $S(C(\mathbb{G}))\times S(C(\mathbb{G}))\to S(C(\mathbb{G}))$.

Where $\varsigma\in S(C(\mathbb{G}))$ and $f\in C(\mathbb{G})$, this implies that as $\varsigma(f)$ is an expectation, only probabilistic data can be got from measuring a quantum permutation. In the classical, commutative setting of the algebra of functions $F(G)$ on a finite group G, where states $\varsigma\in S(F(G)) =: M_p(G)$ are just probabilities on G, this approach says do not consider just the set of deterministic permutations G, but rather the full set of random permutations, $M_p(G)$.

Returning to Gelfand's Theorem: as soon as an algebra of functions $C(\mathbb{G})$ is noncommutative, it is often remarked that it obviously cannot be the algebra of functions on a compact Hausdorff space (with the commutative pointwise multiplication). However the elements of $C(\mathbb{G})$ viewed through the lens of Kadison's function representation are affine functions on a compact Hausdorff space. This is a well worn path, see [29, 36] for discussion and further references.

Note firstly that $S(C(\mathbb{G}))$ is a weak-* compact Hausdorff space [45], and embed $\imath:C(\mathbb{G})\hookrightarrow C(\mathbb{G})^{**}$, $f\mapsto \imath(f)$:

$$\imath(f)(\rho):=\rho(f)\qquad (\rho\in C(\mathbb{G})^*).$$

Finally weak*-convergence of a net of states $\varphi_\lambda\to\varphi$, that for $f\in C(\mathbb{G})$

$$\varphi_\lambda(f)\to\varphi(f),$$

gives continuity to $\imath(f)$:

$$\imath(f)(\varphi_\lambda)\to\imath(f)(\varphi).$$

The multiplication $\imath(C(\mathbb{G}))\otimes\imath(C(\mathbb{G}))\to\imath(C(\mathbb{G}))$ is not the pointwise multiplication, but inherited from $C(\mathbb{G})$:

$$\imath(f)\imath(g)=\imath(fg)\neq\imath(gf)=\imath(g)\imath(f).$$

Through this lens the elements of $C(\mathbb{G})$ are affine functions $S(C(\mathbb{G}))\to\mathbb{C}$, that is for $\rho_1,\rho_2\in S(C(\mathbb{G}))$, and $\lambda\in[0,1]$:

$$\imath(f)(\lambda\,\rho_1+(1-\lambda)\,\rho_2)=\lambda\,\imath(f)(\rho_1)+(1-\lambda)\,\imath(f)(\rho_2).$$

An element of $\imath(C(\mathbb{G}))\subsetneq C(S(C(\mathbb{G})))$ is therefore completely determined by its values on the weak-$*$ closure of the set of pure states, the *pure state space* $\mathcal{P}(C(\mathbb{G})):=\overline{PS(C(\mathbb{G}))}^{w^*}$. This is precisely how a function on a finite group $f_0:G\to\mathbb{C}$ extends to a function on the set of random permutations on G, $f_1:M_p(G)\to\mathbb{C}$:

$$(\lambda\ \text{ev}_{g_1}+(1-\lambda)\ \text{ev}_{g_2})(f_1)=\lambda\,f_0(g_1)+(1-\lambda)\,f_0(g_2).$$

While Gelfand's Theorem says, through the fact that the character space and pure state space coincide for unital commutative C*-algebras, that for a finite group G, the embedded $\imath(F(G))$ is the full algebra of continuous functions $F(\mathcal{P}(F(G)))$, in the noncommutative case, restricting even to $\mathcal{P}(C(\mathbb{G}))$, $\imath(C(\mathbb{G}))$ is a proper subset of $C(S(C(\mathbb{G})))$, so, while tempting, it is abuse of notation to define $\mathbb{G}:=\mathcal{P}(C(\mathbb{G}))$ as the compact Hausdorff space, and continue to use the $C(\mathbb{G})$ notation. If in the classical case the algebra of functions, $F(G)$ is understood as the algebra of affine functions on the random permutations $M_p(G) \to \mathbb{C}$, and an algebra of continuous functions on a quantum permutation group $C(\mathbb{G})$ is understood as an algebra of affine functions $S(C(\mathbb{G})) \to \mathbb{C}$, then the relationship between $\mathcal{P}(C(\mathbb{G}))$ and $S(C(\mathbb{G}))$ reflects in the quantum case the relationship between G and $M_p(G)$ in the classical case.

These analogies are well captured by the following schematic:

$$\begin{array}{ccc} \text{Deterministic} & \hookrightarrow & \text{Random} \\ \downarrow & & \downarrow \\ \text{Pure Quantum} & \hookrightarrow & \text{Quantum} \end{array}$$

The objects on the left are pure states on C*-algebras; while the objects on the right are mixed states. The objects on top are states on commutative algebras; while the objects on the bottom are states on noncommutative algebras. The focus of this work is on the mixed states. This pair of dichotomies is discussed in [18].

Therefore with the focus on mixed rather than pure states, the set of quantum permutations will be denoted by $\mathbb{G} := S(C(\mathbb{G}))$, *a* quantum permutation written an element $\varsigma \in \mathbb{G}$, but the $C(\mathbb{G})$ notation will be kept, but with the implicit understanding that it is a proper subset of $C(S(C(\mathbb{G})))$ (not to mention the fact that for non-coamenable $\mathbb{G}$ there are different C*-completions of $\mathcal{O}(G)$, and thus different state spaces). To emphasise that an element $f \in C(\mathbb{G})$ is a continuous *function*, for a quantum permutation $\varsigma \in \mathbb{G}$, freely associate

$$f(\varsigma) := \imath(f)(\varsigma) = \varsigma(f).$$

There is confusion if $f \in B(\mathsf{H})$ and $\varsigma \in P(\mathsf{H})$, but in such cases $\pi(f)(\xi_\varsigma)$ will be used. Particularly if $\varsigma \in \mathbb{G}$ is deterministic, the notation $\varsigma = \text{ev}_\sigma$ and $\text{ev}_\sigma(f) = f(\sigma)$ will be used. In this sense, the counit can also be denoted $\varepsilon := \text{ev}_e$.

The *Gelfand–Weaver picture* calls the elements of $\mathbb{G} := S(C(\mathbb{G}))$ quantum permutations. The Weaver picture is the fixing instead a faithful Hilbert representation $\pi(C(\mathbb{G})) \subset B(\mathsf{H})$ and calling elements of $P(\mathsf{H})$ quantum permutation vectors. In either case, unless working with a von Neumann algebra, in general only sequential measurements of finite spectrum observables should be considered. At various stages it is easier to invoke the Gelfand–Weaver picture rather than the Weaver picture, and vice versa.

6. More about quantum permutations

Time to give some concrete meaning to all the above discussion.

6.1. **The Birkhoff Slice.**

Given a quantum permutation group $\mathbb{G}$ on N symbols generated by a magic unitary $(u_{ij}^{\mathbb{G}})_{i,j=1}^N \in M_N(C(\mathbb{G}))$, via the Gelfand–Weaver picture, an element $\varsigma \in \mathbb{G}$ is a quantum permutation. In this picture, the projections $u_{ij}^{\mathbb{G}}$ are Bernoulli observables. Make the following interpretation:

$$\mathbb{P}[\varsigma(j) = i] := \mathbb{P}[u_{ij}^{\mathbb{G}} = 1 \,|\, \varsigma] = u_{ij}^{\mathbb{G}}(\varsigma). \tag{6.1}$$

These probabilities can be collected in a matrix:

$$\Phi(\varsigma)_{ij} := u_{ij}^{\mathbb{G}}(\varsigma).$$

That $(u_{ij}^{\mathbb{G}})_{i,j=1}^N$ is a magic unitary implies that $\Phi(\varsigma)$ is a doubly stochastic matrix, i.e. $\Phi(\varsigma)$ is in the Birkhoff polytope $\mathcal{B}_N$, and call the map $\Phi:\mathbb{G}\to\mathcal{B}_N$ the *Birkhoff slice*. It is called a slice as it only captures an ephemeral aspect of a quantum permutation; and is not injective.

In the case of compact matrix quantum groups, there is a natural generalisation of the Birkhoff slice, $\Phi : S(C(\mathbb{G}))\to M_N(\mathbb{C})$. The restriction of this map to characters, an injective map $\Phi:\Omega(C(\mathbb{G}))\to M_N(\mathbb{C})$, has been studied previously. Immediately Woronowicz uses this map to prove Theorem 2.3 [55]. Kalantar and Neufang [32], who associate to a (locally) compact quantum group $\mathbb{G}$, a (locally) compact classical group $\tilde{\mathbb{G}}$, use the map to show that in the case of a compact matrix quantum group, $\tilde{\mathbb{G}}$ is homeomorphic to $\Phi(\Omega(C(\mathbb{G})))$.

Assuming that $\mathbb{P}[\varsigma(k)=\ell]\neq 0$, the quantum permutation ς can be conditioned on $\varsigma(k)=\ell$, and conditional probabilities can be collected in a Birkhoff slice. Recall state conditioning:

$$\begin{aligned}\widetilde{u_{\ell k}^{\mathbb{G}}}(\varsigma) &:= \frac{\varsigma(u_{\ell k}^{\mathbb{G}}\cdot u_{\ell k}^{\mathbb{G}})}{\varsigma(u_{\ell k}^{\mathbb{G}})}\\ \Rightarrow \Phi(\widetilde{u_{\ell k}^{\mathbb{G}}}(\varsigma)) &= \left[\frac{\varsigma(u_{\ell k}^{\mathbb{G}}u_{ij}^{\mathbb{G}}u_{\ell k}^{\mathbb{G}})}{\varsigma(u_{\ell k}^{\mathbb{G}})}\right]_{i,j=1}^N\\ &= [\mathbb{P}[\varsigma(j)=i\,|\,\varsigma(k)=\ell]]_{i,j=1}^N\end{aligned}$$

PROPOSITION 6.1. *Let $\mathbb{G}$ be a quantum permutation group on N symbols. For $\varsigma\in\mathbb{G}$, if $u_{ij}^{\mathbb{G}}(\varsigma)$ is non-zero, the matrix $\Phi(\widetilde{u_{ij}^{\mathbb{G}}}(\varsigma))$ has a one in the (i,j)-th entry. If the algebra of functions is represented by $\pi(C(\mathbb{G}))\subset B(\mathsf{H})$, and ς given by a vector state $\xi_\varsigma\in P(\mathsf{H})$, $\xi_\varsigma\in\operatorname{ran}(\pi(u_{ij}^{\mathbb{G}}))$ if and only if $\Phi(\varsigma)_{ij}=1$* •

Proposition 6.1 then implies that, e.g.:

$$\Phi(\varsigma)=\begin{bmatrix}\Phi(\varsigma)_{11} & \Phi(\varsigma)_{12} & \cdots & \Phi(\varsigma)_{1N}\\ \Phi(\varsigma)_{21} & \Phi(\varsigma)_{22} & \cdots & \Phi(\varsigma)_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ \Phi(\varsigma)_{N1} & \Phi(\varsigma)_{N2} & \cdots & \Phi(\varsigma)_{NN}\end{bmatrix}$$

$$\Rightarrow \Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))=\begin{bmatrix}\Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))_{11} & 0 & \Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))_{13} & \cdots & \Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))_{1N}\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & \Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))_{33} & \cdots & \Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))_{3N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))_{N1} & 0 & \Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))_{N3} & \cdots & \Phi(\widetilde{u_{22}^{\mathbb{G}}}(\varsigma))_{NN}\end{bmatrix}$$

Inductively, assuming $u_{i_nj_n}^{\mathbb{G}}\cdots u_{i_1j_1}^{\mathbb{G}}(\varsigma)\neq 0$

$$\Phi(\widetilde{u_{i_nj_n}^{\mathbb{G}}\cdots \widetilde{u_{i_1j_1}^{\mathbb{G}}}}(\varsigma))_{ij}=\mathbb{P}[\varsigma(j)=i\,|\,(\varsigma(j_n)=i_n)\succ\cdots\succ(\varsigma(j_1)=i_1)].$$

Indeed

$$\mathbb{P}[(\varsigma(j)=i)\succ(\varsigma(j_n)=i_n)\succ\cdots\succ(\varsigma(j_1)=i_1)]=$$
$$\Phi(\widetilde{u_{i_nj_n}^{\mathbb{G}}\cdots \widetilde{u_{i_1j_1}^{\mathbb{G}}}}(\varsigma))_{ij}\cdot\Phi(\widetilde{u_{i_{n-1}j_{n-1}}^{\mathbb{G}}\cdots \widetilde{u_{i_1j_1}^{\mathbb{G}}}}(\varsigma))_{i_nj_n}\cdots\Phi(\varsigma)_{i_1j_1}.$$

EXAMPLE 6.2. *No quantum permutations on three symbols* One hope for this Gelfand–Weaver picture is that it might inspire new results. As a toy example consider the theorem that $S_3^+ = S_3$. This is just to say that $C(S_3^+)$, the universal C*-algebra generated by a 3×3 magic unitary $u^{S_3^+} = (u_{ij}^{S_3^+})_{i,j=1}^3$ is commutative. This was known by Wang [53], but Banica, Bichon, & Collins [8] describe the Fourier-type proof as "quite tricky". Lupini, Mančinska, & Roberson however give a more elementary proof [38].

By allowing talk of *a* quantum permutation. the Gelfand–Weaver picture suggests *why* there are no quantum permutations on three symbols. Without assuming $C(S_3^+)$ commutative, consider the observable

$$x(1) = u_{11}^{S_3^+} + 2u_{21}^{S_3^+} + 3u_{31}^{S_3^+} \in C(S_3^+),$$

which asks of a quantum permutation $\varsigma \in S_3^+$ what it maps one to. Measure ς with $x(1)$ and the denote the result by $\varsigma(1)$. The intuition might be that as soon as $\varsigma(1)$ is known, $\varsigma(2)$ and $\varsigma(3)$ are entangled in the sense that measurement of $x(2)$ cannot be made without affecting $x(3)$. This is only intuition: mathematically, it might still be possible to exhibit e.g. the non-classical event:

$$(\varsigma(3) = 3) \succ (\varsigma(2) = 1) \succ (\varsigma(1) = 3), \tag{6.2}$$

but pausing before measuring $\varsigma(2)$ allows the noting of a relationship between the events $(\varsigma(2) = 1) \succ (\varsigma(1) = 3)$ and $(\varsigma(3) = 2) \succ (\varsigma(1) = 3)$ that implies (6.2) cannot happen. This can be illustrated with the help of the Birkhoff slice and the Weaver picture.

Represent $\pi(C(S_3^+)) \subset B(\mathsf{H})$ with the universal GNS representation so that $\varsigma \in S_3^+$ is a vector state. Assume without loss of generality that measuring ς with $x(1)$ gives $\varsigma(1) = 3$ with non-zero probability $u_{31}^{S_3^+}(\varsigma)$, and the quantum permutation transitions to the vector state $\pi(u_{31}^{S_3^+})(\xi_\varsigma) \in P(\mathsf{H})$.

Now consider, for *any* vector state quantum permutation $\xi \in P(\mathsf{H})$, using the fact that $u_{21}^{S_3^+} u_{31}^{S_3^+} = 0 = u_{32}^{S_3^+} u_{31}^{S_3^+}$, and the rows and columns of u are partitions of unity:

$$\pi(u_{31}^{S_3^+})(\xi) = \pi(u_{12}^{S_3^+} + u_{22}^{S_3^+} + u_{32}^{S_3^+})\pi(u_{31}^{S_3^+})(\xi) = \pi(u_{21}^{S_3^+} + u_{22}^{S_3^+} + u_{23}^{S_3^+})\pi(u_{31}^{S_3^+})(\xi)$$

$$\Rightarrow \pi(u_{12}^{S_3^+} u_{31}^{S_3^+})(\xi) = \pi(u_{23}^{S_3^+} u_{31}^{S_3^+})(\xi) \tag{6.3}$$

That is conditioning on $(\varsigma(2) = 1) \succ (\varsigma(1) = 3)$ is the same as conditioning on $(\varsigma(3) = 2) \succ (\varsigma(1) = 3)$. That is measurement of $\pi(u_{31}^{S_3^+})(\xi_\varsigma) \in P(\mathsf{H})$ with $\pi(x(2)) = \pi(u_{12}^{S_3^+} + 2u_{22}^{S_3^+} + 3u_{33}^{S_3^+})$ produces $x(2) = 1$ with some probability, then there is projection to $\pi(u_{12}^{S_3^+} u_{31}^{S_3^+})(\xi_\varsigma) \in P(\mathsf{H})$, but this is equal to $\pi(u_{23}^{S_3^+} u_{31}^{S_3^+})(\xi_\varsigma) \in P(\mathsf{H})$, the same quantum permutation that is projected to after $x(3) = 2$ is observed (after $\varsigma(1) = 3$).

Associating $\pi(u_{12}^{S_3^+} u_{31}^{S_3^+})(\xi_\varsigma) \in P(\mathsf{H})$ with the quantum permutation $\widetilde{u_{12}^{S_3^+}}\,\widetilde{u_{31}^{S_3^+}}(\varsigma)$

$$\Phi(\widetilde{u_{12}^{S_3^+}}\,\widetilde{u_{31}^{S_3^+}}(\varsigma)) = \begin{bmatrix} 0 & 1 & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}$$

Using (6.3)

$$\Phi(\widetilde{u_{12}^{S_3^+}}\,\widetilde{u_{31}^{S_3^+}}(\varsigma))_{23}=\left\langle \frac{\pi(u_{12}^{S_3^+}u_{31}^{S_3^+})(\xi_\varsigma)}{\|\pi(u_{12}^{S_3^+}u_{31}^{S_3^+})(\xi_\varsigma)\|},\pi(u_{23}^{S_3^+})\frac{\pi(u_{12}^{S_3^+}u_{31}^{S_3^+})(\xi_\varsigma)}{\|\pi(u_{12}^{S_3^+}u_{31}^{S_3^+})(\xi_\varsigma)\|}\right\rangle$$
$$=\left\langle \frac{\pi(u_{23}^{S_3^+}u_{31}^{S_3^+})(\xi_\varsigma)}{\|\pi(u_{23}^{S_3^+}u_{31}^{S_3^+})(\xi_\varsigma)\|},\pi(u_{23}^{S_3^+})\frac{\pi(u_{23}^{S_3^+}u_{31}^{S_3^+})(\xi_\varsigma)}{\|\pi(u_{23}^{S_3^+}u_{31}^{S_3^+})(\xi_\varsigma)\|}\right\rangle=1$$
$$\Rightarrow \Phi(\widetilde{u_{12}^{S_3^+}}\,\widetilde{u_{31}^{S_3^+}}(\varsigma))=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \Phi(\widetilde{u_{12}^{S_3^+}}\,\widetilde{u_{31}^{S_3^+}}(\varsigma))_{31} & 0 & 0\end{bmatrix},$$

and as Φ maps to doubly stochastic matrices, $\Phi(\widetilde{u_{12}^{S_3^+}}\,\widetilde{u_{31}^{S_3^+}}(\varsigma))$ is equal to the permutation matrix (132). This implies that $\widetilde{u_{12}^{S_3^+}}\,\widetilde{u_{31}^{S_3^+}}(\varsigma)\in S_3^+$ is deterministic, that is its Birkhoff slice is a permutation matrix.

Not convinced? Here is a proper proof inspired by the above.

THEOREM 6.3. *$C(S_3^+)$ is commutative.*

Proof. It suffices to show that $u_{11}^{S_3^+}u_{22}^{S_3^+}=u_{22}^{S_3^+}u_{11}^{S_3^+}$ by showing:

$$u_{11}^{S_3^+}u_{22}^{S_3^+}=u_{11}^{S_3^+}u_{33}^{S_3^+}=u_{22}^{S_3^+}u_{33}^{S_3^+}=u_{22}^{S_3^+}u_{11}^{S_3^+}.$$

The first equality follows from:

$$u_{11}^{S_3^+}(u_{21}^{S_3^+}+u_{22}^{S_3^+}+u_{23}^{S_3^+})=u_{11}^{S_3^+}(u_{13}^{S_3^+}+u_{23}^{S_3^+}+u_{33}^{S_3^+}),$$

the second from

$$(u_{11}^{S_3^+}+u_{21}^{S_3^+}+u_{31}^{S_3^+})u_{33}^{S_3^+}=(u_{21}^{S_3^+}+u_{22}^{S_3^+}+u_{23}^{S_3^+})u_{33}^{S_3^+},$$

and the third from

$$u_{22}^{S_3^+}(u_{31}^{S_3^+}+u_{32}^{S_3^+}+u_{33}^{S_3^+})=u_{22}^{S_3^+}(u_{11}^{S_3^+}+u_{21}^{S_3^+}+u_{31}^{S_3^+})\quad\bullet$$

6.2. **Deterministic Permutations.** Let $\jmath:S_N\hookrightarrow M_N(\mathbb{C})$ be the embedding that sends a permutation to its permutation matrix. A *deterministic permutation in* $\mathbb{G}$ is a quantum permutation $\varsigma\in\mathbb{G}$ such that $\Phi(\varsigma)=\jmath(\sigma)$ for some $\sigma\in S_N$. In this case write $\varsigma=\text{ev}_\sigma$. If a deterministic permutation $\text{ev}_\sigma\in\mathbb{G}$, then ev_σ is a character and thus a pure state (see Section 6.2.1). On the other hand, by the homomorphism property of a character $\varsigma\in\mathbb{G}$

$$u_{ij}(\varsigma)=u_{ij}^2(\varsigma)=u_{ij}(\varsigma)^2\Rightarrow u_{ij}(\varsigma)=0\text{ or }1,$$

that is ς is deterministic. The set of deterministic permutations therefore coincides with the set of characters $G_{\mathbb{G}}:=\Omega(C(\mathbb{G}))\subset\mathbb{G}$, and is either empty or a finite group (Corollary 7.3). This implies that the set $G_{\mathbb{G}}$ coincides with the set $\widetilde{\mathbb{G}}$ of Kalantar and Neufang. A *random permutation in* $\mathbb{G}$ is a convex combination of deterministic permutations, and the convex hull of $G_{\mathbb{G}}$ is the set of random permutations in $\mathbb{G}$.

The study of maximal classical subgroups such as $G_{\mathbb{G}} = \widetilde{\mathbb{G}} \leq \mathbb{G}$ has a long history. Another approach, seen for example in Banica and Skalski [13], is to quotient $C(\mathbb{G})$ by its commutator ideal. Formally these approaches can give empty sets: if the character space is empty, or if the commutator ideal coincides with the algebra of functions. Studying instead the universal version $C_u(\mathbb{G})$ at least guarantees a counit, so that $e \in G_{\mathbb{G}}$, and $G_{\mathbb{G}}$ is a group (Corollary 7.3).

In Definition 6.8, with an algebra of continuous functions $C(\mathbb{G})$ represented by $\pi(C(\mathbb{G})) \subset B(\mathsf{H})$ there is a natural definition of a *truly quantum* permutation (vector). At the algebraic level things are less concrete, but working with the enveloping von Neumann algebra $C(\mathbb{G})^{**}$, a deterministic permutation $\text{ev}_\sigma \in \mathbb{G}$ extends to a normal state ω_σ on $C(\mathbb{G})^{**}$, and thus has a support projection $p_\sigma \in C(\mathbb{G})^{**}$ such that $\omega_\sigma(p_\sigma) = 1$. Furthermore, if $\omega_\sigma(p) = 1$, then $p_\sigma \leq p$, and also for all $f \in C(\mathbb{G})^{**}$:

$$\omega_\sigma(f) = \omega_\sigma(p_\sigma f) = \omega_\sigma(f p_\sigma) = \omega_\sigma(p_\sigma f p_\sigma). \tag{6.4}$$

If $G_{\mathbb{G}}$ is empty, define all quantum permutations as truly quantum. Otherwise, define:

$$p_C = \sum_{\sigma \in G_{\mathbb{G}}} p_\sigma, \tag{6.5}$$

and define a quantum permutation $\varsigma \in \mathbb{G}$ as *truly quantum* if its normal extension $\omega_\varsigma \in S(C(\mathbb{G})^{**})$ has the property that $\omega_\varsigma(p_C) = 0$.

A quick consideration of the form of projections on direct sums shows that if an algebra of functions on a quantum permutation is a direct sum with a one-dimensional factor $\mathbb{C}f_i$, then the state $f^i : f_i \mapsto 1$ is deterministic.

Example 6.4. The *Kac–Paljutkin quantum group of order eight*, $\mathfrak{G}_0$, has algebra of functions structure:

$$F(\mathfrak{G}_0) = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3 \oplus \mathbb{C}f_4 \oplus M_2(\mathbb{C}). \tag{6.6}$$

Where $I_2 \in M_2(\mathbb{C})$ the identity, and the projection

$$p := \left(0, 0, 0, 0, \begin{pmatrix} \frac{1}{2} & \frac{1}{2}e^{-i\pi/4} \\ \frac{1}{2}e^{+i\pi/4} & \frac{1}{2} \end{pmatrix}\right),$$

a concrete exhibition of $\mathfrak{G}_0 < S_4^+$ (Th. 1.1 (8), [7]) comes by the magic unitary:

$$u^{\mathfrak{G}_0} := \begin{bmatrix} f_1 + f_2 & f_3 + f_4 & p & I_2 - p \\ f_3 + f_4 & f_1 + f_2 & I_2 - p & p \\ p^T & I_2 - p^T & f_1 + f_3 & f_2 + f_4 \\ I_2 - p^T & p^T & f_2 + f_4 & f_1 + f_3 \end{bmatrix}. \tag{6.7}$$

The one dimensional factors give deterministic permutations, $f^1 = \text{ev}_e$, $f^2 = \text{ev}_{(34)}$, $f^3 = \text{ev}_{(12)}$ and $f^4 = \text{ev}_{(12)(34)}$, so that $G_{\mathfrak{G}_0} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 < \mathfrak{G}_0$. Given a quantum permutation $\varsigma \in \mathfrak{G}_0$, measurement with an $x(j)$ will see collapse to either a random permutation or a state on the $M_2(\mathbb{C})$ factor: a *truly quantum* permutation.

Given a truly quantum permutation $\varsigma \in \mathfrak{G}_0$, the random variables $\varsigma(1)$ and $\varsigma(2)$ are entangled in the following sense: if measurement of $x(1)$ gives $\varsigma(1) = 3$, then subsequent measurement of $x(2)$ will yield $\varsigma(2) = 4$ with probability one. Similarly $\varsigma(3)$ and $\varsigma(4)$ are entangled in this way: this reminds also of S_3^+ given the measurement of $\varsigma(1)$, the random variables $\varsigma(2)$

and $\varsigma(3)$ are entangled. However as soon as e.g. $\varsigma(1)$ is known, there is complete uncertainty about $\varsigma(3)$ and $\varsigma(4)$, for example, for every truly quantum permutation $\varsigma\in\mathfrak{G}_0$,

$$\Phi(\widetilde{u_{31}^{\mathfrak{G}_0}(\varsigma)})=\begin{bmatrix} 0_2 & \frac12 J_2 \\ I_2 & 0_2\end{bmatrix}.$$

With $F(\mathfrak{G}_0)\subset B(\mathbb{C}^6)$, nonclassical behaviour can be exhibited with e.g. the vector state $\varsigma\in\mathfrak{G}_0$ defined by $\xi_\varsigma:=e_6\in\mathbb{C}^6$:

$$\mathbb{P}[(\varsigma(1)=4)\succ(\varsigma(3)=1)\succ(\varsigma(1)=3)]=\frac18.$$

For a truly quantum permutation, certain sequential measurements cannot reveal quantum behaviour. Consider a sequential measurement

$$x(j_n)\succ\cdots\succ x(j_2)\succ x(j_1);$$

if the constituent measurements are all $x(1)$ and $x(2)$ measurements; or all $x(3)$ and $x(4)$ measurements; or a number of $x(1)$ and $x(2)$ measurements followed by $x(3)$ and $x(4)$ measurements (or vice versa), then quantum behaviour will not be observed. Instead these sequential measurements will incorrectly suggest that a quantum permutation $\varsigma\in\mathfrak{G}_0$ is a random permutation (deterministic if there is a mix of $x(1)/x(2)$ and $x(3)/x(4)$ measurements) in the complement of $\mathbb{Z}_2\times\mathbb{Z}_2$ in the dihedral group of order eight.

6.2.1. *Deterministic permutations in the Weaver picture.* For $\mathbb{G}$ a quantum permutation group on N symbols represented by $\pi(C(\mathbb{G}))\subset B(\mathsf{H})$, define

$$\mathsf{H}_\sigma:=\bigcap_{j=1}^N \operatorname{ran}\pi(u^{\mathbb{G}}_{\sigma(j),j}),$$

the subspace whose associated vector states have Birkhoff slice $\jmath(\sigma)$, and are thus equal to ev_σ. Elements of different H_σ are orthogonal. Let $\xi_\varsigma\in P(\mathsf{H}_\sigma)$ define a quantum permutation $\varsigma\in\mathbb{G}$:

$$\pi(u_{ij}^{\mathbb{G}})(\xi_\varsigma)=\begin{cases}\xi_\varsigma, & \text{if } \Phi(\varsigma)_{ij}=1\\ 0, & \text{otherwise.}\end{cases}$$

Hence for all $f\in C(\mathbb{G})$, $\pi(f)(\xi_\varsigma)$ is in the linear span of ξ_ς, so that for a scalar $f_\varsigma\in\mathbb{C}$

$$\pi(f)(\xi_\varsigma)=f_\varsigma\xi_\varsigma.$$

Proposition 6.5. *For all $f\in C(\mathbb{G})$, $\pi(f)|_{\mathsf{H}_\sigma}$ is scalar.*

Proof. If H_σ is (zero- or) one-dimensional the claim holds. Suppose that there are two linearly independent vectors $x_1,\,x_2\in\mathsf{H}_\sigma$. For $f\in C(\mathbb{G})$:

$$\begin{aligned}\pi(f)(x_1+x_2)&=\pi(f)(x_1)+\pi(f)(x_2)\\ \Rightarrow f_{x_1+x_2}(x_1+x_2)&=f_{x_1}(x_1)+f_{x_2}(x_2)\\ \Rightarrow (f_{x_1+x_2}-f_{x_1})(x_1)+(f_{x_1+x_2}-f_{x_2})(x_2)&=0.\end{aligned}$$

This implies that $f_{x_1+x_2}=f_{x_1}=f_{x_2}:=f_\sigma$ and so $\pi(f)|_{\mathsf{H}_\sigma}=f_\sigma I_{\mathsf{H}_\sigma}$ •

Corollary 6.6. *For each $f\in C(\mathbb{G})$,*

$$\pi(f)|_{\bigoplus\mathsf{H}_\sigma}=\sum_{\sigma\in S_N}f_\sigma\,I_{\mathsf{H}_\sigma},$$

and the Hilbert spaces H_σ can be taken to be zero- or one-dimensional •

Corollary 6.7. *Every deterministic permutation is a character.*

Proof. Suppose that $\varsigma = \text{ev}_\sigma$ is deterministic. Consider the GNS representation $(\mathsf{H}_\varsigma, \pi_\varsigma, \xi_\varsigma)$ associated to ς. Arguing as before, that

$$u_{ij}(\varsigma) = \langle \xi_\varsigma, \pi_\varsigma(u_{ij}^{\mathbb{G}})(\xi_s)\rangle = 0 \text{ or } 1$$

implies that for all $f \in C(\mathbb{G})$, there exists $f_\varsigma \in \mathbb{C}$ such that $\pi_\varsigma(f)(\xi_s) = f_\varsigma(\xi_s)$. Therefore

$$\begin{aligned} gf(\varsigma) &= \langle \xi_\varsigma, \pi_\varsigma(gf)\xi_\varsigma\rangle = \langle \xi_\varsigma, \pi_\varsigma(g)\pi_\varsigma(f)(\xi_\varsigma)\rangle \\ &= f_\varsigma\langle \xi_\varsigma, \pi_\varsigma(g)\xi_\varsigma\rangle = g(\varsigma)f(\varsigma) \quad \bullet \end{aligned}$$

Definition 6.8. Let $\mathbb{G} < S_N^+$ be faithfully represented by $\pi(C(\mathbb{G})) \subset B(\mathsf{H})$ with any non-zero H_σ taken to be one-dimensional. The subset $G_{\mathbb{G},\pi} \subset S_N$ made up of $\sigma \in S_N$ for which H_σ is non-zero is the set of *deterministic permutation vectors in* $\pi(C(\mathbb{G}))$, and the set $P\left(\ell^2(G_{\mathbb{G},\pi})\right)$ the *random permutation vectors in* $\pi(C(\mathbb{G}))$. A quantum permutation in $P(\ell^2(G_{\mathbb{G},\pi})^\perp)$ is a *truly quantum permutation vector.*

Suppose that $\sigma \in G_{\mathbb{G},\pi}$ for a faithful representation $\pi(C(\mathbb{G})) \subset B(\mathsf{H})$. The theme of

$$\text{Deterministic} \subset \text{Random} \subset \text{Quantum},$$

is then captured by:

$$P(\mathsf{H}_\sigma) \subset P(\ell^2(G_{\mathbb{G},\pi})) \subset P(\mathsf{H}),$$

and $\mathsf{H} = \ell^2(G_{\mathbb{G},\pi}) \oplus \ell^2(G_{\mathbb{G},\pi})^\perp$: classical, random permutations vectors are orthogonal to truly quantum permutation vectors.

Assume $G := G_{\mathbb{G},\pi}$ is non-empty, and define the $\pi(C(\mathbb{G}))$-invariant subspace

$$F(G) := \bigoplus_{\sigma\in G} \mathsf{H}_\sigma.$$

These random permutation vectors are all random permutations, and so $G_{\mathbb{G},\pi} \subset G_{\mathbb{G}}$. It will be seen in Section 7.2.1, that the random permutation vectors can form a proper or even empty subset.

Question. Given a quantum permutation group $\mathbb{G}$ represented by $\pi(C(\mathbb{G})) \subset B(\mathsf{H})$, if $G_{\mathbb{G},\pi}$ is non-empty, does it form a (classical) finite group?

The property of $C(\mathbb{G})$ having deterministic permutation vectors is *not* unitarily invariant, and it follows from Davis (Lemma 5.1, [21]), that $\dim \mathsf{H}_\sigma$, is unitarily invariant (for example $\dim \mathsf{H}_\sigma = 0$ vs $\dim \mathsf{H}_\sigma = 1$).

7. Quantum group law and identity

7.1. **Quantum Group Law.** In the classical case of a finite group $G \leq S_N$, for g_2, $g_1 \in G$, the group law is encoded within the convolution of the pure states ev_{g_1} and ev_{g_2}:

$$\mathbb{1}_{j\to i}(g_2 \star g_1) = \Delta(\mathbb{1}_{j\to i})(g_2 \otimes g_1) = \sum_k (\mathbb{1}_{k\to i} \otimes \mathbb{1}_{j\to k})(g_2 \otimes g_1) = \mathbb{1}_{j\to i}(g_1 g_2).$$

The same game can be played with quantum permutations:

Definition 7.1. The *quantum group law* $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$ is the convolution, $\varsigma_2 \star \varsigma_1 := (\varsigma_2 \otimes \varsigma_1)\Delta$.

If $C(\mathbb{G})$ admits a counit, it plays precisely the role of the identity. Restricted to $G_{\mathbb{G}}$, precomposing ev_σ with the antipode $S : C(\mathbb{G}) \to C(\mathbb{G})$ is an inverse:

$$\text{ev}_\sigma \circ S = \text{ev}_{\sigma^{-1}} \Rightarrow \text{ev}_{\sigma^{-1}} \star \text{ev}_\sigma = \text{ev}_e = \varepsilon.$$

For a general quantum permutation, it might be more accurate to call $\varsigma^{-1} := \varsigma \circ S$ the *reverse* of ς in the sense that

$$|u_{i_nj_n} \cdots u_{i_1j_1}|^2(\varsigma^{-1}) = |u_{j_ni_n} \cdots u_{j_1i_1}|^2(\varsigma)$$
$$\Rightarrow \mathbb{P}[(\varsigma^{-1}(i_n) = j_n) \succ \cdots \succ (\varsigma^{-1}(i_1) = j_1)] = \mathbb{P}[(\varsigma(j_n) = i_n) \succ \cdots \succ (\varsigma(j_n) = i_n)]$$

PROPOSITION 7.2. *Let ς_2, $\varsigma_1 \in \mathbb{G}$ be quantum permutations. The Birkhoff slice is multiplicative:*

$$\Phi(\varsigma_2 \star \varsigma_1) = \Phi(\varsigma_2)\Phi(\varsigma_1).$$

Proof. Calculate

$$\Phi(\varsigma_2 \star \varsigma_1)_{ij} = (\varsigma_2 \star \varsigma_1)\Delta(u_{ij}^{\mathbb{G}}) = \sum_k (u_{ik}^{\mathbb{G}} \otimes u_{kj}^{\mathbb{G}})(\varsigma_2 \otimes \varsigma_1) = \sum_k u_{ik}^{\mathbb{G}}(\varsigma_2)u_{kj}^{\mathbb{G}}(\varsigma_1) = [\Phi(\varsigma_2)\Phi(\varsigma_1)]_{ij} \bullet$$

COROLLARY 7.3. *The set of deterministic permutations $G_{\mathbb{G}}$ is either empty or a group. It is a group if and only if $\varepsilon \in \mathbb{G}$. Therefore if a quantum permutation group $\mathbb{G}$ is coamenable, or the algebra of continuous functions $C(\mathbb{G}) \cong C_u(\mathbb{G})$, then $G_{\mathbb{G}}$ is a group. In particular, if $\mathbb{G} \leq S_4^+$, then $G_{\mathbb{G}}$ is a group* $\bullet$

Remarkably for a piece about compact quantum groups, the *Haar state* has not yet been introduced. The following is equivalent to more conventional definitions.

DEFINITION 7.4. A quantum permutation group $\mathbb{G}$ has a quantum permutation $h_{\mathbb{G}}$ called the *Haar state* that is the unique annihilator for the quantum group law, that is for all $\varsigma \in \mathbb{G}$

$$h_{\mathbb{G}} \star \varsigma = h_{\mathbb{G}} = \varsigma \star h_{\mathbb{G}}.$$

The non-zero elements of Birkhoff slice $\Phi(h_{\mathbb{G}})$ are equal along rows and columns. The Haar state can be thought of as the "maximally random" quantum permutation: in the classical case of S_N it corresponds to the equal superposition of deterministic permutations, i.e. the uniform measure on S_N.

7.2. **Abelian Quantum Permutation Groups.** Given a compact group G, the algebra of continuous functions analogue of "*G is abelian*" is that "*$C(G)$ is cocommutative*". In this sense an abelian compact quantum group is given by a cocommutative algebra of continuous functions $C(\widehat{\Gamma})$, that is an algebra of continuous functions on the dual of a discrete group Γ. As the quantum permutations in $\widehat{\Gamma}$ are in the state space of $C(\widehat{\Gamma})$, which is some class of positive definite functions on Γ with pointwise, commutative multiplication, this idea that duals of discrete groups are abelien is trivial through Definition 5.2.

Consider a cyclic group of order N, $\langle\gamma\rangle$. For $\omega := \exp(2\pi i/N)$, consider the following vector in $F(\widehat{\langle\gamma\rangle})^N$:

$$(u^{\widehat{\langle\gamma\rangle}})_{,1} := \frac{1}{N} \begin{bmatrix} e + \gamma + \gamma^2 + \cdots + \gamma^{N-1} \\ e + \omega\gamma + \omega^2\gamma^2 + \cdots + \omega^{N-1}\gamma^{N-1} \\ e + \omega^2\gamma + (\omega^2)^2\gamma^2 + \cdots + (\omega^2)^{N-1}\gamma^{N-1} \\ \cdots \\ e + \omega^{N-1}\gamma + (\omega^{N-1})^2\gamma^2 + \cdots + (\omega^{N-1})^{N-1}\gamma^{N-1} \end{bmatrix}. \tag{7.1}$$

A magic unitary for $\widehat{\langle\gamma\rangle}\cong\widehat{\mathbb{Z}_N}\cong\mathbb{Z}_N<S_N^+$ is the circulant matrix defined by this vector:

$$[u^{\widehat{\langle\gamma\rangle}}]_{i,j}:=\frac{1}{N}\sum_{\ell=1}^{N}\omega^{(i-j)\ell}\gamma^{\ell}.$$

Note that

$$\gamma=u_{11}+\omega^{N-1}u_{21}+\omega^{N-2}u_{31}+\cdots+\omega u_{N,1}.$$

Let $\Gamma=\langle\gamma_1,\dots\gamma_k\rangle$ be a finitely generated discrete group, with generators of finite order $N_1,N_2,\dots,N_k$. Then the dual $\widehat{\Gamma}$ is a quantum permutation group on $N:=\sum_p N_p$ symbols via the block magic unitary:

$$u^{\widehat{\Gamma}}=\begin{pmatrix} u^{\widehat{\gamma_1}} & 0 & \cdots & 0\\ 0 & u^{\widehat{\gamma_2}} & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & u^{\widehat{\gamma_k}}\end{pmatrix}.$$

Note that due to the fact that the $u^{\widehat{\gamma_p}}$ are circulant matrices, the entries of each $u^{\widehat{\gamma_p}}$ commute. Such a construction was used to prove Theorem 4.4. It is long known that duals of finite groups G are quantum permutation groups, but earlier references such as [9] placed $\widehat{G}<S^+_{|G|^2}$. This result can be shown by considering each group element an order $|G|$ generator, so making for each g_i a $|G|\times|G|$ magic unitary $u^{\widehat{\langle g_i\rangle}}$, and forming the block matrix $\operatorname{diag}(u^{\widehat{\langle g_1\rangle}},\dots,u^{\widehat{\langle g_{|G|}\rangle}})$. Smaller embeddings of duals of finite groups abound. An induction on $|G|$ shows that $\widehat{G}\leq S^+_{|G|}$. A much smaller embedding $\widehat{S_N}<S^+_{N+2}$ is provided by $S_N=\langle(12),(12\dots N)\rangle$. Slightly better, for $N\geq 3$, is $\widehat{S_N}<S^+_{N+1}$ via $S_N=\langle(12),(23\dots N)\rangle$. In fact, except for $N=5,6,8$, S_N is generated by an element of order two and an element of order three [43], and so S_5^+ contains all the duals $\widehat{S_N}$ for $N\geq 9$. The dual of any dihedral group, including the infinite dihedral group, is a quantum subgroup of S_4^+. On the other hand, the smallest embedding of the dual of the quaternion group is $\widehat{Q}\leq S_8^+$ via $Q=\langle j,k\rangle$.

Let G be a finite group. Where $\operatorname{Irr}(G)$ is an index set for a maximal set of pairwise inequivalent *unitary* irreducible representations $\rho_\alpha:G\to\operatorname{GL}(V_\alpha)$, and $d_\alpha:=\dim V_\alpha$, the algebra of functions on the dual can be given by

$$F(\widehat{G})=\bigoplus_{\alpha\in\operatorname{Irr}(G)}M_{d_\alpha}(\mathbb{C}).$$

When looking at concrete examples, sometimes it is easy to look at the regular representation:

$$\pi(F(\widehat{G}))\subset B(\mathbb{C}^{|G|}),\quad g:e_h\mapsto e_{gh}.$$

Each one dimensional representation gives a deterministic permutation. The quantum permutations are positive definite functions on G. The function $\mathbb{1}_G\in\widehat{G}$ is the counit $F(\widehat{G})\to\mathbb{C}$, and has Birkhoff slice $\jmath(e)$. That $\widehat{G}$ is abelian implies that the group of deterministic permutations is abelian. If G is a simple group, either there are no truly quantum permutations, and $G\cong\mathbb{Z}_p$ for a prime p, and $\widehat{G}=G$, or $G_{\widehat{G}}$ is the trivial group. The dual of the symmetric group for $N\geq 2$ has only two deterministic permutations: one is the counit $\varepsilon=\mathbb{1}_G$, and the other is the sign representation, an order two deterministic permutation: $\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\delta_\sigma$.

The dual of the quaternion group has four deterministic permutations $G_{\widehat{Q}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the dual of a dihedral group has either two or four depending on the degree.

For every odd prime p, the semi-direct product of $\mathbb{Z}_p$ and the multiplicative group $(\mathbb{Z}_p)^\times$, acting by multiplication on $\mathbb{Z}_p$, is a group of order $p(p-1)$ with p characters and thus the dual has p deterministic permutations. In this picture, Pontryagin duality for a finite abelian group G is nothing but $\widehat{G}$ having no truly quantum permutations: all the representations are one dimensional, and hence deterministic.

Suppose that $\Gamma = \langle \gamma_1, \dots, \gamma_k \rangle$ is a discrete group such that $\widehat{\Gamma} < S_N^+$. Partition the symbols $1, \dots, N$ into blocks $B_1, \dots, B_k$ of length $|\gamma_p|$. The fact that the blocks of $u^{\widehat{\Gamma}}$ are circulant matrices implies that if for $j \in B_p$, the measurement of a quantum permutation $\varsigma \in \widehat{\Gamma}$ with $x(j)$ will see collapse of the wave function such that the restriction of the state to B_p is now deterministic in the sense that if, for $r, s \in B_p$, $u_{rs}^{\widehat{\Gamma}}(\varsigma) \neq 0$, then the matrix $\left[\Phi(\widetilde{u_{rs}^{\widehat{\Gamma}}}(\varsigma))\right]_{i,j \in B_p}$ is a permutation matrix. For example, for $N \geq 9$, and generators σ_1 of order two, and σ_2 of order three, consider $\widehat{S_N} < S_5^+$ with blocks $B_1 = \{1, 2\}$ from $u^{\widehat{\langle \sigma_1 \rangle}}$ and $B_2 = \{3, 4, 5\}$ from $u^{\widehat{\langle \sigma_2 \rangle}}$. Let $\varsigma \in \widehat{S_N}$ be a quantum permutation. Measure with

$$x(4) = 3u_{34}^{\widehat{S_N}} + 4u_{44}^{\widehat{S_N}} + 5u_{54}^{\widehat{S_N}}.$$

Suppose that the measurement yields $x(4) = 5$. Then the quantum permutation collapses to:

$$\widetilde{u_{54}^{\widehat{S_N}}}(\varsigma) = \frac{\varsigma(u_{54}^{\widehat{S_N}} \cdot u_{54}^{\widehat{S_N}})}{\varsigma(u_{54}^{\widehat{S_N}})},$$

and the circulant nature of $u^{\widehat{\langle \sigma_2 \rangle}}$ implies that the Birkhoff slice

$$\Phi(\widetilde{u_{54}^{\widehat{S_N}}}(\varsigma)) = \left(\begin{array}{cc|ccc} \alpha & 1-\alpha & 0 & 0 & 0 \\ 1-\alpha & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right),$$

that is the measurement of $x(4)$ collapse the state in such a way that it is deterministic on B_2. Unlike in the case of S_3^+, while $\widetilde{u_{54}^{\widehat{S_N}}}(\varsigma)(1)$ and $\widetilde{u_{54}^{\widehat{S_N}}}(\varsigma)(2)$ are entangled, the measurement of $x(1)$ can disturb the state in such a way that the full knowledge about B_2 is now gone. The same phenomenon occurs for the infinite dihedral group. The circulant nature of the blocks implies that for all $j_1, j_2 \in B_p$, the measurement of $x(j_1)$ determines $x(j_2)$, and all such measurements could be denoted $x(B_p)$. It could be speculated that the dual of a discrete group $\Gamma = \langle \gamma_1, \dots, \gamma_k \rangle$ could model a k particle "entangled" quantum system, where the p-th particle, corresponding to the block B_p, has $|\gamma_p|$ states, labelled $1, \dots, |\gamma_p|$. Full information about the state of all particles is in general impossible, but measurement with $x(B_p)$ will see collapse of the pth particle to a definite state. Only the deterministic permutations in $\widehat{\Gamma}$ would correspond to classical states.

7.2.1. *Infinite Dihedral Group.* As promised in Section 4.2, a further study of the dual of the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2 = \widehat{D_\infty} < S_4^+$ can be undertaken. As D_∞ is an amenable group, $C(\widehat{D_\infty})$ is coamenable [15], and thus has deterministic permutations. However $C(\widehat{D_\infty})$ has faithful representations $\pi(C(\widehat{D_\infty})) \subset B(\mathsf{H})$ *without* deterministic permutations *vectors*.

The first representation uses the isomorphism (attributed in [49] to Pedersen)

$$C(\widehat{D_\infty}) \cong \mathrm{C}^*(p,q) \cong \{f \in C([0,1], M_2(\mathbb{C})) \,:\, f(0),\, f(1) \text{ diagonal}\}\,.$$

Recalling the magic unitary $u^{\widehat{D_\infty}}$, the explicit isomorphism maps $u_{21}^{\widehat{D_\infty}}$ to

$$p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $u_{43}^{\widehat{D_\infty}}$ to

$$q(t) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Thus $C(\widehat{D_\infty})$ can be represented faithfully by $\pi_1(C(\widehat{D_\infty})) \subset B(L^2([0,1],\mathbb{C}^2))$. Elements of $P(L^2([0,1],\mathbb{C}^2))$ give vector quantum permutations $\varsigma \in \widehat{D_\infty}$. For example, choose

$$\xi_\varsigma(t) = \begin{pmatrix} it \\ \sqrt{1-t} \end{pmatrix} \in P(L^2([0,1],\mathbb{C}^2)), \text{ then}$$

$$\Phi(\varsigma) = \begin{bmatrix} 3/5 & 2/5 & 0 & 0 \\ 2/5 & 3/5 & 0 & 0 \\ 0 & 0 & 3/10 & 7/10 \\ 0 & 0 & 7/10 & 3/10 \end{bmatrix} \quad\text{and}\quad \Phi(\widetilde{u_{11}^{\widehat{D_\infty}}}(\varsigma)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix},$$

$$\Rightarrow \mathbb{P}[\varsigma(1)=1] = \frac{3}{5} \text{ and } \quad \mathbb{P}[(\varsigma(3)=4) \succ (\varsigma(1)=1)] = \frac{2}{5},$$

$$\Phi(\widetilde{u_{43}^{\widehat{D_\infty}}}\widetilde{u_{11}^{\widehat{D_\infty}}}(\varsigma)) = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \Phi(\widetilde{u_{21}^{\widehat{D_\infty}}}\widetilde{u_{43}^{\widehat{D_\infty}}}\widetilde{u_{11}^{\widehat{D_\infty}}}(\varsigma)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3/5 & 2/5 \\ 0 & 0 & 2/5 & 3/5 \end{bmatrix},$$

$$\Rightarrow \mathbb{P}[(\varsigma(3)=3) \succ (\varsigma(1)=2) \succ (\varsigma(3)=4) \succ (\varsigma(1)=1)] = \frac{3}{50}.$$

However every pair from $\pi_1(u_{11}^{\widehat{D_\infty}})$, $\pi_1(u_{21}^{\widehat{D_\infty}})$, $\pi_1(u_{33}^{\widehat{D_\infty}})$, $\pi_1(u_{43}^{\widehat{D_\infty}})$ has trivial range intersection, and so $G_{\widehat{D_\infty},\pi_1}$ is empty. It can also be shown that the regular representation $\pi_2(C(\widehat{D_\infty}))$ has no deterministic permutation vectors (a vector in the range intersection of $\pi_2(u_{11}^{\widehat{D_\infty}})$ and $\pi_2(u_{33}^{\widehat{D_\infty}})$ would be fixed for the whole of $\widehat{D_\infty}$).

In the universal GNS representation $\pi_{\mathrm{GNS}}(C(\widehat{D_\infty})) \subset B(\mathsf{H}_{\mathrm{GNS}})$ the cyclic vector of the counit ξ_ε gives a deterministic permutation vector. More concretely, by the universal property of $\mathrm{C}^*(p,q)$ there is a *-homomorphism, $\pi_V : C(\widehat{D_\infty}) \to C(\widehat{\mathbb{Z}_2 \times \mathbb{Z}_2})$, mapping

$$p \mapsto \frac{e-(1,0)}{2} \text{ and } q \mapsto \frac{e-(0,1)}{2}.$$

Furthermore $C(\widehat{\mathbb{Z}_2\times\mathbb{Z}_2})$ can be represented in $B(\mathbb{C}^4)$ via:

$$(1,0)\mapsto \text{diag}(1,1,-1,-1) \text{ and } (0,1)\mapsto \text{diag}(1,-1,1,-1).$$

Taking the direct sum of π_1 with π_V gives a faithful representation

$$(\pi_1\oplus\pi_V)(C(\widehat{D_\infty}))\subset B(L^2([0,1],\mathbb{C}^2)\oplus\mathbb{C}^4).$$

This has deterministic permutations vectors in the second factor: $e_1,\dots,e_4$ giving, respectively ev_e, $\text{ev}_{(34)}$, $\text{ev}_{(12)}$ and $\text{ev}_{(12)(34)}$. It is clear from $u^{\widehat{D_\infty}}$ that there are no more characters (a rather heavy handed way of showing that D_∞ has just four one dimensional representations), and so $G_{\widehat{D_\infty}}\cong\mathbb{Z}_2\times\mathbb{Z}_2$, a dihedral group. This is an illustration of the fact that quotients $\Gamma\to\Lambda$ linearly extend to quotients $C(\widehat{\Gamma})\to C(\widehat{\Lambda})$, which give rise to quantum subgroups $\widehat{\Lambda}<\widehat{\Gamma}$

$$\pi: C(\widehat{\Gamma})\to C(\widehat{\Lambda}),\quad \sum_{\gamma\in\Gamma}\alpha_\gamma\gamma\mapsto\sum_{\gamma\in\Gamma}\alpha_\gamma[\gamma];$$

and indeed $\mathbb{Z}_2\times\mathbb{Z}_2$ is a quotient of D_∞ [41]. Using representations $\pi_{\mathbb{Z}_1}: C(\widehat{D_\infty})\to C(\widehat{\mathbb{Z}_1})$ and $\pi_{\mathbb{Z}_2}: C(\widehat{D_\infty})\to C(\widehat{\mathbb{Z}_2})$, there are faithful representations of $C(\widehat{D_\infty})$ with sets of deterministic permutation vectors $\mathbb{Z}_1$ and $\mathbb{Z}_2$. This adding of deterministic permutation vectors seems too easy, but note that it uses the universal property of $C(\widehat{D_\infty})=C^*(D_\infty)$, and universal completions always have at least one character. The question of whether or not the deterministic permutation vectors form a group still stands. It can be shown that there is a representation $\pi(C(\mathbb{G}))\subset B(\mathsf{H})$ with deterministic permutation vector $\sigma\in P(\mathsf{H})$ precisely when:

$$\||u_{\sigma(N)N}\cdots u_{\sigma(1)1}|^2\|=1.$$

This section can be finished with a question.

Question. Suppose that Γ is a finitely generated discrete group with no finite order generator. Is $\widehat{\Gamma}$ a quantum permutation group?

Perhaps an alternative approach to such a compact quantum group could be found in Goswami and Skalski [28].

8. Phenomena & Ideas

8.1. **No Quantum Cyclic Group.** Consider the embedding of the cyclic group $\langle g\,|\,g^N=e\rangle=\mathbb{Z}_N\hookrightarrow S_N\hookrightarrow M_N(\mathbb{C})$ defined by $g\hookrightarrow(123\dots N)$. While a random permutation $\varsigma\in M_p(S_N)$ requires $N-1$ measurements from $\{x(k)\}_{k=1}^N$ to collapse to some deterministic permutation, an element of $M_p(\mathbb{Z}_N)$ requires only one: finding $x(1)=k$ will see collapse to the pure state associated with g^k. This suggests why there is no quantum cyclic group. There is a long standing difficulty in defining non-trivial quantum versions of compact matrix groups, such as the alternating group A_N, whose definition involves determinants. Is there a similar argument to the above which suggests why there is no quantum alternating group (more on a quantum alternating group in Section 8.4)?

8.2. **Quasi-subgroups of finite quantum permutation groups.** In the Gelfand-Weaver picture a random walk on a quantum permutation group is a sequence $(\varsigma^{\star k})_{k\geq 0}$ in $\mathbb{G}$ (see [42] for more). Of particular interest are *ergodic* random walks, those random walks such that the sequence converges in the weak-* topology to the *Haar state*, $h_{\mathbb{G}}$. A necessary (but dramatically far from sufficient) condition for the convolution powers $(\varsigma^{\star k})_{k\geq 0}$ to converge to the Haar state are that:

$$\Phi(\varsigma^k) = \Phi(\varsigma)^k \to \Phi(h_{\mathbb{G}}).$$

It seems like this observation could help achieve some first partial results in the extension of the finite quantum group random walk ergodic theorem [44] to the case of quantum permutations, alas there are many quantum permutations $\varsigma \neq h_{\mathbb{G}}$ such that $\Phi(\varsigma) = \Phi(h_{\mathbb{G}})$. There are even examples of random permutations whose associated random walk is not ergodic, for example $\nu \in M_p(S_3)$ given by:

$$\nu = \frac{1}{3}(\mathrm{ev}_{(12)} + \mathrm{ev}_{(13)} + \mathrm{ev}_{(23)}) \Rightarrow \Phi(\nu) = \frac{1}{3}J_3 = \Phi(h_{S_3}).$$

Very crude results are available though: for example, if $\mathbb{G} < S_N^+$ and any non-trivial N-th root of unity is in $\sigma(\Phi(\varsigma))$, then the associated random walk is not ergodic. A qualitative aspect would be that if $\Phi(\varsigma)$ is such that the convergence of $\Phi(\varsigma)^k \to \Phi(h_{\mathbb{G}})$ is slow, then the associated random walk on $\mathbb{G}$ should take a long time to converge to the Haar state.

If G is a classical finite group, subsets $S \subset G$ that are closed under the group law are subgroups. More generally, for the algebra of continuous functions $C(G)$ on a classical compact group, where the comultiplication is given by precomposition with the group law $m : G \times G \to G$, $\Delta(f) = f \circ m$, states $C(G) \to \mathbb{C}$ correspond via integration to Borel probability measures in $M_p(G)$, and in the context of this work could be called *random permutations* (of an infinite set if G is infinite). In this context the quantum group law might be called the random group law, and the Kawada–Itô theorem says that random permutations idempotent with respect to the random group law are Haar measures/states on compact subgroups of G [34]. Suppose that $\mathbb{H} \leq \mathbb{G}$ by $\pi : C(\mathbb{G}) \to C(\mathbb{H})$. Quantum permutations $\varsigma_2', \varsigma_1' \in \mathbb{H}$ are quantum permutations $\varsigma_2, \varsigma_1 \in \mathbb{G}$ by $\varsigma_i := \varsigma_i' \circ \pi$. Note that

$$\varsigma_i(\ker \pi) = (\varsigma_i')\pi(\ker \pi) = 0,$$

similarly

$$(\varsigma_2 \star \varsigma_1) \ker \pi = (\varsigma_2 \otimes \varsigma_1)\Delta(\ker \pi) = (\varsigma_2' \otimes \varsigma_1')(\pi \otimes \pi)\Delta(\ker \pi) = (\varsigma_2' \otimes \varsigma_1')\Delta\pi(\ker \pi) = 0,$$

so that $\mathbb{H}$ is closed under the quantum group law of $\mathbb{G}$, and the Haar state of $\mathbb{H}$ in $\mathbb{G}$, $h_{\mathbb{H}} \circ \pi$, is an idempotent state in $\mathbb{G}$, called a *Haar idempotent.*

Compact quantum groups, however, can have non-Haar idempotents. These are quantum permutations $\varsigma \in \mathbb{G}$ such that $\varsigma \star \varsigma = \varsigma$, that are not the Haar state on any compact subgroup. In the Gelfand picture, idempotent states correspond to measures uniform on virtual objects called *quasi-subgroups* [33]. Pal's idempotents in the Kac–Paljutkin quantum group provide counterexamples in the quantum case [47]. As the current work allows us to talk of a *set* of quantum permutations $\mathbb{S} \subset \mathbb{G}$, where f^1, f^4 are dual to f_1, $f_2 \in F(\mathfrak{G}_0)$, and E^{11}, E^{22} dual to E_{11}, E_{22} in the $M_2(\mathbb{C})$ factor of $F(\mathfrak{G}_0)$, the convex hulls $\mathbb{S}_i := \mathrm{co}(\{f_1, f_4, E^{ii}\})$ could be called *quasi-subgroups* of $\mathfrak{G}_0$ with associated idempotent states $\frac{1}{4}f^1+\frac{1}{4}f^4+\frac{1}{2}E^{ii}$. Consider $\varsigma\in\mathbb{S}_i$. The Birkhoff slice is given by, for some $\alpha,\beta\in[0,1]$:

$$\Phi(\varsigma)=\begin{pmatrix}\beta M_\alpha & \frac{(1-\beta)}{2}J_2\\ \frac{(1-\beta)}{2}J_2 & \beta M_\alpha\end{pmatrix};\qquad \text{where } M_\alpha=\begin{pmatrix}\alpha & 1-\alpha\\ 1-\alpha & \alpha\end{pmatrix}.$$

Quasi-subgroups behave very much like quantum subgroups: they are closed under the quantum group law (Prop. 3.12, [44]), they contain the identity ε, and they are closed under precomposition with the antipode ([37]).

Let $\Gamma=\langle\gamma_1,\dots,\gamma_k\rangle$ be a discrete group with generators of finite order. The quasi-subgroups $S_\Lambda\subset\widehat{\Gamma}$ are given by non-trivial subgroups $\Lambda\leq\Gamma$:

$$S_\Lambda:=\{\varsigma\in\widehat{\Gamma}:\delta_\lambda(\varsigma)=1\text{ for all }\lambda\in\Lambda\}.$$

The associated idempotent state is $\mathbb{1}_\Lambda$. Suppose that $\gamma_p\in\Lambda$. Then, where $\Phi(\cdot)_{B_p}$ refers to the $u^{\widehat{\langle\gamma_p\rangle}}$ block of $u^{\widehat{\Gamma}}$:

$$\Phi(\mathbb{1}_H)_{B_p}=I_{|\gamma_p|},$$

but as Φ is multiplicative, for $k\in\mathbb{N}$:

$$\Phi(\mathbb{1}_H^{\star k})_{B_p}=I_{|\gamma_p|}\not\longrightarrow\frac{1}{|\gamma_p|}J_{|\gamma_p|}=\Phi(h_{\widehat{\Gamma}})_{B_p},$$

In the finite case the support projection is $\chi_\Lambda=\sum_{\lambda\in\Lambda}\lambda/|\Lambda|$, and the walk remains concentrated on S_Λ in the sense that $\chi_\Lambda(\mathbb{1}_\Lambda^{\star k})=1$ for all $k\in\mathbb{N}$. In the finite case (to guarantee that $|\lambda|<\infty$), if no $\gamma_p\in\Lambda$, and $\Gamma\leq S_N^+$, then let $\lambda\in\Lambda$ and consider $\Gamma<S_{N+|\lambda|}^+$ via:

$$u^{\widehat{\Gamma}<S^+_{N+|\lambda|}}=\begin{pmatrix}u^{\widehat{\Gamma}} & 0\\ 0 & u^{\widehat{\langle\lambda\rangle}}\end{pmatrix},$$

then the non-ergodicity can be seen in the λ block: $\Phi(\mathbb{1}_\Lambda)_{B_\lambda}=I_{|\lambda|}$. Note that it is not required for $\varsigma\in S_\Lambda$ that $\delta_\gamma(\varsigma)=0$ for γ in the complement of Λ in Γ. If $\delta_\lambda(\varsigma)=1$ for all $\lambda\in\Lambda$, then for the projection $\overline{\chi_\Lambda}:=e-\chi_\Lambda\in F(\widehat{\Gamma})$, it is the case that $\overline{\chi_\Lambda}(\varsigma^{\star k})=0$ for all $k\in\mathbb{N}$.

A very good question is: why are quasi-subgroups not always quantum subgroups? The conventional analysis for an idempotent state $\phi\in\mathbb{G}$ on a finite quantum group $\mathbb{G}$ is to consider the ideals of the associated idempotent states:

$$N_\phi:=\{g\in F(\mathbb{G}):\phi(g^*g)=0\}.$$

Franz & Skalski (Th. 4.5, [24]) show that ϕ is a Haar idempotent precisely when N_ϕ is two-sided, equivalently self-adjoint, equivalently S-invariant.

This can be explained in the language of measurement. For example for Pal's quasi-subgroup $\mathbb{S}_1:=\operatorname{co}(\{f_1,f_4,E^{11}\})\subset\mathfrak{G}_0$, note first that for any $\varsigma\in\mathbb{S}_1$,

$$E_{12}(\varsigma)=0. \tag{8.1}$$

Consider $\varsigma:=E^{11}\in\mathbb{S}_1$:

$$\mathbb{P}[\varsigma(3)=2]=\frac{1}{2},$$

and so the quantum permutation conditioned on $\varsigma(3)=2$ is:

$$\widetilde{u_{13}^{\mathfrak{G}_0}(\varsigma)}=\frac{\varsigma(u_{13}^{\mathfrak{G}_0}\cdot u_{13}^{\mathfrak{G}_0})}{u_{13}^{\mathfrak{G}_0}(\varsigma)}=2E^{11}(u_{13}^{\mathfrak{G}_0}\cdot u_{13}^{\mathfrak{G}_0}).$$

But:

$$E_{12}\left(\widetilde{u_{13}^{\mathfrak{G}_0}(\varsigma)}\right)=2E^{11}(u_{13}^{\mathfrak{G}_0}E_{12}u_{13}^{\mathfrak{G}_0})=\frac{1}{2},$$

and this implies with (8.1) that measurement with $u_{13}^{\mathfrak{G}_0}$ has transitioned the quantum permutation outside the quasi-subgroup. This cannot happen in the same way for a quantum subgroup $\mathbb{H}\leq\mathbb{G}$ with associated Haar idempotent $\phi\in\mathbb{H}$. Suppose that $u_{ij}^{\mathbb{G}}(\phi)>0$ and $f\in N_\phi$. Suppose that $\phi(j)=i$ has been observed and the quantum permutation transitions to $\widetilde{u_{ij}^{\mathbb{G}}(\phi)}$. Then

$$|f|^2(\widetilde{u_{ij}^{\mathbb{G}}(\phi)})=\frac{\phi(u_{ij}^{\mathbb{G}}|f|^2u_{ij}^{\mathbb{G}})}{u_{ij}(\phi)}=\frac{\phi(|fu_{ij}^{\mathbb{G}}|^2)}{u_{ij}(\phi)}=0,$$

as N_ϕ is a two-sided ideal.

As another example, consider $\widehat{S_3}\leq S_N^+$ given by $u^{\widehat{S_N}}=\operatorname{diag}(u^{\widehat{\langle(12)\rangle}},\dots)$. The quasi-subgroup $S_{\langle(23)\rangle}\subset\widehat{S_3}$ is the set of quantum permutations such that $e(\varsigma)=(23)(\varsigma)=1$. Take $\mathbb{1}_{\langle(23)\rangle}\in S_{\langle(23)\rangle}$ so that $\mathbb{P}[\mathbb{1}_{\langle(23)\rangle}(1)=1]=1/2$. Then

$$\delta_{(23)}(\widetilde{u_{11}^{\widehat{S_3}}\mathbb{1}_{\langle(23)\rangle}})=\frac{\mathbb{1}_{\langle(23)\rangle}(u_{11}^{\widehat{S_3}}\delta_{23}u_{11}^{\widehat{S_3}})}{u_{11}^{\widehat{S_3}}(\mathbb{1}_{\langle(23)\rangle})}=2\,\mathbb{1}_{\langle(23)\rangle}\left(\frac{1}{4}((13)+(23)+(123)+(132))\right)=\frac{1}{2}\neq 1,$$

so that $\widetilde{u_{11}^{\widehat{S_3}}\mathbb{1}_{\langle(23)\rangle}}\notin S_{\langle(23)\rangle}$.

Certainly in these examples, quasi-subgroups are just like quantum subgroups: until you start measuring them and it is seen that they are not stable under wave function collapse.

Question. Suppose that $\mathbb{G}$ is a quantum permutation group and $\mathbb{S}\subset\mathbb{G}$ a quasi-subgroup that is not a quantum subgroup. Does there exist a generator $u_{ij}^{\mathbb{G}}\in C(\mathbb{G})$ such that $\widetilde{u_{ij}^{\mathbb{G}}(\varsigma)}\notin\mathbb{S}$ for some $\varsigma\in\mathbb{S}$?

In the finite case, an idempotent state $\phi_{\mathbb{S}}$ is associated to a *group-like projection* $\mathbb{1}_{\mathbb{S}}$ (see [44] for more including original references), and therefore it is tenable to define:

Definition 8.1. Let $\mathbb{G}$ be a finite quantum permutation group with an idempotent state $\phi_{\mathbb{S}}\in\mathbb{G}$ and associated group-like projection $\mathbb{1}_{\mathbb{S}}\in C(\mathbb{G})$. The associated *quasi-subgroup* $\mathbb{S}\subset\mathbb{G}$ is given by:

$$\mathbb{S}:=\{\varsigma\in\mathbb{G}:\mathbb{1}_{\mathbb{S}}(\varsigma)=1\}.$$

8.3. **Cyclic Cosets.** Being concentrated on a quasi-subgroup is one barrier to a random walk $\varsigma^{\star k}$ converging to the Haar state. Another is periodicity. In the finite case, assuming that a random walk is not concentrated on a quasi-subgroup, it is concentrated on a cyclic coset of quasi-subgroup:

DEFINITION 8.2. [44] Let $\mathbb{G}$ be a finite quantum permutation group. A quantum permutation $\varsigma \in G$ is *supported on a cyclic coset of a proper quasi-subgroup* if there exists a pair of projections p_0, p_1, such that $p_0p_1 = 0$, $p_0 + p_1 \leq \mathbb{1}_{\mathbb{G}}$, $\varsigma(p_1) = 1$, p_0 is a group-like projection, $(\varsigma \otimes I_{F(\mathbb{G})})\Delta(p_1) = p_0$, and there exists $d > 1$ such that $((\varsigma \otimes I_{F(\mathbb{G})})\Delta)^d(p_1) = p_1$.

In the classical case, p_0, p_1 are indicator functions $\mathbb{1}_N$, $\mathbb{1}_{Ng}$ for some $N \lhd H$, $H \leq G$, such that, for some $d > 1$, $H/N \cong \mathbb{Z}_d$. In the *irreducible* case, where ς is not concentrated on a subgroup, $N \lhd G$, and the p_0, p_1 are elements of a full partition of $\mathbb{1}_G$.

In the 'abelian', dual group case a result of Freslon [26] can be illustrated using the Birkhoff slice. Freslon considers the case where $\varsigma \in \widehat{\Gamma}$ coincides with a non-trivial character on $\Lambda < \Gamma$. In the fashion of the previous section, let $\lambda \in \Lambda$ give a function $\lambda \in F(\widehat{\Gamma})$ and consider $\widehat{\Gamma} \leq S_N^+$ by including λ in the set of generators forming the magic unitary $u^{\widehat{\Gamma}}$. If $\lambda(\varsigma) = 1$ then ς is concentrated on the quasi-subgroup S_Λ. Therefore let $\lambda(\varsigma) = e^{i\theta}$, say $e^{2\pi i m/|\lambda|}$. Considering (7.1), and recalling ς restricted to Λ is a character, it is the case that:

$$\Phi(\varsigma)_{B_\lambda} = \jmath((12\dots|\lambda|)^{|\lambda|-m+1}),$$

and the multiplicative nature of the Birkhoff slice implies that $\Phi(\varsigma^{\star k})_{B_\lambda}$ is periodic, not converging to $\Phi(h_{\widehat{\Gamma}})_{B_\lambda}$.

Note that measurement of $\varsigma \in \widehat{\Gamma}$ with $x(B_p)$ gives a character on $\langle \gamma_p \rangle$. This follows from the fact that $|\delta_\gamma(\varsigma)| \leq 1$ and the form of (7.1). In the classical case, a random walk given by $\nu \in M_p(G)$ which is irreducible but periodic is concentrated on a coset Ng of a proper normal subgroup, and the subsequence $\nu^{\star(k|g|+1)}$ converges to the uniform measure on, Ng, that is the convolution of an idempotent h_N and a character/deterministic permutation ev_g:

$$\nu^{\star(k|g|+1)} \to h_H \star \text{ev}_g \text{ , and } \nu^{\star(k|g|+s)} \to h_H \star \text{ev}_{g^s} \text{ .}$$

In this case, ev_g commutes with the idempotent h_H and so:

$$(h_H \star \text{ev}_g)^{\star k} = h_H \star \text{ev}_{g^k} \text{ .}$$

In the case of irreducible random walks on quantum permutation groups it is not always the case that periodicity comes from a character/deterministic permutation commuting with an idempotent. It can be shown that the random walk on $\widehat{Q}$ given by:

$$\varsigma = \delta_1 - \delta_{-1} - i\delta_j + i\delta_{-j}$$

is irreducible and periodic, but is not equal to any character/deterministic permutation on $\widehat{Q}$ times an idempotent state. Instead passing to $\langle j \rangle$, the character $\langle j \rangle \to \mathbb{C}$, $\chi : j^s \mapsto \exp(3\pi i s/2)$, is such that $\varsigma = \mathbb{1}_{\langle j \rangle}\chi$, but χ is not equal to a restriction of a character on Q. Therefore, a character $\text{ev}_g \in \mathbb{G}$ commuting with an idempotent is sufficient but not necessary for periodic behaviour. One final remark, in the classical case, it is *not* the case that periodic implies concentrated on a coset of a normal subgroup. What is the case is that periodic implies concentrated on a coset of a proper normal subgroup $N \lhd H$ of a subgroup $H \leq G$. In this case, there is the convolution of a deterministic permutation $\text{ev}_g \in H$ and an idempotent h_H, but every deterministic permutation $\text{ev}_g \in H$ is a deterministic permutation $\text{ev}_g \in G$. This is not generally true in the quantum case. The final possibility might be a quantum or even random permutation $\varsigma \in \mathbb{G}$ commuting with an idempotent $\phi_{\mathbb{S}}$ such that $\varsigma^{\star k} = \varepsilon$, however this is impossible because such a quantum permutation satisfies:

$$\Phi(\varsigma^{\star k}) = \Phi(\varsigma)^k = I_N,$$

but the only invertible doubly stochastic matrices come from deterministic permutations.

8.4. **Fixed Points Phenomena and Quantum Transpositions.** Given a quantum permutation group $\mathbb{G} \leq S_N^+$ with algebra of functions $C(\mathbb{G})$, and fundamental representation $u^{\mathbb{G}} \in M_N(C(\mathbb{G}))$, define the number of fixed points observable:

$$\text{fix}^{\mathbb{G}} := \sum_{j=1}^{N} u_{jj}^{\mathbb{G}}.$$

In general, the spectrum of $\text{fix}^{\mathbb{G}}$ contains non-integers: indeed $\sigma(\text{fix}^{\widehat{D_\infty}}) = [0,4]$. When the spectrum of $\text{fix}^{\mathbb{G}}$ is finite, such as in the case of a finite quantum permutation group, there is a spectral decomposition in $F(\mathbb{G})$:

$$\text{fix}^{\mathbb{G}} = \sum_{\lambda \in \sigma(\text{fix}^{\mathbb{G}})} \lambda\, p_\lambda, \tag{8.2}$$

and if $\varsigma \in \mathbb{G}$ is such that

$$\mathbb{P}[\text{fix}^{\mathbb{G}}(\varsigma) = \lambda] = p_\lambda(\varsigma) > 0,$$

then

$$\widetilde{p_\lambda \varsigma} := \frac{\varsigma(p_\lambda \cdot p_\lambda)}{p_\lambda(\varsigma)} \in \mathbb{G},$$

is a quantum permutation with λ fixed points, as is any quantum permutation with $\widetilde{p_\lambda \varsigma} = \varsigma$. Note that if a quantum permutation has λ fixed points, the trace of $\Phi(\varsigma)$ is λ.

Definition 8.3. Where $\text{fix}^{\mathbb{G}} = \sum_{i=1}^N u_{ii}^{\mathbb{G}}$ has spectral decomposition (8.2), a quantum permutation in a finite quantum permutation group $\mathbb{G} < S_N^+$ *has λ fixed points* if $p_\lambda(\varsigma) = 1$. A quantum permutation in $\mathbb{G} < S_N^+$ with $N-2$ fixed points is a quantum *transposition in* $\mathbb{G} < S_N^+$.

Define a magic unitary for $\widehat{S_3} < S_4^+$ by

$$u^{\widehat{S_3}} = \begin{bmatrix} u^{\widehat{\langle (12)\rangle}} & 0 \\ 0 & u^{\widehat{\langle (13)\rangle}} \end{bmatrix}.$$

The spectrum $\sigma(\text{fix}^{\widehat{S_3}}) = \{0,1,3,4\}$. The deterministic permutations ev_e (given by the trivial representation) and $\text{ev}_{(12)(34)}$ (given by the sign representation) have four and zero fixed points. A quantum permutation with three fixed points is:

$$\varsigma = \delta_e + \frac{1}{2}\delta_{(12)} + \frac{1}{2}\delta_{(13)} - \delta_{(23)} - \frac{1}{2}\delta_{(123)} + \delta_{(132)}.$$

It has Birkhoff slice:

$$\Phi(\varsigma) = \begin{pmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 1/4 & 3/4 \end{pmatrix}.$$

Placing $\widehat{S_3} < S_8^+$ via $u^{\widehat{S_3}<S_8^+} = \operatorname{diag}(u^{\widehat{S_3}}, u^{\widehat{S_3}})$ is one way to get a transposition, however note, reflective of $A_8 \lhd S_8$, there is a periodicity to $\varsigma \in S_8^+$:

$$\lim_{k\to\infty} \varsigma^{\star 2k} = \phi_0 := \delta_e + \delta_{(23)} \text{ and } \lim_{k\to\infty} \varsigma^{\star(2k+1)} = \phi_1 := \delta_e - \delta_{(23)}.$$

Note $\phi_0 = \delta_e + \delta_{(23)} \in \widehat{S_3}$ is an idempotent state with support projection $p_0 := (e + (23))/2 \in F(\widehat{G})$, and ϕ_1 has support projection $p_1 := (e - (23))/2$. It is the case that:

$$\operatorname{ev}_{(12)(34)} = \mathbb{1}_{\langle(123)\rangle} - \mathbb{1}_{\{(12),(13),(23)\}}$$

and so

$$\phi_1 = \operatorname{ev}_{(12)(34)} \phi_0 = \phi_0 \operatorname{ev}_{(12)(34)} .$$

This implies that:

$$\phi_1^{\star k} = (\operatorname{ev}_{(12)(34)} \phi_0)^{\star k} = \operatorname{ev}_{(12)(34)^k} \phi_0 = \begin{cases} \phi_0, & \text{if } k \text{ even} \\ \phi_1, & \text{if } k \text{ odd.} \end{cases}.$$

The quantum permutation ς is concentrated on p_1, a cyclic coset of the quasi-subgroup $S_{\langle(23)\rangle} \subset \widehat{S_3}$ encountered in the previous section. This implies that the support of the irreducible random walk associated with ς bounces between p_0 and p_1. On the technical level, this is unlike the periodicity of the state uniform on permutations of odd parity because ϕ_0 is not the Haar state on a quantum subgroup of $\widehat{S_3}$, so it doesn't make sense to say that $\langle(23)\rangle$ is normal in S_3. See [44] for more.

There is some evidence that random walks on quantum permutation groups behave like random walks on finite groups, for example Freslon, Teyssier, and Wang [27], find that a quantum transposition walk has a cutoff at time $N \ln N/2$, the same as the classical random transposition walk. One possible heuristic might be that the distributions of u_{ij} for unobserved/unmeasured quantum permutations are much like random permutations: it is only after measurement that the theories diverge. So, for example, both the quantum and random transposition walks converges to the respective Haar quantum permutations ς_h^+ and ς_h, and

$$u_{ij}^{S_N^+}(\varsigma_h^+) = \frac{1}{N} = u_{ij}^{S_N}(\varsigma_h),$$

but

$$0 < |u_{31}^{S_N^+} u_{22}^{S_N^+} u_{11}^{S_N^+}|^2(\varsigma_h^+) \neq |u_{31}^{S_N} u_{22}^{S_N} u_{11}^{S_N}|^2(\varsigma_h) = 0.$$

However this heuristic can only be pushed so far. Suppose that $\nu \in M_p(S_N)$ is supported on the set of transpositions. Then the associated random walk cannot converge as there is a periodicity associated with the parity: the random walk will jump between sets of permutations with even and odd sign. In the quantum case this does not happen, and indeed a random walk given by a quantum permutation corresponding to the uniform measure on transpositions not only converges, but converges to the Haar state (Th. 4.3, [27]). This suggests that there is no quantum alternating group.

Another quantum phenomenon is that there are quantum permutations with $N - 1$ fixed points which are *not* the identity. Consider the finite quantum group $\widehat{S_4} < S_5^+$ given by the magic unitary:

$$u^{\widehat{S_4^+}}=\begin{pmatrix} u^{\widehat{\langle(12)\rangle}} & 0 \\ 0 & u^{\widehat{\langle(234)\rangle}}\end{pmatrix}\in M_5(F(\widehat{S_4})).$$

Representing $F(\widehat{S_4})\subset B(\mathbb{C}^{24})$ with the regular representation, and employing a CAS, it can be found that

$$\sigma(\text{fix}^{\widehat{S_4}})=\left\{0,\frac{5-\sqrt{17}}{2},1,2,3,4,\frac{5+\sqrt{17}}{2},5\right\},$$

so that the phenomenon of quantum permutations with a non-integer number of fixed points occurs for $\widehat{S_4}$. Define subsets of S_4:

$$X_1:=\langle(34)\rangle,\ X_2:=(12)\langle(34)\rangle,\ X_3:=\{\sigma\neq e:\sigma(1)=1\}\backslash X_1,\ X_4:=\{\sigma\neq e:\sigma(2)=2\}\backslash X_1,$$

$$X_5:=\{(13)(24),(14)(23),(1423),(1324)\},\text{ and }X_6:=S_4\Bigg\backslash\left(\bigcup_{\ell=1}^{5}X_\ell\right).$$

Then the following quantum permutation has four fixed points and is not the identity/counit:

$$\varsigma:=\mathbb{1}_{X_1}+\frac{1}{3}\mathbb{1}_{X_2}+\frac{5}{6}\mathbb{1}_{X_3}-\frac{1}{2}\mathbb{1}_{X_4}-\frac{2}{3}\mathbb{1}_{X_5}-\frac{1}{6}\mathbb{1}_{X_6}, \tag{8.3}$$

and has Birkhoff slice:

$$\Phi(\varsigma)=\begin{pmatrix} 2/3 & 1/3 & 0 & 0 & 0\\ 1/3 & 2/3 & 0 & 0 & 0\\ 0 & 0 & 8/9 & 1/18 & 1/18\\ 0 & 0 & 1/18 & 8/9 & 1/18\\ 0 & 0 & 1/18 & 1/18 & 8/9\end{pmatrix}$$

As the convolution power in $\widehat{S_4}$ is pointwise multiplication, $\varsigma^{\star k}\to\mathbb{1}_{X_1}$, and there is convergence to a non-Haar idempotent.

For any integer $\ell\geq 2$, quantum permutation groups $\mathbb{G}\leq S_N^+$ are also quantum permutation groups $\mathbb{G}<S_{\ell N}^+$ via:

$$u^{\mathbb{G}<S_{\ell N}^+}:=\text{diag}(u^{\mathbb{G}},u^{\mathbb{G}},\dots,u^{\mathbb{G}})\in M_{\ell N}(C(\mathbb{G})),$$

and so if $\varsigma\in\mathbb{G}$ has $N-\frac{2}{\ell}$ fixed points, $\varsigma\in\mathbb{G}\leq S_{\ell N}$ is a transposition, that is it has $\ell N-2$ fixed points. Therefore, $\varsigma\in\widehat{S_4}$ given by (8.3) is a random transposition $\varsigma\in\widehat{S_4}<S_{10}^+$ whose convolution powers do not exhibit periodicity.

This suggests a conjecture. Following an earlier paper of Freslon, [27] defines a state on $C(S_N^+)$ that is the quantum analogue of uniform on transpositions. Similarly a *central* state φ_{N-1} "uniform on quantum permutations with $N-1$ fixed points" is given by defining its values on characters of irreducible representations of S_N^+:

$$\varphi_{N-1}(\chi_n)=Q_n(N-1).$$

CONJECTURE. The random walk on S_N^+ given by the measure uniform on quantum permutations with $N-1$ fixed points has a cut-off in some norm at time $N\ln N$.

Two heuristics suggest why this might be true. Firstly quantum permutations $\varsigma \in S_N^+$ with one fixed point are quantum transpositions in S_{2N}^+. Freslon, Teyssier, and Wang show that the random walk uniform on quantum transpositions in S_{2N}^+ takes $\mathcal{O}(N\ln N)$ steps to converge to uniform. There may or not be a gap here: the analogy may fail to carry through correctly, or the fact that not all quantum transpositions in S_{2N}^+ come from quantum permutations with one fixed point in S_N^+ may also play a role.

The second idea is more speculative again. Freslon, Teyssier, and Wang cleverly implement classical probabilistic methods in their study of quantum transpositions. Where $\overline{u_{ii}} = \mathbb{1}_{S_N^+} - u_{ii}$, it is a little exercise to show that:

$$\sum_{i=1}^{N} \overline{u_{ii}}(\varphi_{N-1}) = 1,$$

and so if the $\overline{u_{ii}}(\varphi_{N-1})$ are equal, consider the random walk given by:

$$\varphi = \frac{1}{N}\sum_{i=1}^{N} \widetilde{\overline{u_{ii}}}\varphi_{N-1} = \sum_{i=1}^{N} \varphi_{N-1}(\overline{u_{ii}}\cdot\overline{u_{ii}}).$$

The idea here is that first a quantum permutation with $N-1$ fixed points is chosen, and then a label $\{1,2,\dots,N\}$ is chosen at random to *not* be fixed. It may be the case that $\phi_i := \widetilde{\overline{u_{ii}}}\varphi_{N-1} = N\varphi_{N-1}(\overline{u_{ii}}\cdot\overline{u_{ii}})$ will suitably randomise the label i in such a way that once all the labels $1,2,\dots,N$ are 'selected', the distribution of the random walk will be uniform:

Conjecture. The N distinct states ϕ_i provide, for any $\sigma \in S_N$, a convolution factorisation of the Haar state:

$$h_{S_N^+} = \phi_{\sigma(1)} \star \phi_{\sigma(2)} \star \cdots \star \phi_{\sigma(N)}.$$

How long it takes to select all the labels might be, along the lines of Aldous & Diaconis [1], some class of quantum 'stopping time', and there might be a distance $\|\cdot\|_\alpha$ such that:

$$\|\varphi^{\star k} - h_{S_N^+}\|_\alpha \leq \mathbb{P}[T > k].$$

The time taken to choose all the labels is known as the coupon collector's problem, and if T is the (stopping, random) time taken to have selected all the labels, it is well known (e.g. [22]) that, for $c > 0$:

$$\mathbb{P}[T > N\ln N + cN] \leq e^{-c}.$$

The final ingredient would be $N \to \infty$ asymptotic equivalence with respect to $\|\cdot\|_\alpha$ between the random walk given by φ_{N-1} and the random walk given by φ.

To record the relationship between quantum permutations in a finite quantum permutation group $\mathbb{G} < S_N^+$ and the quantum transposition studied in [27] is to ask what is a quantum transposition when $\sigma(\text{fix})$ is no longer finite. Recall the universal enveloping von Neumann algebra $C(\mathbb{G})^{**}$ of $C(\mathbb{G})$, which contains $\imath : C(\mathbb{G}) \hookrightarrow C(\mathbb{G})^{**}$ and the spectral projections of elements of $C(\mathbb{G})$. Consider $\imath(\text{fix}^{\mathbb{G}}) \in C(\mathbb{G})^{**}$, with spectral projections $\mathbb{1}_S(\imath(\text{fix}^{\mathbb{G}}))$, in particular $p_{N-2} := \mathbb{1}_{\{N-2\}}(\imath(\text{fix}^{\mathbb{G}}))$. Where ω_ς is the normal extension of $\varsigma \in \mathbb{G}$, define:

$$\mathbb{P}[\varsigma \text{ is a quantum transposition}] := \omega_\varsigma(p_{N-2}),$$

and say that ς is a quantum transposition if $\omega_\varsigma(p_{N-2}) = 1$.

The quantum transposition $\varphi_{\text{tr}} := \varphi_{N-2}$ is a *central state*, and it is the only central quantum transposition in S_N^+. *Central states* such as φ_{tr} have some nice properties: that for any irreducible representation $n \in \mathbb{N}_{\geq 0}$, $\varphi_{\text{tr}}(\rho_{ij}^{(n)}) = \varphi_{\text{tr}}(n)\delta_{i,j}$, and as the matrix elements of the irreducible representations form a basis of $\mathcal{O}(S_N^+)$, they are completely determined by their restriction to the central algebra $\mathcal{O}(S_N^+)_0$ generated by the characters as:

$$\varphi_{\text{tr}}(\chi_n) = \sum_{i=1}^{d_n} \varphi_{\text{tr}}(\rho_{ii}^{(n)}) = d_n \varphi_{\text{tr}}(n).$$

The central algebra is commutative, and it follows from spectral theory that:

$$C(S_N^+)_0 \cong C([0, N]).$$

The isomorphism from the characters to $C([0, N])$ is given by $\chi_n \mapsto (t \mapsto U_{2n}(\sqrt{t}/2))$, where U_n are the Chebyshev polynomials of the second kind, and therefore, restricted to the central algebra, $\chi_0 + \chi_1 = \text{fix} \cong \text{id}_{[0,N]}$. The state φ_{tr} is given by $\text{ev}_{(N-2)} \in C([0, N])^*$. In the case of $C([0, N])$ the spectral projections of fix are not in $C([0, N])$, and so one must pass to $C([0, N])^{**} = \ell^\infty([0, N])$ which contains $C([0, N])$ and acts on $\ell^2([0, N])$ by multiplication. The spectral measures of fix $\in \ell^\infty([0, N])$ are just $S \mapsto M_{\mathbb{1}_S}$, and therefore $\mathbb{1}_{\{N-2\}}(\text{fix}) = M_{\delta_{\{N-2\}}}$, and so $\mathbb{1}_{\{N-2\}}(\text{fix})$ is the projection:

$$f \mapsto f(N-2)\delta_{N-2} \in \ell^2([0, N]). \tag{8.4}$$

Denote this spectral projection by $p_{N-2} \in B(\ell^2([0, N]))$. The normal extension of φ_{tr} is also ev_{N-2}, and indeed $\text{ev}_{N-2}(p_{N-2}) = 1$, and because of (8.4), ev_{N-2} is the unique central quantum transposition in S_N^+.

Related to this "no quantum alternating group" phenomenon is the fact that, unlike characters, where $\text{ev}_{g_1} \star \text{ev}_{g_2} = \text{ev}_{g_1 g_2}$, the product of pure, deterministic, permutations is pure, the product of pure quantum permutations need not be pure. As an example, consider the quantum permutation $\varsigma \in \widehat{S_3}$ given by:

$$\varsigma = \delta_e + \frac{\sqrt{2}+1}{3}\delta_{(12)} - \frac{2\sqrt{2}}{3}\delta_{(13)} + \frac{\sqrt{2}-1}{3}\delta_{(23)} - \frac{2}{3}\delta_{(123)} - \frac{1}{3}\delta_{(132)}.$$

Its powers converge to δ_e, the Haar state, but δ_e is not pure. More on this phenomenon can be read about in Section 4.3.2 of [44], where it is argued that while in the classical case the powers of *random permutations* ς supported on a coset Ng of a proper normal subgroup bounce around the cosets of N, and thus never converges to the Haar state, this is not the case in the quantum world.

8.5. **Alternating Measurement.** Define a magic unitary

$$u^{\widehat{S_3}} = \begin{pmatrix} u^{\widehat{\langle(12)\rangle}} & 0 \\ 0 & u^{\widehat{\langle(13)\rangle}} \end{pmatrix}.$$

There is a probability that a finite number of measurements does not detect quantum behaviour. For example, using the vector state $\xi = e_{(123)}$ in the regular representation $\pi(F(\widehat{S_3})) \subset B(\ell^2(S_3))$:

$$\mathbb{P}[(\varsigma_\xi(4) = 4) \succ (\varsigma_\xi(3) = 3) \succ (\varsigma_\xi(2) = 1) \succ (\varsigma_\xi(1) = 2)] = \frac{1}{4},$$

there is no quantum behaviour observed. It might even be believed that $\varsigma_\xi=\text{ev}_{(12)}$ except the Birkhoff slice $\Phi\left(\widetilde{u_{44}^{\widehat{S_3}}}\widetilde{u_{33}^{\widehat{S_3}}}\widetilde{u_{12}^{\widehat{S_3}}}\widetilde{u_{21}^{\widehat{S_3}}}\varsigma_\xi\right)$ shows that the probability that ς_ξ *now* fixes one, after previously $\varsigma_\xi(1)=2$ was observed, is $1/4$. What is the probability that repeated measurements of $\cdots\succ x(2)\succ x(1)\succ x(4)\succ x(3)\succ x(2)\succ x(1)$ *never* reveal quantum behaviour? Here a result of von Neumann [57] concerning alternating projections can be used:

THEOREM 8.4. *Let* $p,\,q\in B(\mathsf{H})$ *be projections and define* $S:=\operatorname{ran}p\cap\operatorname{ran}q$. *Then the sequence* $((pq)^k)_{k\geq 1}$ *converges strongly to* p_S*, the orthogonal projection onto* S.

If p and q are projections in a C*-algebra A, the intersection defined in the theorem S can be trivial in one faithful representation $\pi_1(A)\subset B(\mathsf{H}_1)$, and not in another, necessarily unitarily inequivalent, $\pi_2(A)\subset B(\mathsf{H}_2)$.

In finite dimensional $F(\widehat{S_3})$, a sequence of the form $((pq)^k)_{k\geq 1}$ converges, and as elements of $\mathbb{Z}_2\times\mathbb{Z}_2<S_4$ are the only permutations that measurements of quantum permutations in $\widehat{S_3}$ can possibly *appear* to reveal, recalling the symmetries in $u^{\widehat{S_3}}$, the only infinite sequential measurements that have to be considered involve alternating projections of $\pi(u_{11}^{\widehat{S_3}})$ and $\pi(u_{33}^{\widehat{S_3}})$ (for $e\in S_4$); $\pi(u_{11}^{\widehat{S_3}})$ and $\pi(u_{34}^{\widehat{S_3}})$ (for (34)); $\pi(u_{12}^{\widehat{S_3}})$ and $\pi(u_{33}^{\widehat{S_3}})$ (for (12)); and $\pi(u_{12}^{\widehat{S_3}})$ and $\pi(u_{34}^{\widehat{S_3}})$ (for $(12)(34)$). Using the notation

$$(E_2\succ E_1)^{\succ n}:=\underbrace{(E_2\succ E_1)\succ\cdots\succ(E_2\succ E_1)}_{n\text{ copies of }(E_2\succ E_1)},$$

the probability that repeated measurements of a vector state ς_ξ with $\pi(u_{11}^{\widehat{S_3}})$ and $\pi(u_{33}^{\widehat{S_3}})$ suggests that $\varsigma_\xi=\text{ev}_{(12)}$ is:

$$\lim_{n\to\infty}\mathbb{P}[((\varsigma_\xi(3)=3)\succ(\varsigma_\xi(1)=2))^{\succ n}]=\lim_{n\to\infty}\left\|\left(\pi(u_{33}^{\widehat{S_3}})\pi(u_{21}^{\widehat{S_3}})\right)^k(\xi)\right\|^2.$$

As $\operatorname{ran}\pi(u_{33}^{\widehat{S_3}})\cap\operatorname{ran}\pi(u_{21}^{\widehat{S_3}})$ is trivial, this probability is zero. The intersections for ev_e and $\text{ev}_{(12)(34)}$ are *not* trivial, and indeed the probability that repeated measurements of e.g. for $\xi=e_{(123)}$ the quantum permutation ς_ξ, with $\pi(x(1))$ and $\pi(x(3))$ suggests that $\varsigma_\xi=\text{ev}_e$ is $1/6$ — with equal probability those measurements suggest that $\varsigma_\xi=\text{ev}_{(12)(34)}$. Indeed where p_C is the projection used to measure how classical a quantum permutation is

$$p_C(\varsigma_\xi)=(p_e+p_{(12)(34)})(\varsigma_\xi)=\frac{1}{3}.$$

Let $\mathbb{G}\leq S_N^+$ be a quantum permutation group and $\varsigma_0\in\mathbb{G}$. Let $M\subset C(\mathbb{G})$ be a set of observables, perhaps finite spectrum like the N^2 measurements $\{u_{ij}\}_{i,j=1}^N$, or the N measurements $\{x(j)\}_{j=1}^N$, and choose a probability distribution ρ on M (or perhaps a sequence of distributions $(\rho_k)_{k\geq 0}$). Starting with ς_0, use ρ to choose an observable $f_1\in M$, and measure ς_0 with f_1. The measurement will produce an eigenvalue of f_1, say $f_{1,i}$, and the quantum permutation will transition:

$$\varsigma_0\mapsto\varsigma_1:=\frac{\varsigma_0(p^{f_{1,i}}\cdot p^{f_{1,i}})}{p^{f_{1,i}}(\varsigma_0)}.$$

Continue to sample observables $f_2, f_3, \dots \in M$ producing a sequence of quantum permutations:

$$\varsigma_0 \mapsto \varsigma_1 \mapsto \varsigma_2 \mapsto \cdots .$$

For a given triple (M, ς_0, ρ) what do the dynamics look like? For $1_{C(\mathbb{G})^{**}}$ denoted (by duplication of notation) by $\mathbb{1}_{\mathbb{G}}$, define $q_C := \mathbb{1}_{\mathbb{G}} - p_C$ (p_C defined in (6.5)), then the distance that a quantum permutation ς_k is from being classical could be given by, where ω_{ς_k} is its normal extension to $C(\mathbb{G})^{**}$:

$$q_C(\omega_{\varsigma_k}).$$

Then the sequence $(\mathbb{E}[q_C(\omega_{\varsigma_k})])_{k\geq 0}$ could be studied, and would be of conceptional interest if it converges to zero, for it would be a model of how measurement could allow classical objects to emerge from quantum objects. In the examples that have been studied here this phenomenon does not occur. For the Kac–Paljutkin quantum group, the nature of the magic unitary is that, certainly working with $M = \{x(1), x(2), x(3), x(4)\}$ a single measurement will see collapse to a classical random permutation or truly quantum permutation, and, certainly in the case of finite quantum permutation groups, classical respectively truly quantum permutations remain classical respectively truly quantum under measurement. In the case of $\widehat{S_3}$, starting with the Haar state $\varsigma_0 = h_{\widehat{S_3}}$, measured alternately with $x(1)$ and $x(3)$, it appears that the sequence $(\mathbb{E}[q_C(\varsigma_k)])_{k\geq 0}$ is constant, equal to 2/3. Let $(\varsigma_k)_{k\geq 0}$ be such a sequence. A quantum permutation ς_k has been conditioned on k measurements of $x(1)$ and $x(3)$, if, for example, $\varsigma_{\ell+2m}(j) \neq \varsigma_\ell(j)$, then there must be collapse to a truly quantum permutation. The only way that ς_k is truly quantum is if the measurements of $x(1)$ and $x(3)$ appear classical, so that

$$\varsigma_k = \cdots \widetilde{u_{33}^{\widehat{S_3}}}\widetilde{u_{11}^{\widehat{S_3}}}\widetilde{u_{33}^{\widehat{S_3}}}\widetilde{u_{11}^{\widehat{S_3}}}\varsigma_0 \text{ or } \widetilde{u_{43}^{\widehat{S_3}}}\widetilde{u_{21}^{\widehat{S_3}}}\widetilde{u_{43}^{\widehat{S_3}}}\widetilde{u_{21}^{\widehat{S_3}}}\varsigma_0.$$

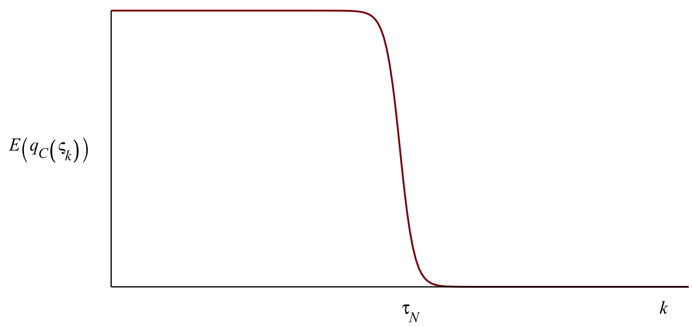
The phenomenon then is that because $(\widetilde{u_{33}^{\widehat{S_3}}}\widetilde{u_{11}^{\widehat{S_3}}})^k(\varsigma_0) \to \text{ev}_e$, once a quantum permutation has been observed e.g. fixing one and three a number of times, the probability that it fixes one/three gets larger, and if measurement keeps seeing one/three fixed, the sequence $q_C(\varsigma_k) \to 0$. It is seen that, because the only deterministic permutations in $\widehat{S_3}$ are e and $(12)(34)$ that

$$q_C(\widetilde{u_{34}^{\widehat{S_3}}}\widetilde{u_{11}^{\widehat{S_3}}}\varsigma_0) = 1 \text{ and } q_C(\widetilde{u_{33}^{\widehat{S_3}}}\widetilde{u_{21}^{\widehat{S_3}}}\varsigma_0) = 1.$$

This is rather the opposite to conjectured phenomenon of $\mathbb{E}[q_C(\varsigma_k)] \to 0$, and so rather bullish to conjecture that the following phenomenon might occur: for a 'natural' family of quantum permutation groups $C(\mathbb{G}_N)$ indexed by a parameter N, a family of initial quantum permutations $\varsigma_0^N \in \mathbb{G}_N$, measurements M_N chosen according to a fixed probability distribution ρ^N (or sequence of probability distributions $(\rho_k^N)_{k\geq 0}$), a *decoherence time* τ_N such that as $N \to \infty$ the *window* $w_N/\tau_N \to 0$, and for $c > 1$:

$$1 \underset{k<\tau_N - cw_N}{\approx} \mathbb{E}[q_C(\varsigma_k)] \underset{k>\tau_N + cw_N}{\approx} 0.$$

The idea would be that as things go macroscopic ($N \to \infty$), random measurements can 'suddenly' reveal, from a quantum object, a classical object. This of course is inspired by the cut-off phenomenon studied by Diaconis and coauthors in the classical case (e.g [23] and Freslon and coauthors in the compact quantum group case (e.g. [27]). There is a whole industry of work on alternating projections that might have applications here. See for example [2]



8.6. **Maximality of $S_N < S_N^+$.** As mentioned in Section 4.2, there is the following maximality result:

THEOREM 8.5. *For $N \leq 5$, there is no intermediate quantum permutation group $S_N < \mathbb{G} < S_N^+$* •

The result is conjectured to be true for $N > 5$ also. A strong way to interpret the conjecture is to say that all that has to be added to S_N to get the whole of S_N^+ is a truly quantum permutation. Alternatively, weaker, start with a genuinely quantum permutation group $\mathbb{G}$, and add all the deterministic permutations to get the whole of S_N^+. As was seen in Section 8.2, quantum permutations in quantum subgroups $\mathbb{G} \leq S_N^+$ are quantum permutations in S_N^+, and deterministic permutations can be used to generate new quantum permutations. Here the $u_{ij} \in C(S_N^+)$ and, in general without making it notationally explicit, all quantum permutations are assumed elements of S_N^+, e.g.

$$\varsigma_0(u_{ij}) = \varsigma_0' \circ \pi_{\mathbb{G}}(u_{ij}) = u_{ij}^{\mathbb{G}}(\varsigma_0'), \text{ and } \operatorname{ev}_\sigma(u_{ij}) = \operatorname{ev}_\sigma' \circ \pi_{S_N}(u_{ij}) = \mathbb{1}_{j\to i}(\sigma).$$

PROPOSITION 8.6. *Suppose that $\varsigma_0 \in \mathbb{G} < S_N^+$ is such that:*

$$\alpha := \mathbb{P}[(\varsigma_0(j_n) = i_n) \succ \cdots \succ (\varsigma_0(j_1) = i_1)] = |u_{i_nj_n} \cdots u_{i_1j_1}|^2(\varsigma_0) > 0,$$

then $\varsigma := \operatorname{ev}_{\sigma_2} \star \varsigma_0 \star \operatorname{ev}_{\sigma_1^{-1}}$ is such that for $\alpha > 0$:

$$\mathbb{P}[(\varsigma(\sigma_1(j_n)) = \sigma_2(i_n)) \succ \cdots \succ (\varsigma(\sigma_1(j_1)) = \sigma_2(i_1))] = |u_{\sigma_2(i_n)\sigma_1(j_n)} \cdots u_{\sigma_2(i_1)\sigma_1(j_1)}|^2(\varsigma) = \alpha,$$

Proof. Calculate $|u_{\sigma_2(i_n)\sigma_1(j_n)}\cdots u_{\sigma_2(i_1)\sigma_1(j_1)}|^2(\varsigma)$

$$\begin{aligned}
&= (\mathrm{ev}_{\sigma_2}\otimes\varsigma_0\otimes\mathrm{ev}_{\sigma_1^{-1}})\Delta^{(2)}(u_{\sigma_2(i_1)\sigma_1(j_1)}\cdots u_{\sigma_2(i_n)\sigma_1(j_n)}\cdots u_{\sigma_2(i_1)\sigma_1(j_1)})\\
&= (\mathrm{ev}_{\sigma_2}\otimes\varsigma_0\otimes\mathrm{ev}_{\sigma_1^{-1}})\Delta^{(2)}(u_{\sigma_2(i_1)\sigma_1(j_1)})\cdots\Delta^{(2)}(u_{\sigma_2(i_n)\sigma_1(j_n)})\cdots\Delta^{(2)}(u_{\sigma_2(i_1)\sigma_1(j_1)})\\
&= (\mathrm{ev}_{\sigma_2}\otimes\varsigma_0\otimes\mathrm{ev}_{\sigma_1^{-1}})\left(\sum_{k_1,k_2=1}^N u_{\sigma_2(i_1)k_1}\otimes u_{k_1k_2}\otimes u_{k_2\sigma_1(j_1)}\right)\times\\
&\cdots\left(\sum_{k_{2n-1},k_{2n}=1}^N u_{\sigma_2(i_n)k_{2n-1}}\otimes u_{k_{2n-1}k_{2n}}\otimes u_{k_{2n}\sigma_1(j_n)}\right)\times\\
&\cdots\left(\sum_{k_{4n-3},k_{4n-2}=1}^N u_{\sigma_2(i_1)k_{4n-3}}\otimes u_{k_{4n-3}k_{4n-2}}\otimes u_{k_{4n-2}\sigma_1(j_1)}\right)\\
&= \sum_{k_1,\dots,k_{4n-2}=1}^N \mathrm{ev}_{\sigma_2}(u_{\sigma_2(i_1)k_1}\cdots u_{\sigma_2(i_n)k_{2n-1}}\cdots u_{\sigma_2(i_1)k_{4n-3}})\times\\
&\varsigma_0(u_{k_1k_2}\cdots u_{k_{2n-1}k_{2n}}\cdots u_{k_{4n-3}k_{4n-2}})\,\mathrm{ev}_{\sigma_1^{-1}}(u_{k_2\sigma_1(j_1)}\cdots u_{k_{2n}\sigma_1(j_n)}\cdots u_{k_{4n-2}\sigma_1(j_1)})
\end{aligned}$$

The deterministic permutations are characters and moreover $\mathrm{ev}_\sigma(u_{ij}) = 1$ if and only if $\sigma(j) = i$, and this implies $k_1 = i_1, \dots, k_{2n-1} = i_n, \dots, k_{4n-3} = i_1$ and also $k_2 = j_1, \dots, k_{2n} = j_n, \dots, k_{4n-2} = j_1$ and so

$$|u_{\sigma_2(i_n)\sigma_1(j_n)}\cdots u_{\sigma_2(i_1)\sigma_1(j_1)}|^2(\varsigma) = \varsigma_0(u_{i_1j_1}\cdots u_{i_nj_n}\cdots u_{i_1j_1}) = \alpha > 0 \qquad \bullet$$

Consider $S_6 < S_6^+$ and $\mathfrak{G}_0 < S_6^+$ given by $u^{\mathfrak{G}_0<S_6^+} = \operatorname{diag}(u^{\mathfrak{G}_0}, \mathbb{1}_{\mathfrak{G}_0}, \mathbb{1}_{\mathfrak{G}_0})$. The quantum permutations in $\mathfrak{G}_0$ can be combined with deterministic permutation to get quantum permutations in S_6^+, for example, where ς_{e_5} is the vector state on $\mathfrak{G}_0$, for $\varsigma := \mathrm{ev}_{(14)}\,\varsigma_{e_5}\,\mathrm{ev}_{(16)}$

$$\mathbb{P}[(\varsigma(6) = 3) \succ (\varsigma(4) = 2) \succ (\varsigma(6) = 1)] = |u_{36}u_{24}u_{16}|^2(\varsigma) = u_{41}u_{24}u_{31}u_{24}u_{41}(\varsigma_{e_5}) = \frac{1}{8} > 0.$$

The convolution of quantum permutations in a subgroup $\mathbb{G} < S_N^+$ with deterministic permutations can do even more. For example, measurement of $\varsigma_{e_5} \in \mathfrak{G}_0$ can see $3 \mapsto 1$ and subsequently $3 \mapsto 2$, however the probability that it ever subsequently maps three to anything other than one or two is zero. However the quantum permutation $\varsigma := \mathrm{ev}_{(23)}\star\varsigma_{e_5}\star\mathrm{ev}_{(24)}\star\varsigma_{e_5}\star\mathrm{ev}_{(23)} \in S_4^+$ is such that:

$$\mathbb{P}[(\varsigma(1) = 4) \succ (\varsigma(2) = 2) \succ (\varsigma(1) = 3) \succ (\varsigma(2) = 2) \succ (\varsigma(1) = 1)] > 0.$$

To show this is to show that $|u_{41}u_{22}u_{31}u_{22}u_{11}|^2(\varsigma)>0$.

$$\begin{aligned}|u_{41}u_{22}u_{31}u_{22}u_{11}|^2(\varsigma) &= (\mathrm{ev}_{(23)}\otimes\varsigma_{e_5}\otimes\mathrm{ev}_{(24)}\otimes\varsigma_{e_5}\otimes\mathrm{ev}_{(23)})\Delta^{(4)}(u_{11}u_{22}u_{31}u_{22}u_{41}u_{22}u_{31}u_{22}u_{11})\\ &= \sum_{k_1,\dots,k_{36}=1}^{4}\mathrm{ev}_{(23)}(u_{1k_1}u_{2k_2}u_{3k_3}u_{2k_4}u_{4k_5}u_{2k_6}u_{3k_7}u_{2k_8}u_{1k_9})\\ &\times\varsigma_{e_5}(u_{k_1k_{10}}\cdots u_{k_9k_{18}})\,\mathrm{ev}_{(24)}(u_{k_{10}k_{19}}\cdots u_{k_{18}k_{27}})\varsigma_{e_5}(u_{k_{19}k_{28}}\cdots u_{k_{27}k_{36}})\\ &\times\mathrm{ev}_{(23)}(u_{k_{28}1}u_{k_{29}2}u_{k_{30}1}u_{k_{31}2}u_{k_{32}1}u_{k_{33}2}u_{k_{34}1}u_{k_{35}2}u_{k_{36}1})\\ &= \sum_{k_{10},\dots,k_{27}}^{4}\varsigma_{e_5}(u_{1k_{10}}u_{3k_{11}}u_{2k_{12}}u_{3k_{13}}u_{4k_{14}}u_{3k_{15}}u_{2k_{16}}u_{3k_{17}}u_{1k_{18}})\times\\ &\mathrm{ev}_{(24)}(u_{k_{10}k_{19}}\cdots u_{k_{18}k_{27}})\varsigma_{e_5}(u_{k_{19}1}u_{k_{20}3}u_{k_{21}1}u_{k_{22}3}u_{k_{23}1}u_{k_{24}3}u_{k_{25}1}u_{k_{26}3}u_{k_{27}1})\end{aligned}$$

Now consider that the ordered pairs

$$(k_{10},k_{19}),\dots,(k_{18},k_{27})\overset{!}{\in}\{(1,1),(2,4),(3,3),(4,2)\}.$$

Recall that ς_{e_5} is pre-composed with $\pi_{\mathfrak{G}_0}:C(S_4^+)\to F(\mathfrak{G}_0)$. Considering the algebra structure (6.6) on $F(\mathfrak{G}_0)$, the magic unitary $u^{\mathfrak{G}_0}$ (6.7), and that ς_{e_5} is zero on the one-dimensional factors, there is only one choice for the ordered pairs $(k_{10},k_{19}),\dots,(k_{18},k_{27})$, namely alternating $(3,3)$ and $(1,1)$, except for $(k_{14},k_{23})=(2,4)$. This leaves the strictly positive:

$$|u_{41}u_{22}u_{31}u_{22}u_{11}|^2(\varsigma)=\varsigma_{e_5}(u_{13}u_{31}u_{23}u_{31}u_{42}u_{31}u_{23}u_{31}u_{13})\varsigma_{e_5}(u_{31}u_{13}u_{31}u_{13}u_{41}u_{13}u_{31}u_{13}u_{31}).$$

How much can such calculations be pushed? Can a result like Proposition 8.6 be written down for $\mathrm{ev}_{\sigma_3}\star\varsigma\star\mathrm{ev}_{\sigma_2}\star\varsigma\star\mathrm{ev}_{\sigma_1}$? Can it be shown that for all non-zero monomials in $u_{ij}\in C(S_N^+)$, a quantum permutation $\varsigma\in S_N^+$ can be constructed from $\mathfrak{G}_0$ and S_N such that $f(\varsigma)>0$?

There is a weaker conjecture/question that can be made here, along with some speculation.

Question. For any N, is there an intermediate *quasi-subgroup* $S_N\subsetneq\mathbb{S}\subsetneq S_N^+$?

The kind of objects that might be studied here are, for example, the random walks defined by $h_{S_N}\star h_{\mathbb{G}}\star h_{S_N}$, for example at $\mathbb{G}=\mathfrak{G}_0$ or $\widehat{S_N}$, or even for the quantum permutation $\varsigma_{e_5}\in\mathfrak{G}_0<S_N^+$ what happens with the random walk defined by $h_{S_N}\star\varsigma_{e_5}\star h_{S_N}$? The quantum permutation ς_{e_5}, one could speculate, through the uncertainty phenomenon mentioned in Section 6.4, might be enough to generate with h_{S_N} all of S_N^+ in the sense that it could be conjectured that:

$$(h_{S_N}\star\varsigma_{e_5}\star h_{S_N})^{\star k}\to h_{S_N^+}.$$

It is possible to build a state like ς_{e_5} in any quantum permutation group $\mathbb{G}<S_N^+$. Take non-commuting $u^{\mathbb{G}}_{i_1,j_1},u^{\mathbb{G}}_{i_2j_2}\in C(\mathbb{G})$. Represent $\pi_{\mathrm{GNS}}(C(\mathbb{G}))\subset B(\mathsf{H})$. Take $\xi\in\mathrm{ran}(\pi_{\mathrm{GNS}}(u^{\mathbb{G}}_{i_1j_1}))$. That $u^{\mathbb{G}}_{i_1,j_1},u^{\mathbb{G}}_{i_2j_2}$ do not commute implies that ξ can be chosen to neither be in $\mathrm{ran}(\pi_{\mathrm{GNS}}(u^{\mathbb{G}}_{i_2j_2}))$ nor its orthogonal complement. This implies that:

$$|u_{i_1j_1}|^2(\varsigma_\xi)=1\text{ and }0<|u_{i_2j_2}u_{i_1j_1}|^2(\varsigma_\xi)<1.$$

The use of alternating projections theory might be able to show that for $p>1$:

$$0<|u_{i_2j_2}u_{i_1j_1}|^{2p}(\varsigma)<1.$$

That it is less than one implies that there exists $i_3\neq i_1$ such that:

$$0<|u_{i_3j_1}(u_{i_2j_2}u_{i_1j_1})^p|^2<1,$$

and if it is the case that $(h_{S_N}\star\varsigma_{e_5}\star h_{S_N})^{\star k}\to h_{S_N^+}$, it might be possible to show that $\varsigma_\xi\in\mathbb{G}<S_N^+$ shares the same properties of ς_{e_5} used to prove $(h_{S_N}\star\varsigma_{e_5}\star h_{S_N})^{\star k}\to h_{S_N^+}$, thus giving $(h_{S_N}\star\varsigma_\xi\star h_{S_N})^{\star k}\to h_{S_N^+}$, and proving the maximality of $S_N<S_N^+$. One barrier to natural attacks via such a method is the non-transitivity of *orbitals*.

8.7. **Orbits and Orbitals.** The study of orbits and orbitals was initiated by Lupini, Mančinska, and Roberson [38]. Banica and Freslon [12] independently studied orbits. In this section one-orbitals (or orbits), two-orbitals, and three-orbitals are studied, in the language of quantum permutations, and a new (conjectured) counter-example to the transitivity of the three-orbital relation is given.

For every positive element f in a C*-algebra $C(\mathbb{X})$, there is a pure state ρ such that $\|f\|=\rho(f)$ (Th. 5.1.11, [45]). This implies that in the case of $\mathbb{G}<S_N^+$, for any non-zero $u_{ij}^{\mathbb{G}}\in C(\mathbb{G})$, there is a (pure) quantum permutation $\varsigma\in\mathbb{G}$ such that $\varsigma(j)=i$ with probability one. Working in either the algebraic, $\mathcal{O}(\mathbb{G})$, or universal, $C_u(\mathbb{G})$, setting, define the *orbit relation*, or *one-orbital relation*, on N by $i\sim_1 j$ if $u_{ij}^{\mathbb{G}}\neq 0$. The identity $\varepsilon\in\mathbb{G}$ is a quantum permutation that fixes all points, so $u_{ii}^{\mathbb{G}}(\varepsilon)=1$, therefore $u_{ii}^{\mathbb{G}}\neq 0$, and thus $\sim_1$ is reflexive. Suppose that $i\sim_1 j$ so $u_{ij}^{\mathbb{G}}\neq 0$, and $\varsigma\in\mathbb{G}$ a (pure) quantum permutation such that $u_{ij}^{\mathbb{G}}(\varsigma)=\|u_{ij}^{\mathbb{G}}\|=1$, then $j\sim_1 i$ by the reverse of ς, $\varsigma^{-1}:=\varsigma\circ S$, as

$$\mathbb{P}[\varsigma^{-1}(i)=j]=u_{ji}^{\mathbb{G}}(\varsigma^{-1})=S(u_{ji}^{\mathbb{G}})(\varsigma)=u_{ij}^{\mathbb{G}}(\varsigma)=1.$$

Let ς_2 and ς_1 be such that $u_{i\ell}^{\mathbb{G}}(\varsigma_2)=\|u_{i\ell}^{\mathbb{G}}\|=1$ and $u_{\ell j}^{\mathbb{G}}(\varsigma_1)=\|u_{\ell j}^{\mathbb{G}}\|=1$. Consider the quantum permutation $\varsigma_2\star\varsigma_1\in\mathbb{G}$:

$$u_{ij}^{\mathbb{G}}(\varsigma_2\star\varsigma_1)=\Delta(u_{ij}^{\mathbb{G}})(\varsigma_2\otimes\varsigma_1)=\sum_{k=1}^N u_{ik}^{\mathbb{G}}(\varsigma_2)u_{kj}^{\mathbb{G}}(\varsigma_2).$$

However if $u_{i\ell}^{\mathbb{G}}(\varsigma_2)=1$ and $u_{\ell j}^{\mathbb{G}}(\varsigma_1)=1$, all of the terms with $k\neq\ell$ are zero, so

$$u_{ij}^{\mathbb{G}}(\varsigma_2\star\varsigma_1)=u_{i\ell}^{\mathbb{G}}(\varsigma_2)u_{\ell j}^{\mathbb{G}}(\varsigma_2)=1,$$

and so $u_{ij}^{\mathbb{G}}\neq 0$, so that $\sim_1$ is an equivalence relation on N. This can also be seen by considering $\Phi(\varsigma_2\star\varsigma_1)=\Phi(\varsigma_2)\Phi(\varsigma_1)$.

Higher order orbitals may also be defined as relations on N^k. Say that:

$$(i_m,\cdots,i_1)\sim_m(j_m,\dots j_1)\Leftrightarrow u_{i_mj_m}^{\mathbb{G}}\cdots u_{i_1j_1}^{\mathbb{G}}\neq 0.$$

Similarly to the above, with $\varepsilon\in\mathbb{G}$, and $\mathbb{G}$ closed under reversal, $\sim_m$ is reflexive and symmetric.

PROPOSITION 8.7. *The two-orbital, $\sim_2$, is transitive (Lemma 3.4,* [38]*).*

Proof. Note that for projections $p,q\in C(\mathbb{G})$, the C*-identity gives:

$$\|pq\|^2=\|(pq)^*pq\|=\|qpq\|,$$

and so $pq=0$ exactly when $qpq=0$. Suppose that $u_{i_2\ell_2}^{\mathbb{G}}u_{i_1\ell_1}^{\mathbb{G}}\neq 0$ and $u_{\ell_2j_2}^{\mathbb{G}}u_{\ell_1j_1}^{\mathbb{G}}\neq 0$ so that $u_{i_1\ell_1}^{\mathbb{G}}u_{i_2\ell_2}^{\mathbb{G}}u_{i_1\ell_1}^{\mathbb{G}}\neq 0$ and $u_{\ell_1j_1}^{\mathbb{G}}u_{\ell_2j_2}^{\mathbb{G}}u_{\ell_1j_1}^{\mathbb{G}}\neq 0$, so that $(i_2,i_1)\sim_2(\ell_2,\ell_1)$ and $(\ell_2,\ell_1)\sim_2(j_2,j_1)$. Let ς_2', ς_1' be such that $u_{i_1\ell_1}^{\mathbb{G}}u_{i_2\ell_2}^{\mathbb{G}}u_{i_1\ell_1}^{\mathbb{G}}(\varsigma_2')=\|u_{i_1\ell_1}^{\mathbb{G}}u_{i_2\ell_2}^{\mathbb{G}}u_{i_1\ell_1}^{\mathbb{G}}\|\neq 0$, and $u_{\ell_1j_1}^{\mathbb{G}}u_{\ell_2j_2}^{\mathbb{G}}u_{\ell_1j_1}^{\mathbb{G}}(\varsigma_1')=\|u_{\ell_1j_1}^{\mathbb{G}}u_{\ell_2j_2}^{\mathbb{G}}u_{\ell_1j_1}^{\mathbb{G}}\|\neq 0$. This means that ς_2' maps ℓ_2 to i_2 after mapping ℓ_1 to i_1 with non-zero probability (and similar for ς_1'). Note in particular, by e.g. (5.2), that $u^{\mathbb{G}}_{i_1\ell_1}(\varsigma_2'), u^{\mathbb{G}}_{\ell_1 j_1}(\varsigma_1') \neq 0$. It turns out that conditioning ς_2' by $\varsigma_2'(\ell_1) = i_1$, to $\widetilde{u^{\mathbb{G}}_{i_1\ell_1}}\varsigma_2'$ gives a quantum permutation that maps ℓ_1 to i_1 with probability one, but the property of ς_2' that it can map ℓ_2 to i_2 after mapping ℓ_1 to i_1 is not lost by this conditioning, and a Bayes-type rule emerges:

$$\begin{aligned}\mathbb{P}[\varsigma_2'(\ell_2) = i_2 \,|\, \varsigma_2'(\ell_1) = i_1] &= \widetilde{u^{\mathbb{G}}_{i_1\ell_1}}\varsigma_2'(u^{\mathbb{G}}_{i_1\ell_1}u^{\mathbb{G}}_{i_2\ell_2}u^{\mathbb{G}}_{i_1\ell_1}) \\ &= \frac{\varsigma_2'(u^{\mathbb{G}}_{i_1\ell_1}u^{\mathbb{G}}_{i_1\ell_1}u^{\mathbb{G}}_{i_2\ell_2}u^{\mathbb{G}}_{i_1\ell_1}u^{\mathbb{G}}_{i_1\ell_1})}{u^{\mathbb{G}}_{i_1\ell_1}(\varsigma_2')} \\ &= \frac{\varsigma_2'(u^{\mathbb{G}}_{i_1\ell_1}u^{\mathbb{G}}_{i_2\ell_2}u^{\mathbb{G}}_{i_1\ell_1})}{u^{\mathbb{G}}_{i_1\ell_1}(\varsigma_2')} \\ &= \frac{\mathbb{P}[(\varsigma_2'(\ell_2) = i_2) \succ (\varsigma_2'(\ell_1) = i_1)]}{\mathbb{P}[\varsigma_2'(\ell_1) = i_1]} \\ &\geq \mathbb{P}[(\varsigma_2'(\ell_2) = i_2) \succ (\varsigma_2'(\ell_1) = i_1)] \\ &= u^{\mathbb{G}}_{i_1\ell_1}u^{\mathbb{G}}_{i_2\ell_2}u^{\mathbb{G}}_{i_1\ell_1}(\varsigma_2') = \|u^{\mathbb{G}}_{i_1\ell_1}u^{\mathbb{G}}_{i_2\ell_2}u^{\mathbb{G}}_{i_1\ell_1}\| > 0.\end{aligned} \tag{8.5}$$

Define $\varsigma_2 := \widetilde{u^{\mathbb{G}}_{i_1\ell_1}}\varsigma_2'$ and $\varsigma_1 := \widetilde{u^{\mathbb{G}}_{\ell_1 j_1}}\varsigma_1'$ so that by (8.5) $u^{\mathbb{G}}_{i_2\ell_2}(\varsigma_2) > 0$ and similarly $u^{\mathbb{G}}_{\ell_2 j_2}(\varsigma_1) > 0$.

$$\begin{aligned}u^{\mathbb{G}}_{i_1j_1}u^{\mathbb{G}}_{i_2j_2}u^{\mathbb{G}}_{i_1j_1}(\varsigma_2\star\varsigma_1) &= \Delta(u^{\mathbb{G}}_{i_1j_1}u^{\mathbb{G}}_{i_2j_2}u^{\mathbb{G}}_{i_1j_1})(\varsigma_2\star\varsigma_1) \\ &= \sum_{k,k_1,k_2=1}^{N} u^{\mathbb{G}}_{i_1k_1}u^{\mathbb{G}}_{i_2k}u^{\mathbb{G}}_{i_1k_2}(\varsigma_2)u^{\mathbb{G}}_{k_1j_1}u^{\mathbb{G}}_{kj_2}u^{\mathbb{G}}_{k_2j_1}(\varsigma_1) \\ &= \sum_{k,k_1,k_2=1}^{N} \frac{\varsigma_2'(u^{\mathbb{G}}_{i_1\ell_1}u^{\mathbb{G}}_{i_1k_1}u^{\mathbb{G}}_{i_2k}u^{\mathbb{G}}_{i_1k_2}u^{\mathbb{G}}_{i_1\ell_1})}{u^{\mathbb{G}}_{i_1\ell_1}(\varsigma_2')}\frac{\varsigma_1'(u^{\mathbb{G}}_{\ell_1j_1}u^{\mathbb{G}}_{k_1j_1}u^{\mathbb{G}}_{kj_2}u^{\mathbb{G}}_{k_2j_1}u^{\mathbb{G}}_{\ell_1j_1})}{u^{\mathbb{G}}_{\ell_1j_1}(\varsigma_1')}\end{aligned}$$

All of the terms with $k_1 \neq \ell_1$ or $k_2 \neq \ell_1$ are zero:

$$\begin{aligned}\Rightarrow u^{\mathbb{G}}_{i_1j_1}u^{\mathbb{G}}_{i_2j_2}u^{\mathbb{G}}_{i_1j_1}(\varsigma_2\star\varsigma_1) &= \sum_{k=1}^{N} u^{\mathbb{G}}_{i_2k}(\varsigma_2)u^{\mathbb{G}}_{kj_2} \\ &= \underbrace{u^{\mathbb{G}}_{i_2\ell_2}(\varsigma_2)u^{\mathbb{G}}_{\ell_2j_2}(\varsigma_1)}_{>0} + \underbrace{\sum_{k\neq\ell_2} u^{\mathbb{G}}_{i_2k}(\varsigma_2)u^{\mathbb{G}}_{kj_2}(\varsigma_1)}_{\geq 0} > 0.\end{aligned}$$

Therefore $u^{\mathbb{G}}_{i_1j_1}u^{\mathbb{G}}_{i_2j_2}u^{\mathbb{G}}_{i_1j_1} \neq 0$, so that $u^{\mathbb{G}}_{i_2j_2}u^{\mathbb{G}}_{i_1j_1} \neq 0$ so that $(i_2, i_1) \sim_2 (j_2, j_1)$, that is $\sim_2$ is transitive (and thus an equivalence relation on $N \times N$ •

Lupini, Mančinska, and Roberson [38] (as well as Banica [5]) expressed the belief that $\sim_3$ is not transitive in general. The algebra of functions on a finite quantum group, as a finite dimensional C*-algebra, is a direct sum of A, the direct sum of the one dimensional factors, and B, the direct sum of the higher dimensional factors. Counterexamples to $\sim_3$ transitive can occur in the finite case when for the elements along the diagonal $u^{\mathbb{G}}_{ii}F(\mathbb{G}) \subset A$; that is if a quantum permutation $\varsigma \in \mathbb{G}$ is such that $\mathbb{P}[\varsigma(i) = i] = u^{\mathbb{G}}_{ii}(\varsigma) > 0$, $\widetilde{u^{\mathbb{G}}_{ii}}\varsigma$ is a random permutation that cannot exhibit non-classical behaviour. For example, if $\mathbb{P}[\varsigma(i_1) = i_1] > 0$,

$\widetilde{u^G_{i_1i_1}\varsigma}$ is a random permutation, and so for all $\varsigma \in \mathbb{G}$

$$\mathbb{P}[(\varsigma(i_1) \neq i_1) \succ (\varsigma(i_2) = i_2) \succ (\varsigma(i_1) = i_1)] = 0,$$

which implies that for any $j_3 \neq i_1$, $u^{\mathbb{G}}_{j_3i_1}u^{\mathbb{G}}_{i_2i_2}u^{\mathbb{G}}_{i_1i_1} = 0$. Therefore to find:

$$u^{\mathbb{G}}_{i_1j_3}u^{\mathbb{G}}_{i_2j_2}u^{\mathbb{G}}_{i_1j_1} \neq 0, \text{ and } u^{\mathbb{G}}_{j_3i_1}u^{\mathbb{G}}_{j_2i_2}u^{\mathbb{G}}_{j_1i_1} \neq 0,$$

yields the non-transitivity of $\sim_3$.

This phenomenon occurs in both the Kac-Paljutkin quantum group $\mathfrak{G}_0$ and also the dual $\widehat{Q}$ of the quaternions. In the case of $\mathfrak{G}_0$, the uncertainty phenomenon implies that the quantum permutation given by $\varsigma_2 := \widetilde{u^{\mathfrak{G}_0}_{41}\varsigma_{e_5}}$ satisfies:

$$\mathbb{P}[(\varsigma(1) = 3) \succ (\varsigma(3) = 1) \succ (\varsigma(1) = 4)] = \frac{1}{4} \Rightarrow u^{\mathfrak{G}_0}_{31}u^{\mathfrak{G}_0}_{13}u^{\mathfrak{G}_0}_{41} \neq 0 \Rightarrow (3,1,4) \sim_3 (1,3,1).$$

Similarly $\varsigma_1 := \widetilde{u^{\mathfrak{G}_0}_{14}\varsigma_{e_5}}$ shows that $u^{\mathfrak{G}_0}_{14}u^{\mathfrak{G}_0}_{31}u^{\mathfrak{G}_0}_{14} \neq 0$, and so $(1,3,1) \sim_3 (4,1,4)$. For transitivity, it would have to be the case that $(3,1,4) \sim_3 (4,1,4)$, that is $u^{\mathfrak{G}_0}_{34}u^{\mathfrak{G}_0}_{11}u^{\mathfrak{G}_0}_{44} \neq 0$, but as for all $\varsigma' \in \mathfrak{G}_0$ such that $\mathbb{P}[\varsigma'(4) = 4] > 0$, $\varsigma := \widetilde{u^{\mathfrak{G}_0}_{44}\varsigma'}$ is a random permutation:

$$\mathbb{P}[(\varsigma(4) = 3) \succ (\varsigma(1) = 1) \succ (\varsigma(4) = 4)] = 0 \Rightarrow u^{\mathfrak{G}_0}_{34}u^{\mathfrak{G}_0}_{11}u^{\mathfrak{G}_0}_{44} = 0,$$

and so $\sim_3$ is not transitive for $\mathfrak{G}_0 < S_4^+$.

A counterexample for duals occurs for the quaternion group. Let $\widehat{Q} < S_8^+$:

$$u^{\widehat{Q}} = \begin{pmatrix} u^{\widehat{\langle j\rangle}} & 0 \\ 0 & u^{\widehat{\langle k\rangle}} \end{pmatrix}.$$

Both $u^{\widehat{Q}}_{11}$ and $u^{\widehat{Q}}_{55}$ are orthogonal projection onto direct sums of two one dimensional spaces (respectively the one dimensional factors associated with $\{e,(57)(68)\} \subset G_{\widehat{Q}}$, and $\{e,(13)(24)\} \subset G_{\widehat{Q}}$). Direct calculation shows that $u^{\widehat{Q}}_{78}u^{\widehat{Q}}_{12}u^{\widehat{Q}}_{58} \neq 0$ so $(7,1,2) \sim_3 (8,2,8)$, and $u^{\widehat{Q}}_{85}u^{\widehat{Q}}_{21}u^{\widehat{Q}}_{85} \neq 0$ so $(8,2,8) \sim_3 (5,1,5)$, but $u^{\widehat{Q}}_{75}u^{\widehat{Q}}_{11}u^{\widehat{Q}}_{55} = 0$ as conditioning by $u^{\widehat{Q}}_{55}$ gives a random permutation.

As a final comment, one can imagine how much of this kind of thinking might help intuitions for compact quantum groups which are not quantum permutation groups. For example, can something similar be done for $\text{QAut}(A)$ [53], the quantum symmetry group of the 'finite quantum space' given by a finite dimensional C*-algebra A? Can something be done for the free orthogonal group O_N^+ together with the free real sphere, $S^{N+1}_{\mathbb{R},+}$, almost certainly with the help of the von Neumann algebras $L^\infty(O_N^+)$ and $L^\infty(S^{N+1}_{\mathbb{R},+})$?

Acknowledgement. Given that this is such a non-conventional piece of academic writing, an acknowledgement might be read as a call for endorsement rather than a genuine thank you. There are a number of quantum group theorists I have been in correspondence with over the past year, and I think all of them helped me with this piece to greater and lessor extents, and it certainly could not have come to fruition without their help and encouragement.

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