MATH6055 — Maths for Computer Science

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0.1 Introduction

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Module Description
Mathematics is an important component of Computer Science. This module offers a first introduction to some of the principles that computer scientists will use and apply to solving everyday tasks and introduces students to sets, relations, combinatorial graphs, functions and recursion.

Exercises
There are many ways to learn maths. Two methods which aren’t going to work are

1. reading your notes and hoping it will all sink in
2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.

There are exercises in the notes for your consumption. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be left as an exercise. How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

Starred exercises are optional: either they are harder than examinable or not examinable.

Mathematics Stack Exchange
If you find yourself stuck and for some reason feel unable to ask me the question you could do worse than go to the excellent site math.stackexchange.com. If you are nice and polite, and show due deference to the principles (https://math.stackexchange.com/tour) you will find that your questions are answered promptly. In general, it is good for asking theoretical questions but if you make an effort (i.e. write down what you have tried), people will also help you with specific questions about exercises.
Here are some examples:

**Figure 1:** A Chapter 1 question, someone hoping to clear up their confusion about ‘ordered triples’ (p. 37). [https://tinyurl.com/yy7s7hpr](https://tinyurl.com/yy7s7hpr)

**Figure 2:** A Chapter 2 question, someone is looking for a way to explain ‘functions’ (p. 76). [https://tinyurl.com/y3ymkd5g](https://tinyurl.com/y3ymkd5g)
Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6055, they will not detail all there is to know. Further references are to be found in the library. Good references include:


The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isn’t too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for further maths modules.
0.2 Motivation

Some food for thought.

1. An answer to the question *Is mathematics necessary for programming?*

To answer your question as it was posed I would have to say, “No, mathematics is not necessary for programming”. However, as other people have suggested in this thread, I believe there is a correlation between understanding mathematics and being able to “think algorithmically”. That is, to be able to think abstractly about quantity, processes, relationships and proof.

I started programming when I was about 9 years old and it would be a stretch to say I had learnt much mathematics by that stage. However, with a bit of effort I was able to understand variables, for loops, goto statements (forgive me, I was Vic 20 BASIC and I hadn’t read any Dijkstra yet) and basic co-ordinate geometry to put graphics on the screen.

I eventually went on to complete an honours degree in Pure Mathematics with a minor in Computer Science. Although I focused mainly on analysis, I also studied quite a bit of discrete maths, number theory, logic and computability theory. Apart from being able to apply a few ideas from statistics, probability theory, vector analysis and linear algebra to programming, there was little maths I studied that was directly applicable to my programming during my undergraduate degree and the commercial and research programming I did afterwards.

However, I strongly believe the formal methods of thinking that mathematics demands - careful reasoning, searching for counter-examples, building axiomatic foundations, spotting connections between concepts - has been a tremendous help when I have tackled large and complex programming projects.

Consider the way athletes train for their sport. For example, footballers no doubt spend much of their training time on basic football skills. However, to improve their general fitness they might also spend time at the gym on bicycle or rowing machines, doing weights, etc.

Studying mathematics can be likened to weight-training or cross-training to improve your mental strength and stamina for programming. It is absolutely essential that you practice your basic programming skills but studying mathematics is an incredible mental work-out that improves your core analytic ability.
2. An answer to the question *Do you have to be good at math to be a good programmer?*

I’m going against the grain and saying yes, you need a math mindset. Most people think of math as doing arithmetic or memorizing arcane formulas. This is like asking if you need perfect spelling or an extraordinary vocabulary to be a good writer.

Writing is about communication, and math/programming is about the process of clear, logical thinking (in a way that you can’t make mistakes; the equation doesn’t balance, or the program doesn’t compile). Specifically, that logical thinking manifests in:

(a) Ability to estimate/understand differences between numbers:
   \( \mathcal{O}(n^2) \) vs \( \mathcal{O}(\ln(n)) \), intuitive sense of KB vs MB vs GB, how slow disk is compared to RAM. If you don’t realize how tiny a KB is compared to a GB you’ll be wasting time optimizing things that don’t matter.

(b) Functions/functional programming (is it any coincidence that the equation \( f(x) = x^2 \) is so similar to how you’d write that method? The words “algorithm” and “function” were around in the math world far before the first computer was born.

(c) Basic algebra to create & reorder your own equations, take averages, basic stats.

So, I’ll say you need a math mindset, being able to construct & manipulate mental models of what your program is doing, rather than a collection of facts & theorems. Certain fields like graphics or databases will have certain facts you need also, but to me that’s not the essence of being “good at math”.

Chapter 1

Sets and Relations

*No one shall expel us from the paradise that Cantor has created for us.*

David Hilbert

1.0.1 Number

A *set* is a well-defined collection of distinct objects. The objects can be referred to as members/elements. We will frequently refer to the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. We will introduce them here. What are they? A *natural number* is an ordinary counting number 1, 2, 3,.... Here the dots signify that this list goes on forever.

The set or collection of all natural numbers \{1, 2, 3, \ldots\} we call $\mathbb{N}$ for short:

Perhaps if we want to include zero we might write $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. We will use the following notation to signify the sentence “$n$ is a natural number”:

said “$n$ in $\mathbb{N}$” or “$n$ is an element of $\mathbb{N}$”. The symbol ‘$\in$’ means ‘in’ or ‘is an element of’.

The set of *integers* or *whole numbers* is the set of natural numbers together with 0 and all the negative numbers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. Once again the dots signify that this list continues indefinitely in both directions. We denote the set of “*integer* $\mathbb{Z}$” by $\mathbb{Z}$: the $\mathbb{Z}$ stands for *Zahlen* — the German for ‘numbers’.
The set of fractions (or Quotients) is the set of all ‘numbers’ of the form $n \neq 0$ is shorthand for “$n$ is not zero”. Examples include $1/2, -4/3, 211/24$. We call the ‘top’ number the numerator and the ‘bottom’ number the denominator. We do not let the ‘bottom’ equal zero, because this would be division by zero, which is undefined because $\times 0$ is not 1-1 and so not invertible:

A real number is any number that can be written as a decimal. Examples:

- $1 = 1.0$
- $3 = 3.0$
- $-5 = -5.0$
- $1/2 = 0.5$
- $2/3 = 0.6666...$
- $\sqrt{2} = 1.41421356...$
- $\pi = 3.14159265...$

In the context of this module a real number is just any number at all be it a natural number, negative number, fraction, square root, etc. For those of you looking to jump the gun complex numbers are not real numbers...

Again we write $x \in \mathbb{R}$ to signify ”$x$ is a real number”. 
1.1 Set Notation and Venn Diagrams

We use Venn diagrams to visually represent sets and possible relationships between them. Usually a set is represented by a circle inside a rectangular universal set, $U$:

Allowing the existence of a universal set leads to paradoxes so rather than thinking of it as some kind of universal set I am going to call it an ambient set, and what it is should be clear from context. For example, if I am talking about people, the ambient set $U$ will be the set of all people; or if I am talking about numbers the ambient set might be the set of whole numbers or the set of real numbers as appropriate.

A set may be described by listing its members, for example

$$C = \{\text{Algebra, Sets, Functions I, Graphs, Functions II}\},$$

is the set of chapters in this manual. Alternatively, a set may be described by specifying a property which only the members of the set have:

$$P = \{n \neq 1 \mid n \text{ is a natural number with no factors other than one and itself}\}; \text{ i.e.}$$

$$= \{2, 3, 5, 7, 11, 13, 17, \ldots \} = \text{‘the set of prime numbers’}$$

read

$P$ is the set of $n$ not equal to one, such that $n$ is a natural number, with no factors other than one and itself.

Sometimes to be brief we might just write:

$$P = \{n \text{ is a natural number with no factors other than one and itself}\},$$

or even

$$P = \{\text{natural numbers with no factors other than one and itself}\}.$$ 

The cardinality (measure of size) of a finite set $X$ is the number of elements it contains and is denoted by $|X|$.

A set $B$ is a subset of a set $A$, written $B \subset A$, if every member of $B$ is a member of $A$. For example, given:

$$C = \{\text{people born in Cork}\}, \text{ and}$$

$$M = \{\text{people born in Munster}\},$$

$C$ is a subset of $M$, $C \subset M$.

\[1\text{together with something called the Axiom of Comprehension} \]
In terms of a Venn diagram:

A set can have no members in which case we call it the *empty set*, denoted by \( \emptyset \) or \{\}. Examples of the empty set

1. the set of CIT students that are older than 200 years old;
2. the set of whole numbers *strictly* between one and two;
3. where \( C \) is the set of those born in Cork, and \( W \) is the set of counties containing the letter, ‘w’, \( C \cap W \):

Given a set \( X \), the set of subsets of \( X \) is also a set (whose elements are sets) called the *power set* of \( X \), denoted \( \mathcal{P}(X) \). For example, the power set of \( X = \{a, b, c\} \) is given by:

In general, the cardinality of \( \mathcal{P}(X) \) is given by \( |\mathcal{P}(X)| = 2^{|X|} \).

One way to think about subsets is as follows. Where \( X \) and \( Y \) are sets, a subset \( X \subseteq Y \) corresponds to a choice of elements from \( Y \). For example, where \( Y = \{1, 2, 3, 4\} \), a subset \( X \) is formed by answering the questions

For example, the subset \( \{1, 4\} \subset Y \) corresponds to choosing one and four but not two and three. If we choose all of the elements of \( Y \) we have the full subset and so, as it consists of a choice of elements of \( Y \) — namely all of them — we have that

Now what if we choose *none* of the elements of \( Y \)? This is a choice of elements and so is a subset of \( Y \). It is of course the empty set and in this sense we have

So a subset corresponds to a choice of elements from \( Y \), choosing none is a choice, and therefore
the empty set is always a subset.
When we list all possible subsets of $X$ — all possible choices from $X$ — this is the power set, $\mathcal{P}(S)$.

**Examples**

1. List the elements of the following sets:

   (a) The set of vowels

   (b) $\{x \in \mathbb{N} : 10 \leq x \leq 20 \text{ and } x \text{ is divisible by } 3\}$

**Solution**:

(a) This is

(b) We say that $x - y$ is divisible by 3 if

$$\frac{x - y}{3} \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$  

   This is

2. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8\}$.

   (a) Write down $\overline{B}$.

   (b) Carefully find $A \cup (B \cap \overline{C})$.

   (c) Find, or otherwise write down, $A \cup (\overline{B \cup \overline{C}})$.

**Solution**:

(a) We carefully write:

(b) Find first $\overline{C}$:

$$\overline{B} \cap \overline{C}$$:

Finally:

(c) We can find carefully $B \cup C$ then $\overline{B \cup C}$ and $A \cup (\overline{B \cup C})$. Alternatively note that $\overline{B \cup C} = \overline{B} \cap \overline{C}$ (why?) and so:

3. Let $S = \{x, y, z\}$

   (a) List the elements of $\mathcal{P}(S)$, i.e. list all the subsets of $S$

   (b) Hence, or otherwise, write down the value of $|\mathcal{P}(S)|$. 

Solution:

(a) We have

(b) By counting the answer is eight.

Also

\[ |\mathcal{P}(S)| = 2^{|S|} = 2^3 = 8. \]

1.1.1 Subsets and Binary Strings

As we will learn a little later when we talk about truth tables, a 1 indicates that an element is an element of a set, while a 0 indicates that an element is not an element of a set. Using this idea we can make a connection between the subsets of an ordered set \( S \) and binary strings of length \( |S| \). For example, suppose \( S = \{a, b, c, d\} \), and consider the subset \( A = \{b, d\} \subset S \) \((A \in \mathcal{P}(S))\). The subset \( A \) corresponds to the binary string 0101 (how?).

In this fashion we can see that there are just as many binary strings of length \( n \) as there are subsets of a set of size \( n \). How many?

At this point we can catch a glimpse of some serious mathematics. Consider the (closed) interval \([0, 1]:\)

(Take my word:) every number in this interval can be written as a possibly infinite binary number (in a similar manner to how they can be written as a decimal). For example,

\[
\begin{align*}
0 &= 0.000\ldots_2 \\
\frac{1}{2} &= 0.1000\ldots_2 \\
\frac{3}{4} &= 0.1100\ldots_2 \\
\frac{1}{7} &= 0.001001001001001\ldots_2 \quad \text{(pattern continues)} \\
\frac{1}{\sqrt{2}} &= .1011010100\ldots_2 \quad \text{(no pattern)}
\end{align*}
\]

Now consider the natural numbers \( \mathbb{N} = \{1, 2, 3, 4, \ldots\} \). Subsets of \( \mathbb{N} \) can be identified with infinite strings, for example:

\[
\{3, 4, 5, 10, 17\} \leftrightarrow 00111000010000001000\ldots_2.
\]

In a certain sense this means that, where \(|\cdot|\) denotes cardinality:
1.2 Set Operations and De Morgan’s Laws

Given two sets $X$ and $Y$, the union of $X$ and $Y$, written $X \cup Y$ (non-exclusive OR), is the set whose members belong to either $X$ ‘or’ $Y$. This is a mathematical ‘or’ which includes elements that belong to both $X$ and $Y$ (the inclusive or). For example, if

$$C = \{\text{counties beginning with the letter ‘C’}\}, \quad \text{and} \quad W = \{\text{counties containing the letter ‘w’}\},$$

then

$$C \cup W = \{\text{Carlow, Cavan, Clare, Cork, Down, Galway, Waterford, Westmeath, Wexford, Wicklow}\}.$$

Note that it doesn’t matter that Carlow is an element of $C$ and of $W$ — it is still an element of $C \cup W$. In terms of a Venn Diagram:

The intersection of $X$ and $Y$, written $X \cap Y$, is the set whose members belong to both $X$ and $Y$. For example, for the sets $C$ and $W$ above,

$$C \cap W = \{\text{Carlow}\}.$$

In terms of a Venn Diagram:

The relative complement or difference of sets $A$ and $B$, written $A \setminus B$ or $A - B$, said sometimes $A$ without $B$, consists of all those elements of $A$ that are not elements of $B$:

For example, where $C$ is the set of those born in Cork, and $M$ is the set of those born in Munster, then $M \setminus C$ consists of all those born in Munster but not Cork, so... What about $C \setminus M$?
In the presence of an appropriate ambient set $U$, we can talk of the complement, or NOT, of a set $A$. We write this as $\overline{A}$ or $A'$ or $A^c$ and it is equal to $U \setminus A$:

For example, where $\mathbb{P}$ is the set of prime numbers, and $U = \mathbb{N}$ is assumed, then $\overline{\mathbb{P}} = \mathbb{N} \setminus \mathbb{P}$ consists of natural numbers that are not prime.

Combining these various ideas gives us set algebra. A key feature of set algebra, and intrinsically linked to logic, are set identities (an identity is an equation that is true for all variable values). They have the form

As an example, $\overline{A} \cup B = \overline{A} \cap B$. Set identities combine a number of variable sets and the various operations and are true for all possible variable sets... what this means is that a set identity, such as $\overline{A} \cup B = \overline{A} \cap B$, is true no matter what the sets $A$ and $B$ are.

There are two ways to show that two sets are identical:

1. show that the two sets are a subset of each other: ‘$\subset$’ and ‘$\supset$’;
2. show that the two sets have the same truth table (more on this in a moment).

To show that two sets are not equal, you need to find an example such that there is an element that is in one set but not the other (a Venn diagram might help).

**Examples**

1. Consider $U$ the set of length three binary strings:

   $$U = \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

   Let $A$ be the set of binary strings with exactly two ones, and let $B$ be the set of binary strings that begin and end with a one. Investigate if $\overline{A} \cap \overline{B} = \overline{A} \cup \overline{B}$.

   **Solution:** Note that:

   $$A = \{110, 101, 011\}$$
   $$B = \{101, 111\}$$

   What is $A \cap B$:

   There are 7 elements in $\overline{A} \cap \overline{B}$:

   $$\{000, 001, 010, 011, 100, 110, 111\}$$
Let us keep going:

\[ A = \{000, 001, 010, 100, 111\} \]

and

\[ B = \{000, 001, 010, 011, 100, 110\} \]

Therefore \( A \cup B \) is:

2. Illustrate using Venn Diagrams \( A \cup B = A \cap B \).

Solution: \( A \cup B \) and \( A \cap B \) are given by:

\[ \overline{A}, \overline{B}, \text{ and } A \cap B \] are:

3. Show that \( A \cup B = A \cap B \).

Solution: ‘\( \subset \)’ Let \( x \in A \cup B \). This means that \( x \) is not in \((A \text{ OR } B)\). As \( x \) is not in \( A \), or not in \( B \), \( x \) is in \( A \text{ AND } B \). Therefore \( x \) is in \( A \cap B \).

‘\( \supset \)’ Let \( x \in A \cap B \). This means that \( x \) is not in \( A \) AND not in \( B \), it is in neither \( A \) nor \( B \): i.e. it is not in \( A \text{ OR } B \), i.e. it is in \( A \cup B \).
4. **Winter 2017** Use symbols to describe the shaded area in the following Venn diagram.

![Venn Diagram](image)

*Solution:* Any ideas?

5. **Winter 2017** Suppose \( A = \{1, 2, 3, 4, 5\} \), \( B = \{3, 4, 5, 6, 7\} \), and the ambient set is \( U = \{1, 2, 3, 4, 5, 6, 7, 8\} \). Find \( (A \cap B) \) and \( A \cup B \).

*Solution:* First we write down \( A \cap B \):

Now find the complement:

By De Morgan’s Law the second set is the same... alternatively find first \( \overline{A} \) and \( \overline{B} \):

and find their union:
6. Investigate whether or not \( A \cap B = \overline{A \cap B} \).

\textit{Solution:} Using Venn diagrams. Note that \( A \) and separately \( B \) are given by:

What is in both of these is:

Now looking at \( A \cap B \) and after \( A \cap B \):

we can see that these sets are not equal. Indeed for \( A = C \), counties starting with ‘C’; and \( B = W \), counties starting with ‘w’:

\[
C \cap W = \{ \text{Antrim, Armagh, Derry, Donegal, Dublin, Fermanagh, Kerry, Kildare, Kilkenny, Laois, Leitrim, Limerick, Longford, Louth, Mayo, Meath, Monaghan, Offaly, Roscommon, Sligo, Tipperary, Tyrone} \}.
\]

while \( \overline{C \cap W} = \{ \text{Carlow} \} \), which is every country except for Carlow. These sets are not equal.

1.2.1 Truth Tables

Perhaps it was hard to follow and difficult for you to reproduce the proof that \( \overline{A \cup B} = \overline{A} \cap \overline{B} \)? A way of keeping track of the argument, and not using as much notation, is to use \textit{truth tables}. It is probably best to introduce truth tables by way of an example:

\[
\begin{array}{c|c|c}
A & B & A \cup B \\
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array}
\]

The first row, below the line, reads:
The second (similar to the third) reads:

The last row reads:

Truth tables can be used to prove set identities.

**Examples**

1. Use truth tables to prove that $A \cup B = A \cap B$:

   **Solution:** Consider

   $A$ | $B$ | $A \cup B$ | $A \cap B$ | $A$ | $B$ | $A \cap B$
   --- | --- | --- | --- | --- | --- | ---
   | | | | | | |

   The empty set, $\emptyset$, has no elements and so its ‘membership column’ consists of zeros.

2. Use truth tables to prove that $A \cup \emptyset = A$.

   **Solution:**

   $A$ | $\emptyset$ | $A \cup \emptyset$
   --- | --- | ---
   | 0 | 0
   | 0 | 0

   An ambient set $U$, contains all the elements of interest and so its ‘membership column’ consists of ones:

3. Use truth tables to prove that $A \cap U = A$.

   **Solution:**

   $A$ | $U$ | $A \cap U$
   --- | --- | ---
   | 1 | 1
   | 1 | 1

   Note an identity with three variable sets required a Venn Diagram looking like:

   and a truth table with eight rows.
4. Complete the truth table for \((A \cup B) \cap C\):

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(B)</td>
<td>(C)</td>
<td>((A \cup B) \cap C)</td>
</tr>
</tbody>
</table>

5. Consider three subsets of an ambient set \(U\): \(A, B, C \subset U\).

(a) Produce two copies of a Venn Diagram for \(A, B, C, U\).

(b) In the first, \textit{carefully} shade \((\overline{A} \cup B) \cap C\).

(c) In the second, \textit{carefully} shade \(\overline{A} \cup B\).

\textit{Solution:} Part (b) is a lot harder than part (c). Therefore we can look at (b) using a truth table:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(A)</th>
<th>(\overline{A})</th>
<th>(\overline{A} \cup B)</th>
<th>((\overline{A} \cup B) \cap C)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Therefore we shade:

Part (c) can also be done with truth tables but probably easier just to produce two Venn diagrams:

There are more laws (collated at the back of the manual — and will be on the back of your final exam paper also).

<table>
<thead>
<tr>
<th>Name</th>
<th>Equality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double Complement Law</td>
<td>( \overline{\overline{A}} = A )</td>
</tr>
<tr>
<td>Identity Laws</td>
<td>( A \cap U = A )</td>
</tr>
<tr>
<td>Annihilation Laws</td>
<td>( A \cup U = U )</td>
</tr>
<tr>
<td>Inverse/Complement Laws</td>
<td>( A \cup \overline{A} = U )</td>
</tr>
<tr>
<td>Idempotent Laws</td>
<td>( A \cup A = A )</td>
</tr>
<tr>
<td>Commutative Laws</td>
<td>( A \cup B = B \cup A )</td>
</tr>
<tr>
<td>DeMorgans Laws</td>
<td>( (A \cup B) = \overline{A} \cap \overline{B} )</td>
</tr>
<tr>
<td>Absorption Laws</td>
<td>( A \cup (A \cap B) = A )</td>
</tr>
<tr>
<td>Associative Laws</td>
<td>( (A \cap B) \cap C = A \cap (B \cap C) )</td>
</tr>
<tr>
<td>Distributive Laws</td>
<td>( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) )</td>
</tr>
</tbody>
</table>
Examples

1. Simplify $P \cup (Q \cap P)$ using only the laws of sets.

   Solution: First we use...

2. Winter 2018 Simplify $\overline{P} \cup (Q \cap P)$. Identify the laws used in each step of your solution.

   Solution: First we use...

3. Winter 2019 Simplify $X \cup (Z \cap \overline{X})$ using the laws of sets. Indicate explicitly which laws are used at each step.

   Solution: First we use...
Further Remark: Four Flavours of De Morgan

We tell four tales of De Morgan. In each case we have something that looks like AND, something that looks like OR, and something that looks like NOT.

1.2.2 Sets

The Collection of Objects

Consider a universe of discourse/universal set/ambient set \( U \). When talking about people this might be the collection of all people. When talking about natural numbers this might be the set \( \mathbb{N} \). When talking about real numbers this might be the set \( \mathbb{R} \). When talking about curves it might be the set of subsets of the plane, \( \mathcal{P}(\mathbb{R} \times \mathbb{R}) \), etc.

The collection of objects in this case is the set of subsets of \( U \), denoted \( \mathcal{P}(U) \). Suppose, for the purposes of illustration, that

\[
U = \{1, 2, 3, 4, 5\}.
\]

The subsets \( A = \{2, 3, 5\} \) and \( B = \{1, 3, 5\} \) are objects.

AND

Note that two objects are contained both in \( A \) AND in \( B \). We call the set of such objects the intersection of \( A \) AND \( B \), \( A \cap B \):

\[
A \cap B = \{3, 5\}.
\]

We can represent the ambient set \( U \), as well as the sets \( A \) and \( B \) — and the fact that they intersect — using a Venn Diagram:

We can demonstrate for a general \( A \) and \( B \) ‘where’ the intersection is:

OR

Another set that can be formed from \( A \) and \( B \) is their union. The union of \( A \) and \( B \), denoted \( A \cup B \), consists of all those objects that are elements of \( A \) OR* elements of \( B \). So, with the instances of \( A \) and \( B \) above:

\[
A \cup B = \{2, 3, 5\} \cup \{1, 3, 5\} = \{1, 2, 3, 5\}
\]
*The “OR” here is a ‘mathematical OR’ rather than an ‘exclusive OR’, the likes of which we sometimes use in every day speech. Imagine the following question put to a ‘normal’ person and a mathematician, both of whom ate a starter and a dessert:

**PERSON 1:** Did you have starter or a dessert?

‘NORMAL PERSON’: I actually had both.

**MATHEMATICIAN:** Yes.

If you really want, you can think of the union as being all objects that are in $A$, OR in $B$, OR in both $A$ AND $B$:

$$A \cup B = (\text{in } A) \text{ OR (in } B) \text{ OR (in both) } = (\text{in } A) \text{ OR (in } B) \text{ OR (in } A \cap B).$$

We can represent the union as follows:

**NOT**

We can also talk about the *complement* of a set — the set of objects that are NOT in a set. Now there are lots of things that are NOT in the set $A$. For example, Chuck Norris is NOT in the set $A$ — but we don’t chuck Chuck Norris into the complement of $A$ — only objects in the ambient set $U$ in which the objects in $A$ ‘live’.

For a set $A$, we denote this set by $\overline{A}$ and so, for example, for the instance of $A$ sitting in $U = \{1, 2, 3, 4, 5\}$ above:

$$\overline{A} = \{2, 3, 5\} = \{1, 4\}$$
We can also represent the complement using a Venn Diagram, for example, $\overline{A}$:

\[ A \cup B = \{1, 2, 3, 5\} = \{4\}. \]

This set contains objects that are NOT in $A$ AND not in $B$:

\[ \overline{A} = \{1, 4\}, \overline{B} = \{2, 4\}, \]

the objects that are in $\overline{A}$ AND $\overline{B}$ are the intersection of these sets:

\[ \overline{A} \cap \overline{B} = \{1, 4\} \cap \{2, 4\} = \{4\}, \]

which is the same as $\overline{A} \cup \overline{B}$ so we have demonstrated that

\[ A \cup B = \overline{A} \cap \overline{B}. \]

This is known as De Morgan’s Law and is always true. Look at what it says:

If an object is NOT in ‘$A$ OR $B$’, then it must NOT be in $A$ AND it must NOT be in $B$.

Similarly,

If an object is NOT in ‘$A$AND$B$’, then it is NOT in $A$ OR it is NOT in $B$.

This is the other De Morgan’s Law

\[ \overline{A} \cap \overline{B} = \overline{A} \cup \overline{B}. \]

### 1.2.3 Logic

**The Collection of Objects**

Now I should really be careful with what I write here, but all we are really doing is analogy and so a lot of subtlety is being swept under the carpet. Consider the set of all sentences $S$. A sentence is either true or false. We will not define further what a sentence is (fairly big carpet used).
Examples:

- The capital of Ireland is Dublin (true)
- Wednesday is not a weekday (false)
- 4 is a prime number (false)
- The word “Earth” contains five letters (true)

AND

If $S_1$ and $S_2$ are sentences, then their conjunction is the sentence:

$$S_1 \land S_2.$$

The conjunction is true whenever $S_1$ AND $S_2$ are true, and so the conjunction $\land$ is AND. For example, with $A = \{2, 3, 5\}$ and $B = \{1, 3, 5\}$,

$$(1 \in A) \land (1 \in B),$$

is a false statement, while

$$(2 \in A) \land (1 \in B),$$

is a true sentence. Therefore the conjunction is AND.

OR

The disjunction of sentences $S_1$ and $S_2$ is the sentence

$$S_1 \lor S_2,$$

which is true whenever $S_1$ OR $S_2$ is true (mathematical OR); that is $S_1 \lor S_2$ is true if and only if

$$S_1 \text{ is true or } S_2 \text{ is true.}$$

NOT

The negation of a sentence $S$ is a sentence $\neg S$, which is true whenever $S$ is false. For example, where $A = \{2, 3, 5\}$,

$$\neg(1 \in A),$$

is true, because $(1 \in A)$ is false. Indeed

$$\neg(1 \text{ is an element of } A) = (1 \text{ is not an element of } A),$$

and so the negation $\neg$ is NOT.
De Morgan’s Laws

Now consider for sentences $S_1, S_2$ the sentence:

$$\neg (S_1 \land S_2).$$

This is true whenever $S_1 \land S_2$ is false. For $S_1 \land S_2$ to be false, either $S_1$ OR $S_2$ (or both) must be false. That is either $\neg S_1$ OR $\neg S_2$ must be true, and so the truth values of

$$\neg (S_1 \land S_2), \text{ and } (\neg S_1) \lor (\neg S_2)$$

are the same, and we say they are equal as sentences:

$$\neg (S_1 \land S_2) \equiv (\neg S_1) \lor (\neg S_2)$$

This is De Morgan’s Law for sentences. We also have a second De Morgan Law:

$$\neg (S_1 \lor S_2) \equiv (\neg S_1) \land (\neg S_2).$$

1.2.4 Switches

The Collection of Objects

Consider a set of $n$ switches.

$$\{S_1, S_2, \ldots, S_n\}.$$

Each switch may be ON OR OFF.

AND

We can combine switches in series:

We denote such a switch by $S_1 \otimes S_2$. We have $S_1 \otimes S_2$ is on if $S_1$ is on AND $S_2$ is on — that is the series switch is on if both constituent switches are on.

OR

We can combine switch in parallel:

We denote such a switch by $S_1 \oplus S_2$. The switch $S_1 \oplus S_2$ is on if $S_1$ is on OR $S_2$ is on.
NOT

So consider the operation of toggling a switch: from on to off or vice versa. For example, if $S$ is on, then $\text{toggle}(S)$ is off, that is:

$$S \text{ is on } \Rightarrow \text{toggle}(S), \text{ is off}.$$  

De Morgan’s Laws

Now ask the question: when is $S_1 \oplus S_2$ off, that is $\text{toggle}(S_1 \oplus S_2)$ is on? This can only happen with both $S_1$ is off AND $S_2$ is off, that is when both $\text{toggle}(S_1)$ is on and $\text{toggle}(S_2)$ is on:

$$\text{toggle}(S_1 \oplus S_2) \text{ is on } \iff (\text{toggle}(S_1) \otimes \text{toggle}(S_2) \text{ is on.}$$

A similar situation holds when the toggled parallel switch is off and so we have that:

$$\text{toggle}(S_1 \oplus S_2) = \text{toggle}(S_1) \oplus \text{toggle}(S_2).$$

It might look different at this time, but this is a De Morgan’s Law for switches. Similarly we have:

$$\text{toggle}(S_1 \otimes S_2) = \text{toggle}(S_1) \oplus \text{toggle}(S_2).$$

1.2.5 Functions

We will meet Functions in Chapter 3. You may have to look back on this after studying Section 3.

Collection of Objects

Consider the set $\{1, 2, \ldots, 5\}$ and consider the collection of all 0-1 valued functions on $\{1, 2, 3, 4, 5\}$. For example, consider $f_1$ defined by

$$f_1(1) = 0, \ f_1(2) = 1, \ f_1(3) = 1, \ f_1(4) = 0, \ f_1(5) = 1$$

which we can write in ordered pair notation:

$$f_1 = \{(1, 0), (2, 1), (3, 1), (4, 0), (5, 1)\}.$$  

Consider also $f_2$ defined by:

$$f_2 = \{(1, 1), (2, 0), (3, 1), (4, 0), (5, 1)\}.$$

We can combine elements $f_1$ and $f_2$ of $F$ in various ways. The first we consider is the $\text{max}$ function $\text{max}(f_1, f_2)$ defined by

$$\text{max}(f_1, f_2)(x) = \text{max}\{f_1(x), f_2(x)\}.$$  

For example, consider $f_1$, $f_2$ as above. Well, anyway,

$$\text{max}(f_1, f_2)(1) = \text{max}\{f_1(1), f_2(1)\} = \text{max}\{0, 1\} = 1.$$  

and we have:

$$\text{max}(f_1, f_2) = \{(1, 1), (2, 1), (3, 1), (4, 0), (5, 1)\}.$$
Similarly we can define the \( \text{min} \) function \( \text{min}(f_1, f_2) \), for example:

\[
\text{min}(f_1, f_2) = \{(1, 0), (2, 0), (3, 1), (4, 0), (5, 1)\}.
\]

Consider for a function \( f \in F \) the function \( \text{toggle}(f) \), given by:

\[
\text{toggle}(f)(x) = 1 - f(x),
\]

so for example we have

\[
\text{toggle}(f_1) = \{(1, 1), (2, 0), (3, 0), (4, 1), (5, 0)\},
\]

so that \( \text{toggle}(f) \) is found by toggling the values \( 0 \leftrightarrow 1 \).

There are De Morgan’s Laws here also; namely:

\[
\text{toggle}(\max(f_1, f_2)) = \min(\text{toggle}(f_1), \text{toggle}(f_2)), \quad \text{and}
\]

\[
\text{toggle}(\min(f_1, f_2)) = \max((\text{toggle}(f_1), \text{toggle}(f_2))).
\]

### 1.2.6 The Inclusion-Exclusion Principle

Recalling that \(|A|\) is the \# elements in the set \( A \), we have the following for any sets \( A \) and \( B \):

\[
|A \cup B| = |A| + |B| - |A \cap B|.
\]  \hspace{1cm} (1.1)

**Proof.** Each element of \( A \cup B \) is either an element of \( A \setminus B \), \( B \setminus A \), or \( A \cap B \):

When we calculate \(|A| + |B|\) we count elements of \( A \cap B \) twice so we must subtract these to find the number of elements in \( A \cup B \).

**Examples**

1. Let \( R \) be the set of red playing cards and \( P \) be the picture playing cards (K, Q, J: we won’t include A). Find \(|R \cup P|\).

   **Solution:** Using

   If we know \(|R \cap P|\) we are away (\(|P|\) and \(|R|\) are easy):
2. In a group of 20 people, all of which drink or smoke, 15 said they drank and 7 said they smoked. How many drank and smoked?

Solution: Let \( D \) be the set of drinkers, and \( S \) the set of smokers:

1.2.7 Elements, Subsets, The Empty Set, and the Natural Numbers*

Let \( X = \{a, b, c, d\} \) be a set. There is a difference between the element \( a \in X \) and the subset \( \{a\} \subset X \), \( \{a\} \in \mathcal{P}(X) \). It is false to say that \( a \subset X \), or \( a \in \mathcal{P}(X) \), or \( \{a\} \in X \) or \( \{a\} \subset \mathcal{P}(X) \). You can write that \( \{\{a\}\} \subset \mathcal{P}(X) \).

The empty set is a subset of every set \( \emptyset \subset S \), but is not an element necessary. The empty set is an element of the power set of any set, \( \emptyset \in \mathcal{P}(S) \). Bizarrely, the empty set is also a subset of the power set, \( \emptyset \subset \mathcal{P}(S) \). If a set contains the empty set, perhaps \( S = \{\emptyset, 1, 2\} \), then it is true that \( \emptyset \in S \).

Another notation for \( \emptyset = \{\} \) — the set with no elements.

We can construct the natural numbers using sets as follows:

\[
\begin{align*}
0 &= \emptyset \\
1 &= \{0\} \\
2 &= \{0, 1\} \\
3 &= \{0, 1, 2\} \\
4 &= \{0, 1, 2, 3\} \\
&\vdots
\end{align*}
\]

Exercises

1. Let \( U = \{1, 2, \ldots, 9\} \), \( T = \{3, 6, 9\} \), \( P = \{2, 3, 5, 7\} \), and \( E = \{2, 4, 6, 8\} \).

   (a) Write down \( \overline{P} \).

   (b) Carefully find \( T \cup (\overline{P} \cap E) \).

   (c) Find, or otherwise write down, \( (T \cup \overline{P}) \cap (T \cup E) \).

2. Show that \((A \cap \overline{B}) \cup B = A \cup B\) for \( A = \{a, b, c, d, e\} \), \( B = \{d, e, f, g, h\} \) where the ambient set is \( U = \{a, b, c, d, e, f, g, h, i, j, k, l\} \).

3. Let \( U = \{0, 1\} \). List the elements of \( \mathcal{P}(U) \) (the set of subsets of \( U \)).
4. Let $X = \{b, l, d\}$.
   
   (a) List the elements of $\mathcal{P}(X)$.
   
   (b) Hence, or otherwise, find $|\mathcal{P}(X)|$.
   
   (c) The set $X$ represents the meals of breakfast, lunch, and dinner. Suppose three more meals were added: elevensies, tea, and supper to give $Y = \{b, e, l, t, d, s\}$. Find $|\mathcal{P}(Y)|$.

5. **Winter 2017** Let $A = \{a, b, c, d\}$. How many elements are in $\mathcal{P}(A)$, the power set of $A$?

6. **Autumn 2018** Use symbols to describe the shaded area in the following Venn diagram.

![Venn Diagram](image1)

7. **Autumn 2020** Use symbols to describe the shaded area in the following Venn diagram.

![Venn Diagram](image2)

8. Let $U = \{1, 2, \ldots, 10\}$. If $C \subset U$ has bit string representation 0110100011 and $D$ has bit string representation 0101001000, determine the bit string representations of (i) $C \cap D$ (ii) $C \cup D$ (iii) $\overline{D}$

9. Let $U = \{a, b, c, d, e, f, g, h\}$. Suppose $A \subset U$ has bit string representation 10010110 and $B \subset U$ has bit string representation 01101111. Find the bit string representations of
   
   (a) $\overline{A}$
   
   (b) $A \cap B$
   
   (c) $A \cup B$

10. Simplify $(\overline{P} \cap Q)$ using only the laws of sets.

11. Simplify $\overline{P} \cap (P \cup Q)$ using only laws of sets. Quote each law that you use.
12. Simplify $P \cup (\overline{P \cap Q})$ using only laws of sets. Quote each law that you use.

13. In a class of 40 students every student is either in the guild gaming society or the anime society. The class rep. asked everyone in the guild gaming society to raise their hands. Nine hands go up. Then the class rep. asked everyone in the anime society to raise their hands. This time there are 34 hands raised. How many students from this class are in both the guild gaming society and the anime society?

14. Autumn 2020 Simplify $G \cap H \cap \overline{G}$ using the laws of sets. Indicate explicitly which laws are used.

15. Let $A, B, C \subseteq U$ be sets. Illustrate the following using Venn Diagrams:

   (i) $(A \setminus B) \setminus C$
   (ii) $(A \setminus B) \cup (A \setminus C)$.

16. Let $U = \{x \in \mathbb{N} : x < 12\}$, $A = \{x : x \text{ is odd}\}$, $B = \{x : x > 7\}$, and $C = \{x : x \text{ is divisible by 3}\}$. Depict the sets using a Venn Diagram. Hence list the elements of the following sets: (i) $A \cap B$  (ii) $B \cup C$  (iii) $\overline{A}$  (iv) $(A \cup B) \cap C$  (v) $(A \cap C) \cup \overline{C}$  (vi) $\overline{A} \cap \overline{B}$.

17. Let $A, B \subseteq U$ be sets. Produce truth tables for the following:

   (i) $A \cap B$  (ii) $A \setminus B$  (iii) $\overline{A}$ (one row).

18. Use truth tables to prove De Morgan’s Laws

   (i) $\overline{A \cap B} = \overline{A} \cup \overline{B}$  (ii) $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

19. For the empty set $\emptyset$; and any set $A$, prove the following:

   (i) $\emptyset \cap A = \emptyset$,  (ii) $A \cap \overline{A} = \emptyset$.

20. For an ambient set $U$, and any set $A$, use Venn Diagrams and/or truth tables to simplify $A \cup U$ and $A \cup \overline{A}$.

21. Let $A, B, C \subseteq U$ be sets. Complete the following truth table:

\[
\begin{array}{cccc|ccc|c}
A & B & C & A \setminus B & (A \setminus B) \setminus C & B \cup C & A \setminus (B \cup C) \\
\hline
1 & 1 & 1 & & & & \\
1 & 0 & 1 & & & & \\
1 & 1 & 0 & & & & \\
1 & 0 & 0 & & & & \\
0 & 1 & 1 & & & & \\
0 & 0 & 1 & & & & \\
0 & 1 & 0 & & & & \\
0 & 0 & 0 & & & & \\
\end{array}
\]

What can you conclude?
22. Let \( A, B, C \subseteq U \) be sets. Complete the following truth table:

\[
\begin{array}{cccc|c|c|c}
A & B & C & B \setminus C & A \setminus (B \setminus C) & A \setminus B & (A \setminus B) \cup (A \cap C) \\
1 & 1 & 1 & & & & \\
1 & 0 & 1 & & & & \\
1 & 1 & 0 & & & & \\
1 & 0 & 0 & & & & \\
0 & 1 & 1 & & & & \\
0 & 0 & 1 & & & & \\
0 & 1 & 0 & & & & \\
0 & 0 & 0 & & & & \\
\end{array}
\]

What can you conclude?

23. Let \( A, B, C \subseteq U \) be sets. Use truth tables to prove the distributive laws:

\[
A \cap (B \cup C) = (A \cap B) \cup (A \cap C),
\]
\[
A \cup (B \cap C) = (A \cup B) \cap (A \cup C).
\]

24. Let \( A, B, C \subseteq U \) be sets. Use truth tables to show the following:

(a) \( A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \),
(b) \( A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \).

25. * List the elements of the following sets:

\[
A = \{x \in \mathbb{R} : x > 0, x^2 = 81\}, \quad B = \{x \in \mathbb{N} : x \leq 13 \text{ and } 3 \text{ divides } x\}.
\]

26. * Given \( A = \{1, \{1\}, \{2\}, 3\} \), determine which of the following statements are true and which are false: (a) \( 1 \in A \) \quad (b) \( 1 \subset A \) \quad (c) \( \{1\} \in A \) \quad (d) \( \{1\} \subset A \) \quad (e) \( \{\{1\}\} \subset A \)
(f) \( 2 \in A \) \quad (g) \( \{2\} \in A \) \quad (h) \( \{2\} \subset A \) \quad (i) \( \{3\} \in A \) \quad (j) \( \{3\} \subset A \) \quad (k) \( \{3\} \in \mathcal{P}(A) \)
(l) \( \{2\} \in \mathcal{P}(A) \)

27. * Let \( A, B, C \subseteq U \) be sets. The symmetric difference of \( A \) and \( B \) is defined as \( A \Delta B := (A \setminus B) \cup (B \setminus A) \). Use truth tables to show that

\[
(A \cup B \cup C) \setminus (A \cap B \cap C) = (A \Delta B) \cup (B \Delta C).
\]
1.3 Cartesian Products: ALL Ordered Pairs

The Cartesian Product describes a way of combining two sets to generate another. If \(A, B\) are the two sets in question, then the Cartesian product of \(A\) and \(B\) is denoted by \(A \times B\). The elements of \(A \times B\) are ordered pairs, with the first part from \(A\) and the second from \(B\). For example, if \(C\) is the set of counties that begin with ‘C’ and \(W\) is the set of counties that contain a ‘w’, then (Cork, Down) is an element of \(C \times W\).

More precisely, \(A \times B\) is the set:

\[
A \text{ cross/times } B \text{ is the set of all ordered pairs } (a, b) \text{, such that } a \text{ is an element of } A \text{ and } b \text{ is an element of } B.
\]

In other words all possible pairs with the first object in \(A\), and the second in \(B\). Note the word ordered: \(A \times B \neq B \times A\) if \(A \neq B\). A set-theoretic definition of an ordered pair is \((a, b) = \{a, \{a, b\}\}\)

Examples

1. Deck of Cards: Let \(V = \{A, 2, \ldots, 10, J, Q, K\}\) and \(S = \{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\}\). An element of the set \(V \times S\) can be chosen by first picking a value, \(v \in V\) and then a suit, \(\square \in S\), to give \((v, \square) \in V \times S\). We can see that \(V \times S\) is the set of cards in a deck, with for example \(A\heartsuit := (A, \heartsuit)\). If you want to include jokers consider the set \((V \times S) \cup \{J_1, J_2\}\).

Note that, certainly when \(A\) and \(B\) are finite, the cardinality (number of elements) of \(A \times B\) is equal to the cardinality of \(A\) times the cardinality of \(B\): \(|A \times B| = |A| \cdot |B|\):

2. Chessboard: Let \(L = \{a, b, c, \ldots, h\}\) and \(N = \{1, 2, \ldots, 8\}\). The set \(L \times N\) consists of pairs \((\square, n)\), with \(\square \in L\) a letter and \(n \in N\) a number. Using a coordinate system, we can see that \(L \times N\) is the set of chess squares:

![Chessboard Diagram](shutterstock)
3. **Autumn 2018** Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $X = \{a, b\}$. List the elements of $(A \cap B) \times X$, and the elements of $(A \times X) \cap (B \times X)$. What can you conclude?

*Solution:* First we write down $A \cap B$

Now we write down $(A \cap B) \times X$

Now we write down $A \times X$ and $B \times X$:

and find their intersection

Note $(A \cap B) \times X = (A \times X) \cap (B \times X)$.

4. A very basic protocol asks for a letter from $L_1 = \{a, b, c\}$ and a digit from $D_1 = \{0, 1\}$ for a password.

(a) List the elements of $L_1 \times D_1$, the set of possible passwords.

(b) Hence, or otherwise, write down $|L_1 \times D_1|$, the number of possible passwords.

(c) A more advanced protocol asks for a letter from $L_2 = \{a, b, c, \ldots, z\}$ and a digit from $D_2 = \{0, 1, 2, \ldots, 9\}$. The set of possible passwords for this advanced protocol is $L_2 \times D_2$. Write down $|L_2 \times D_2|$, the number of possible passwords for this advanced protocol.

*Solution:*

(a) List:

(b) We have

(c) We have
5. **The Plane** Take two copies of \( \mathbb{R} \). The set \( \mathbb{R} \times \mathbb{R} \) (also denoted \( \mathbb{R}^2 \)) consists of all pairs \((x, y)\) such that \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \). A single copy of \( \mathbb{R} \) may be visualised as a numberline:

The plane is nothing but two numberlines — one for \( x \) and one for \( y \) — and a coordinate system:

![Figure 1.1: Given a scale, each element of \( \mathbb{R} \times \mathbb{R} \) can be associated to a point on the plane; and each point on the plane can be associated to an ordered pair, \((x_0, y_0)\).](image)

6. It is also possible to take a Cartesian product of two sets, say \( A \times B \) and take the Cartesian product of this with another set \( C \):

However this is just the same as the set of ordered triples:

For example, \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 \) is the set of points in space:

![Figure 1.2: If you choose any point \( P \) in space, then this implicitly defines an ordered triple \((a, b, c)\) ∈ \( \mathbb{R}^3 \).](image)

1.3.1 **Tuples: Passwords and Bit-Strings**

As suggested above, we can form the set of (ordered) \( n \)-tuples:

\[
A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i\}.
\]
We use the notation:

\[ A^n = A \times \cdots \times A. \]

For example, consider the alphabet \( A = \{0, 1, 2, \ldots, 9, a, b, \ldots, z, A, B, \ldots, Z\} \). Passwords of length 4 with characters from the alphabet \( A \) are described by elements of:

\[ \text{for example } JP85 \cong (J, P, 8, 5) \in A^4. \]

Note that the number of passwords here is:

and in Chapter 4 we will answer questions such as

\[ \text{given an alphabet } A \text{ of size 26, what should be the minimum length of password be to ensure that there are 1,000,000 different passwords.} \]

Binary- or Bit-Strings also fall under this framework. Take length three binary strings. Viewing, say, \( 010 \cong (0, 1, 0) \), we have:

\[ \text{length three binary strings} = \{0,1\} \times \{0,1\} \times \{0,1\} = \{0,1\}^3. \]

Recall that subsets of \( A \) are paired with length \(|A|\) binary strings:

\[ \mathcal{P}(A) \cong \{0,1\}^{\{|A|\}}. \]

This is another way to find the cardinality of \( \mathcal{P}(A) \):

\[ \text{Exercises:} \]

1. Let \( C = \{H,T\} \). List the elements of \( C \times C \) and \( C \times C \times C \).

2. \textbf{Autumn 2020} Let \( S = \{a,b,c\} \). Write down all the elements in the following sets.

   i. \( S \times S \)

   ii. the power set of \( S \), \( \mathcal{P}(S) \).

3. List the elements of \( A \times B \) where \( A = \{1, 2, 3\} \) and \( B = \{a, b\} \).

4. How many elements in the set \( D \times D \), where \( D = \{1, 2, \ldots, 6\} \)? What is the set \( D \times D \) the same as?

5. Let \( X = \{B, W\} \) and \( Y = \{H, S, D, C\} \).

   (a) List the elements of \( X \times Y \).

   (b) Hence, or otherwise, write down the value of \(|X \times Y|\). \textbf{Ans:} eight.

   (c) Write down \(|\mathcal{P}(X \times Y)|\). \textbf{Ans:} 256.
6. **Winter 2019** Let $A = \{2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and suppose the universal set is $U = \{1, 2, \ldots, 9\}$. List all the elements in the following sets.

(a) $A \cup B$;
(b) $(A \cap B) \times A$;
(c) the power set of $B \cap A$, the set $\mathcal{P}(B \cap A)$.

7. * Draw a graphical representation of the set $B \times \mathbb{R}$, where $B = \{0, 1\}$.

8. * Given a set $A := \{a, b, c\}$, what is $A \times \emptyset$ equal to?

### 1.4 Relations: Databases — *SOME* Ordered Pairs

If $A$, $B$ are sets, a *relation* $R$ between $A$ and $B$ is a subset of $A \times B$. If we want to say that $a \in A$ is related to $b \in B$ we can either write $(a, b) \in R$ or $aRb$. For example, let $P = \{\text{people in this room}\}$ and $A = \mathbb{N}$ and we can define a relation $R \subseteq P \times A$:

Now for example, I am related to 35: this can be written as $(\text{JP}, 35) \in R$ or $\text{JP}R35$.

We can also represent relations graphically. For example, let $A = \{2, 3, 5\}$ and define a relation $R$ on $A \times A$ by:

This can be represented by a *digraph*, or an arrow diagram:

Relations between ordered sets\(^2\) can also be represented using a (coordinate) graph.

\(^2\)any set with an *order*
Simply set up a pair of axes and plot all the points that appear as ordered pairs in the relation:

![Graph](image-url)

Figure 1.3: We always put the first set (source), $A$ in this case, along the horizontal, $x$-axis; and the second (target), $B$ in this case, along the vertical, $y$-axis.

If $A$ and $B$ are equal, say $A = B = S$, then $R$ is a subset of $S \times S$, and we call $R$ a relation on $S$.

For example, let $S = \{a, b, c\}$. A relation on $S = \{a, b, c\}$ is any subset of, $S \times S$:

For example, the following subset of $S \times S$ is a relation on $S$:

Relations can be a very convenient language in mathematics. How many relations on a set $S$? How many subsets has $S \times S$... if $|S| = 3$? How large is $|\mathcal{P}(S \times S)|$?

**Examples**

1. Let $M = \{1, 2, \ldots, 10\}$. Define a relation $R$ on $M$ (a subset of $M \times M$) such that $(x, y) \in R$ if and only if there is a positive integer $k$ such that $y$ ‘is a multiple of $x$’. Find the elements of $R$.

   *Solution:* What is one related to? What is a multiple of one? All of $M$. So $(1, x) \in R$ for all $x \in M$ (i.e. 1 is related to everything). What is two related to?

   What three related to?
Four? Five?

Six, Seven, Eight, Nine, Ten?

There are 27 elements in the relation $R$. This is too unwieldy to draw an arrow diagram for but a (coordinate) graph is perfect:

![Graph](image)

Figure 1.4: Recall the first coordinate is the $x$-coordinate: the horizontal coordinate. Should I ‘join the dots’?

2. Let $U = \{0, 1\}$.
   
   (a) Let $\text{SubsetOf} = \{(X, Y) : X, Y \subseteq U \text{ and } X \subseteq Y\}$ be a relation on $\mathcal{P}(U)$ (i.e. $X$ is related to $Y$ if $X$ is a subset of $Y$). List all ordered pairs in $\text{SubsetOf}$.
   
   (b) Let $\text{StrictSubsetOf} = \{(X, Y) : X, Y \subseteq U \text{ and } X \neq Y\}$. List all ordered pairs in $\text{StrictSubsetOf}$.

   Solution: Probably the first thing to do here is to note that these are relations on $\mathcal{P}(U)$ – the set of subsets of $U$. It is worth listing the elements of $\mathcal{P}(U)$

   (a) Here we list all the elements of $\text{SubsetOf}$:
This can be represented well by an arrow diagram:

(b) This is what we would call a sub-relation of SubsetOf. Elements of the form \((A, A)\) are not included so we have:

This can be represented well by a digraph:

3. *Implicitly Defined Curves* Consider the following relation on \(\mathbb{R}\) (subset of \(\mathbb{R} \times \mathbb{R}\) = the plane):

\[(x, y) \in R \text{ if and only if } x^2 + 3xy^2 + y^3 = 5.\]

In other words, \(xRy\) is \((x, y)\) satisfies the equation. We could call this the equation of the relation. The graph of this relation can be plotted using a computer software package:
Figure 1.5: Points such as (1, 1), which satisfy the equation of the relation.

Exercises

1. **Autumn 2018** Let \( A = \{a, b, c, d, e\} \) and \( xRy \) if and only if both \( x \) and \( y \) are vowels, or both consonants. List the elements of the relation \( R \).

2. Let \( X = \{1, 2, 3, 4\} \). Define a relation \( R \) on \( X \) by \((x, y) \in R \) if \( x < y \). List the elements of \( R \) and represent the relation graphically.

3. * If \( S \) has three elements, find the number of relations on \( S \).

4. * Recall that \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \) is the plane. Plot the following relations:
   
   (a) \( xR_1y \) if \( x = 1 \);
   
   (b) \( xR_2y \) if \( y = 2 \).

5. * Define a relation on the set of all points in the plane \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \) by saying \((p_1, p_2) \in R \) if there is a line through \( p_1 \) and \( p_2 \) that also contains the origin. Is \((1, 2), (2, 3) \in R\)? Give an element of \( R \).

6. * Let \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \) be the plane. Call a triple of points \( \{p_1, p_2, p_3\} \subseteq \mathbb{R}^2 \) a triangle if they do not lie on a single line. Let \( T \) be the set of triangles and define a relation \( R \) on \( T \) by \( T_1RT_2 \) if the set of internal angles of \( T_1 \) is the same as the set of internal angles of \( T_2 \). Suppose that \((T_1, T_2) \in R\). Does it follow that \( T_1 = T_2 \)? Give two ways in which this fails to occur (i.e. find two essentially different ways in which \( T_1 \) is related to \( T_2 \) but the triangles are not the same).
1.4.1 Properties of Relations

Relations on a set $A$ are relations between $A$ and $A$ in other words subsets of $A \times A$. If $A$ is finite, such relations may be represented by a directed graph/digraph. Each element of $A$ is given a node and if $(a_1, a_2) \in R$, then an arrow is drawn between $a_1$ and $a_2$. For example, the relation $R$ on $A = \{a, b, c\}$ given by $\{(a, a), (a, b), (b, b), (c, b), (c, a)\}$ is given by:

If a relation $R$ is such that every element of $A$ is related to itself (with a loop), so that $aRa$, $(a, a) \in R$, then $R$ is said to be reflexive. For example, the relation above is not reflexive as $(c, c) \notin R$. Consider the relation on $S = \{1, 2, 3, 4\}$ given by $aRb$ if $\frac{a}{b} \in \mathbb{N} = \{1, 2, 3, \ldots\}$:

This is an example of reflexive relation.

If a relation $R$ is such that whenever $aRb$ that $bRa$ — so that either nodes are unconnected or a double edge connects nodes — then $R$ is said to be symmetric. For example, neither of the two relations presented in this section are symmetric. For the first, $cRb$ but $(b, c) \notin R$. In the second, $2R1$ but $(1, 2) \notin R$. An example of a symmetric relation is siblinghood. Take any set of people $P$, for example $P := \{\text{parent 1, parent 2, child 1, child 2}\}$, and say that $p_1Rp_2$ is $p_1$ and $p_2$ are siblings. The digraph should (!) look like:

Note this relation is not reflexive.
If we have a relation $R$ such that whenever $aRb$ and $bRc$ then $R$ is said to be transitive. In this case, whenever a pair of arrows $a \rightarrow b \rightarrow c$, we also have $a \rightarrow c$. Siblinghood is NOT a transitive relation. It is an exercise to determine whether the other two relations in this section are. A good example of a transitive relation is less than. Let $S = \{1, 2, 3, 4\}$ and let us draw the digraph for $aRb$ if $a < b$:

Note this relation is neither reflexive nor symmetric. The relation $\leq$ is reflexive but not symmetric.

Consider the following $R = \{(P, R), (R, S), (S, P)\}$ on $X = \{R, P, S\}$:

Is this transitive? What is this?

**Further Remark: Non-Transitive Dice**

Let $X = \{D_1, D_2, \ldots, D_n\}$ be $n$ dice and define a relation $R$ on $X$ by:

$$(D_i, D_j) \in R \iff P[D_i > D_j] > \frac{1}{2}.$$ 

If the three dice are identical, then this is the empty relation. However consider three dice with sides:

- $D_1 \sim 2, 2, 4, 4, 9, 9$
- $D_2 \sim 1, 1, 6, 6, 8, 8$
- $D_3 \sim 3, 3, 5, 5, 7, 7$

It can be shown that

$$R = \{(D_1, D_2), (D_2, D_3), (D_3, D_1)\} \text{ so } (D_1, D_3) \notin R.$$ 

The digraph of this relation is isomorphic (same-shape) to the digraph of Rock-Paper-Scissors.
Now, consider the following game, which is played with a set of dice\textsuperscript{2}.

1. The first player chooses a die from the set.
2. The second player chooses one die from the remaining dice.
3. Both players roll their die; the player who rolls the higher number wins.

If this game is played with a transitive set of dice, it is either fair or biased in favor of the first player, because the first player can always find a die that will not be beaten by any other dice more than half the time. If it is played with the set of dice described above, however, the game is biased in favor of the second player, because the second player can always find a die that will beat the first player’s die with probability $\frac{5}{9}$.

Relations that are reflexive, symmetric AND transitive are said to be equivalence relations. This is because they resemble equality:

1.4.2 Equivalence Relations & Partitions

It can be shown that if $R$ is an equivalence relation on $X$ that $R$ partitions the set $X$ into disjoint subsets ($A$ and $B$ are disjoint if $A \cap B = \emptyset$). For example, let $X$ be the set of students in this class and let $p_1Rp_2$ if $p_1$ is born in the same year as $p_2$.

- $R$ is reflexive because a person is born in the same year as themselves.
- $R$ is symmetric because if $p_1$ is born in the same year as $p_2$ then $p_2$ is born in the same year as $p_1$.
- $R$ is transitive because if $p_1$ is born in the same year as $p_2$ (say 1998), and $p_2$ is born in the same year as $p_3$, then $p_3$ must also be born in 1998 and so $p_1$ is born in the same year as $p_3$, i.e. $p_1Rp_3$.

If we represent the set $X$ using a diagram then we see how this forms a partition:

This works the other way in that a partition defines an equivalence relation in a natural way. These disjoint subsets are called the equivalence classes, so for example take $X$ again and partition it into programmes (or rather disjoint subsets) ‘Web’, ‘ITM’, etc.

\textsuperscript{2}robbed from Wikipedia
Now if we define a relation $R$ by

Note that if $R$ is a relation on $X$ then $E(a) \times E(a) \subset R$, and the equivalence classes exhaust $R$ in this fashion. For example, say $E(a) = \{a,d,e\}$, $E(b) = \{b,f\}$ and $E(c) = \{c\}$. Then $R$ is given by:

**Examples**

1. **Winter 2019** Given $A = \{1, 3, 5, 9, 11\}$ and the relation $R$ on $A$ given by:

   $$R = \{(1, 1), (3, 3), (3, 1), (3, 9), (9, 3), (5, 5), (11, 5), (5, 11), (11, 11)\}.$$

   (a) Draw a graph that represents $R$.
   (b) Is $R$ reflexive? Give a reason for your answer.
   (c) Is $R$ symmetric? Give a reason for your answer.
   (d) Is $R$ transitive? Give a reason for your answer.

   **Solution:** The graph that is best for studying a relation is a **digraph**:
2. Autumn 2020 Let $T$ be the set of all actors. For $x, y \in T$, define $(x, y) \in R$ if there is some movie that both $x$ and $y$ appear in. Which properties of equivalence relations does the relation $R$ satisfy. Justify your answers.

Solution:

3. Given $E(1) = [1] = \{1, 3, 5\}$, $E(2) = [2] = \{2, 4\}$, $E(6) = [6] = \{6\}$ are the equivalence classes for a relation $R$ on $A = \{1, 2, 3, 4, 5, 6\}$.

(a) Write out the set $R$ in terms of ordered pairs.

(b) Draw a graph that represents $R$.

Solution:

(a) Recall that everything in an equivalence class is related to everything else in that class. Therefore:

(b) We can use a coordinate graph

![Figure 1.6: There is a pattern here but it is hard to spot: the relation is $\bigcup_{i \in A}(E[i] \times E[i])$.](image)
An arrow diagram is nasty but not too bad if we write the elements in the order 1, 3, 5, 2, 4, 6:

A digraph

And, in the special case of an equivalence relation, a partition diagram:

A very natural example is value: that elements of a set are considered equivalent if they have the same value. For example, let $S$ be the set of all objects in a shop, and suppose you are only allowed purchase one of each object. Then the set of shopping trolleys would be equal to $X := \mathcal{P}(S)$. We can define a symmetric relation on $X$ by saying that two shopping trolleys are related if their (euro)-value is equal. This is an equivalence relation. Similarly, we could say that two sums of cash, any currency, are related if they both convert to the same amount in euros. This is also an equivalence relation.

Exercises:

1. Autumn 2020 Let $A = \{a, b, c, d\}$ and the relation $R$ on $A$ is given by:

   $R = \{(a, a), (a, c), (b, b), (b, d), (c, a), (c, d), (d, a), (d, b), (d, d)\}$.

   i. Draw a directed graph that represents $R$.
   ii. Is $R$ symmetric? Give a reason for your answer.
   iii. Is $R$ transitive? Give a reason for your answer.
2. **Winter 2019** Given \( A = \{2, 4, 6, 8, 10\} \), and the relation \( R \) on \( A \) given by
\[
R = \{(2, 2), (2, 4), (4, 4), (4, 8), (8, 2), (8, 6), (8, 10), (10, 6), (10, 10)\};
\]
(a) Draw a directed graph that represents \( R \).
(b) Is \( R \) reflexive? Give a reason for your answer.
(c) Is \( R \) transitive? Give a reason for your answer.

3. **Autumn 2019** Let \( X = \{2, 4, 6, 8\} \) and \( R \) be a relation on \( X \):
\[
\{(2, 2), (2, 8), (4, 4), (4, 8), (6, 2), (6, 6), (6, 8), (8, 2), (8, 4), (8, 8)\}
\]
(a) Is \( R \) a symmetric relation? Explain.
(b) Is \( R \) a transitive relation? Explain.
(c) Is \( R \) a equivalence relation? Explain.

4. **Autumn 2020** The following set \( R \) defines an equivalence relation on the set \( \{1, 2, 3\} \).
\[
R = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}.
\]
i. Draw a graph that represents \( R \).
ii. List the equivalence classes.

5. Let \( X = \{1, 2, 3, 4, 5\} \) and define a relation \( R \subset X \times X \) by
\[
(x, y) \in R \text{ if and only if } x - y \text{ is divisible by 2.}
\]
Write out the elements of \( R \). This is an equivalence relation. What does this mean? Determine all the equivalence classes of \( R \).

6. Let \( A = \{\text{dog, cat, goose, lemur, rabbit}\} \). Define a relation \( R \) on \( A \) by the following:
\[
(w_1, w_2) \in R \iff w_1Rw_2 \iff \text{the word } w_1 \text{ shares a letter with the word } w_2.
\]
So, for example, (rabbit, cat) \( \in R \), ‘rabbit’ \( R \) ‘cat’ because both words have an ‘a’.
(a) Graphically represent the relation \( R \) using a digraph.
(b) Hence, or otherwise, determine if \( R \) is:
   i. reflexive. Justify your answer.
   ii. symmetric. Justify your answer.
   iii. transitive. Justify your answer.
(c) Is \( R \) an equivalence relation? Justify your answer.

7. Given \( E(a) = [a] = \{a, b\} \), and \( E(c) = [c] = \{c, d\} \) are the equivalence classes for a relation \( R \) on \( A = \{a, b, c, d\} \).
(a) Write out the set \( R \) in terms of ordered pairs.
(b) Draw a graph that represents \( R \).
8. **Winter 2018** Let $X = \mathcal{P}(S)$ be the power set of the set $S = \{a, b, c, d\}$.

(a) Determine how many subsets of $S$ contain exactly three elements.

(b) The relation

$$R = \{(A, B) : A, B \in X \text{ and } |X| = |Y|\}$$

is an equivalence relation. Write out the equivalence class $E(\{a, b, c\})$.

9. Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation $R$ on $A$ by the following:

$$xRy \iff x \text{ is a factor of } y \iff x \text{ divides } y \iff \frac{y}{x} \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots \}.$$ 

For example, $(1, 1) \in R$ because $\frac{1}{1} = 1 \in \mathbb{Z}$; $(1, 2) \in R$ because $\frac{2}{1} = 2 \in \mathbb{Z}$; but $(3, 1)$ is not in $R$ because $\frac{1}{3}$ is not an element of $\mathbb{Z}$ (i.e. not a whole number). You may assume that $R$ is a *transitive relation*.

(a) Write down all the ordered pairs of $R$.

(b) Hence, graphically represent the relation $R$ using a *digraph*.

(c) Hence, or otherwise, determine if $R$ is:

i. reflexive. Justify your answer.

ii. symmetric. Justify your answer.

(d) Is $R$ an equivalence relation? If you answer yes, write down the equivalence classes of $R$. If you answer no, justify your answer.
Chapter Summary

1. A set is a well-defined collection of distinct\footnote{so \{a, a\} = \{a\} and \{a, b\} = \{b, a\}} objects.

2. An object, $x$, in a set, $A$, is a member or element of $A$; written $x \in A$, said ‘$x$ in $A$’.

3. The cardinality of a finite set, $A$, is the number of elements it contains. It is denoted by $|A|$.

4. A subset, $B$, of a set, $A$, denoted $B \subseteq A$, is a set such that every element of $B$ is also an element of $A$. It can be thought of a selection of some of the elements of $A$. Often $B \subset A$ means that $B \neq A$ is a strict subset. To emphasise this we might write $B \subset A$.

5. The powerset of a set $A$, denoted $\mathcal{P}(A)$, is the set consisting of all of the subsets of $A$. It’s cardinality is $|\mathcal{P}(A)| = 2^{|A|}$.

6. The union of two sets $A$ and $B$, denoted $A \cup B$, is a set consisting of those elements that belong to $A$ or $B$ (or both).

7. The intersection of two sets $A$ and $B$, denoted $A \cap B$, is a set consisting of those elements that belong to $A$ and $B$.

8. The Inclusion-Exclusion Principle is given by:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$ 

9. The empty set, $\emptyset$ or $\{\}$, is the set with no elements.

10. The difference of two sets $A$ and $B$, denoted $A \setminus B$, consists of those elements that belong to $A$ but not $B$.

11. Given an ambient set $U$, the complement of $A$ is the set $U \setminus A$, also denoted $\overline{A}$, not $-A$, $A'$, $A^c$.

12. The Cartesian product of two sets $A$ and $B$, denoted $A \times B$, consists of the set of all ordered pairs, $(a, b)$, such that $a \in A$ and $b \in B$.

13. A relation between two sets $A$ and $B$ is a subset of $A \times B$... some ordered pairs.

14. A relation on $A$ is a relation between $A$ and itself; a subset of $A \times A$; best studied with a digraph.

15. A relation $R$ on $A$ is reflexive if for all $a \in A$, $aRa$, $(a, a) \in R$; if all nodes in the digraph have loops.

16. A relation $R$ on $A$ is symmetric if whenever $aRb$ we also have $bRa$; if all edges in the digraph are double edges.

17. A relation $R$ on $A$ is transitive if whenever $aRb$ and $bRc$ we also have $aRc$; if nodes connected in ‘two steps’ are connected in one.

18. A relation $R$ on $A$ that is reflexive, symmetric, and transitive is said to be an equivalence relation.

19. If $R$ is a relation on $A$, then the equivalence class of $a \in A$ is the set of elements related to $a$:

$$[a] = E(a) = \{x \in A : (a, x) \in R\}.$$
Facebook isn’t helping you make new connections, Facebook doesn’t develop new relationships, Facebook is just trying to be the most accurate model of your social graph. There’s a part of me that feels somewhat bored by all of this.

Sean Parker

Figure 2.1: Paul Erdős was a fascinating mathematician who used to work in the area of networks. He collaborated with more than 500 mathematicians. These people are said to have an Erdős Number of one. Those who collaborated with a collaborator of Erdős have an Erdős Number of two. Einstein has an Erdős Number of two. It is believed the highest Erdős Number is 13. The Kevin Bacon Numbers are a spin-off of this idea.
2.1 Introduction to Networks

2.1.1 The Seven Bridges of Königsberg

At its very basic level, a network is a selection of nodes joined by edges. This theory began with the Bridges of Königsberg Problem. Given the city of Königsberg (modern day Kaliningrad), find a route through the city crossing every bridge exactly once:

Task 1

Figure 2.2: Find a route through the city that crosses every bridge exactly once. If no such route exists, explain why. Define clearly the terms you use.

Hints:

1. Consider this easier problem:

Figure 2.3: Find a route through this city that crossing every bridge exactly once.
2. Consider this problem:

![Network Diagram](attachment://network_diagram.png)

Figure 2.4: Find a route through this city that crossing every bridge exactly once.

3. Notice that the ‘islands’ are like nodes separated by edges:

![Network Diagram](attachment://network_diagram.png)

Figure 2.5: This is reducing the Bridges of Königsberg Problem to its basic constituents. Find a route through this network which goes through every edge exactly once.

4. No closer to a solution? Try finding a route around this network that goes through every edge exactly once:

![Network Diagram](attachment://network_diagram.png)

Figure 2.6: You should be able to find a route that starts at either of the bottom nodes. It is not possible to find a route that starts elsewhere.

5. If you are still no closer to solving the problems:

*Except for the ____ and ____ nodes, every time the route ____ a node via an edge, another edge must ____ the node. Therefore except for two nodes, every node must have an ____ number of edges.*
Task 2

From what you have learned in Task 1, write down a theorem about such routes. We use the language of nodes and edges.

1. come up with a term to describe ‘route’ (if you can do better than route),

2. come up with a term to describe a ‘route’ that visits every edge exactly once,

3. come up with a term to describe how many edges ‘hit’ a node.

It turns out that the barrier to solving the Bridges of Königsberg is a barrier for all networks. It turns out it is the only barrier. That is, whenever this barrier does not exists, there is always a route through all edges that uses an edge only once. The proof of this is beyond the scope of the course.

**Theorem:** A network has a ______ if and only if there are ____ or ____ nodes of ____ ______.

**In-Class Exercise:** Use your theorem to answer the following. Which of graphs 1-10 have a ‘route’ that uses every edge exactly once. Find the routes.

We will learn later that the terms used give the following theorem:

**Theorem (Eulerian Trail)**

A (connected) network has an Eulerian Trail if and only if there are either two or zero nodes of odd degree.

**Remark:** We will see that the degree of a vertex/node is the number of edges that ‘hit’ it (with loops counting twice). If there are zero nodes of odd degree then there is an Euler Circuit — which can start and end at any vertex/node. If there are two nodes of odd degree, then you start at one of these nodes and finish at the other — using all the edges but not ‘looping back’ to where you started.

### 2.2 Basics of Network Theory

#### 2.2.1 Digraphs — Directed Graphs

In Section 1.4.1 Properties of Relations, we saw how a relation on a set $S$ gives rise to a directed graph or a digraph. Recall that a relation on $S$ is a subset $R \subseteq (S \times S)$ (of ordered pairs). The digraph of $R$ has

- each element of $S$ represented by a node, and

- each relationship $(x, y) \in R$ represented by a directed edge/arrow from node $x$ to node $y$.

Note that it is possible to have loops (elements $(x, x) \in R$) but it is not possible to have multiple directed edges (because a relation is a subset, it is a set, and therefore duplication is not allowed).
Examples

1. Autumn 2019 Let $D$ be a digraph with vertices $V = \{a, b, c, d, e\}$ and edges
   \[ E = \{(a, a), (a, b), (a, c), (a, e), (b, a), (b, d), (c, c), (c, e), (d, d), (d, e), (e, d)\} \].
   
   i. Draw this digraph.
   
   ii. The set $E$ is a relation on $V$. Is this relation reflexive? How can this be seen from the digraph you drew for part i.?

   Solution: We draw:

2. Winter 2017 List the elements of $V$ and $E$, where $V$ is the set of vertices, and $E$ is the set of edges of the graph represented in the figure below:

   Figure 2.7: Remark (not on exam paper): the arrows imply this is a directed graph.

   Solution: We have
Exercises:

1. **Winter 2019** Consider $V = \{a, b, c, d, e, f\}$, the set of all vertices in the following graph. Define a relation $R$ on $V$ as follows: for any $x, y \in V$, $(x, y) \in R$ if there is a walk from $x$ to $y$ with an even number of edges.

   (a) Explain why $R$ is:
   
   i. reflexive,
   ii. symmetric,
   iii. transitive.

   (b) What are the equivalence classes?

   ![Graph](image)

2. **Winter 2018** Consider the following undirected graph:

   ![Graph](image)

   Write down the set of vertices $V$ and the set of edges $E$.

3. **Autumn 2018** List the elements of $V$ and $E$, where $V$ is the set of vertices, and $E$ is the set of edges, of the graph represented on the next page.

4. Let $A = \{1, 2, 3, 4\}$ and define a relation $P$ on $A$ by $(a_1, a_2) \in P$ if and only if $a_2 - a_1$ is an integer multiple of two. Draw the digraph of $P$.

5. Let $L = \{a, b, c, d, e\}$ and define a relation $T$ on $L$ by $l_1 T l_2$ if $l_1$ is a vowel and $l_2$ and consonant or $l_1$ is a consonant and $l_2$ a vowel. Draw the digraph of $T$.

6. Let $X = \{0, 1\}$. Define a relation $\subseteq$ on $\mathcal{P}(X)$ by $(S_1, S_2) \in \subseteq$ if and only if $S_1 \subseteq S_2$. Draw the digraph of $\subseteq$. 
2.2.2 Networks

Considered as a network, what is the difference between Twitter and Facebook?

Networks that are symmetric can be represented by undirected graphs or networks or simply graphs (not to be confused with the graph of a function). The difference between a non-symmetric and symmetric graph can be captured by the language of relations. Below we have two digraphs with nodes $V = \{a, b, c\}$. The one on the left is symmetric, the one on the right is not:

If we are dealing with a symmetric relation (symmetric directed graph) we might replace the two edges $(a, b)$ and $(b, a)$ with a single, undirected, edge $\{a, b\}$. Loops $a \rightarrow a$ are represented by $\{a\}$ and not $\{a, a\}$. Why?

Definition

An undirected graph or network or graph consists of a set $V$ of nodes or vertices and a set of edges $E$ given as a set of one- and two-element subsets of $V$. 
Example
Let $V = \{1, 2, 3, 4, 5\}$ (vertices/nodes) and define a relation $R$ on $V$ by

$$(x, y) \in R \iff |x - y| = \text{dist}(x, y) \leq 2.$$ 

The elements of $R$ are given by:

This is a symmetric relation and so can be represented by an undirected graph. What are the edges?

Hence the graph is given by:

Exercises
1. Let $G$ be a graph with vertices $V = \{a, b, c, d\}$ and edges $E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$. Draw this graph.
2.3 Properties of Graphs

2.3.1 Connectedness

An (undirected) graph is *connected* if, for each pair of nodes $v_1, v_2 \in V$, there exists a series of edges $e_1, e_2, \ldots, e_n$ ‘joining’ $v_1$ to $v_2$. For example, the graph on the left is connected, while the graph on the right is not connected:

The separate connected elements of a disconnected (undirected) graph are called *components*. How can we tell whether a graph is connected or not?

**Task 3 A**

Given an undirected graph, describe an algorithm (a series of tasks) that determines whether or not the graph is connected. Here are some graphs to consider to help you find such an algorithm:

Figure 2.8: Come up with an algorithm that determines whether or not a graph is connected.
Here is one possible algorithm:

**Exercise:** Implement the Task 3 A Algorithm to determine which of graphs 1-10 on the exercise sheet are connected.

### 2.3.2 Degree

We already met the concept of degree when looking for a ‘trail’ through an undirected graph that uses all edges. Apart from the starting and ending nodes, each node must have an even number of edges for such a route to exist. This number of edges is the degree of a node. For example, in the following graph:

The nodes have the following degrees:

‘Degree’ is a function deg : V → \( \mathbb{N}_0 \) so this might be written as:
Loops are counted as double edges so that the degree of node $a$ below is three:

**Task 4**

1. Go into a group of size greater than three.

2. Each person in the group is a node so that $V$ is the set of people in the group. Draw the nodes below, using a letter or letters to denote each person:

3. Now *some* people in the group are to shake each others hands (or fist bump, high five or whatever) (a two-element element of $\mathcal{P}(V)$).

4. Every time two people shake hands, draw an edge between them.

5. Calculate for this graph, the degrees of the nodes.

6. Add up the degrees and make an observation about the answer.

7. Count up the number of nodes of odd degree. Make an observation about the answer.
Complete the following.

**Handshaking Lemma**

In an undirected graph:

1. the sum of the degrees is ________
2. the sum of the degrees is equal to twice ________
3. the number of nodes of odd degree is ________

*Exercise:* In Graphs 1-10, give the degree/valency of each vertex.

### 2.4 Trees, Trails & Cycles

Many important applications of network theory involve travelling round the (undirected) graph, in the sense of moving from node to node along incident edges.

#### 2.4.1 Definitions

A *walk* in a graph $G$ is a sequence of edges:

The walk is represented more compactly by:

$v_0$ is the initial node and $v_n$ is the final node. For example, given the following (undirected) graph:

A walk in which all the edges are distinct is a *trail*. For example,

A trail in which all the nodes are distinct is a *path*. A path therefore is a walk in which neither
nodes nor edges are repeated (expect perhaps for the ‘first’ node). For example,

A path in which the initial and final nodes are equal is a *cycle*.

Two graphs \((V_1, E_1)\) and \((V_2, E_2)\) are *isomorphic* (literally “same-shape”) if there is a bijection \(f : V_1 \to V_2\) such that \(\{v_i, v_j\} \in E_1\) implies that \(\{f(v_i), f(v_j)\} \in E_2\) for all edges \(\{v_i, v_j\} \in E_1\). In other words \(f\) is nothing but a relabelling of the nodes. For example the following two graphs are isomorphic:
A tree is a connected simple graph (no loops) with no cycles. Examples include:

Trees with $n$ nodes always have $n - 1$ edges.

Example: Family Tree

What do you call an unconnected graph whose components are trees?

**Task 5: The number of Trees with $n$ Nodes**

Where $T(n)$ is the number is the number of non-isomorphic trees with $n$ nodes, find $T(1)$, $T(2)$, $T(3)$, $T(4)$ and $T(5)$. $T(6) = 6$. Draw the six trees.

**Exercise:** Determine which of graphs 1-10 on the exercise sheet are trees.
Further Remark: Decision Trees and Random Forest

In the very popular field of Data Science/Machine Learning, Trees are used to build binary classifiers. For example, suppose you want to classify an email as spam or not-spam. What you do is you get a large database of emails that are labelled spam and non-spam. You use machine learning techniques to extract a number of predictor variables, $v_1, v_2, \ldots, v_p$, and, where $\text{ran}(v_i)$ is the range of values that $v_i$ can take, together they live in a feature space:

$$\mathcal{F} := \text{ran}(v_1) \times \text{ran}(v_2) \times \cdots \times \text{ran}(v_p),$$

and you build a model based on the data:

$$\text{classifier} : \mathcal{F} \to \{\text{spam, not-spam}\}.$$

One such model is a decision tree:

---

Figure 2.9: This is a decision tree for a ‘Good loan - Bad loan’ classifier.
These decision trees tend to overfit the data. This means that they tend not to predict unseen data well. A better algorithm is the random forest:

![Random Forest Diagram](http://www.nrronline.org/)

Figure 2.10: Rather than building one decision tree, build an ensemble of decision trees: a forest. Each tree in the forest only considers a (small) random subset of the feature variables: hence the name random forest. [Credit: http://www.nrronline.org/].

### 2.5 Eulerian Trails

This concept we met before (Bridges: use every edge exactly one):

**Definitions**

- An **Eulerian Trail** is a trail which contains every edge of the graph.
- An **Eulerian Circuit** is a closed trail \((v_0 = v_n)\) which contains every edge of the graph.

**Theorem**

- A (undirected, connected) graph has an Eulerian Trail if and only if it contains exactly zero or two nodes of odd degree.
- A (undirected, connected) graph has an Eulerian Circuit if and only if every node has even degree.

**Question:** Can there be a (simple undirected) graph with one node of odd degree?

The above theorem tells you exactly when there is an Eulerian Trail/Circuit and exactly when there is not. It does not however tell you how to find such a Trail/Circuit.
There is a theorem called *Fleury’s Algorithm* that does this. First, another definition, a *bridge* is an edge such that if that edge is removed the graph becomes disconnected. For example,

**Fleury’s Algorithm**

1. Make sure that the graph has either 0 or 2 nodes of odd degree. Otherwise, no such Eulerian Trail/Circuit exists.

2. If there are 0 edges of odd degree, then you can pick any node as your starting point. Otherwise, pick one of the odd nodes.

3. Build your path by choosing edges one at a time. If you have a choice between a bridge and a non-bridge, always choose the non-bridge. Do not pick an edge twice.

4. Stop when you run out of edges.

![Figure 2.11: This graph has an Eulerian trail but not an Eulerian Circuit.](image-url)
Example: Winter 2017

i. State a condition that guarantees that a graph with \( n \) vertices will have an Euler Circuit.

ii. Find an Euler Circuit in the graph below.

![Graph Image]

**Solution:**

i. This is just bookwork:

ii. Off ye go:

![Graph Image]

**Exercises:**

1. Autumn 2019

   i. What is the definition of an *Eulerian Cycle*?

   ii. Give a condition on a graph that guarantees the existence of an Eulerian Cycle.

   iii. A computer game consists of a player moving from vertex to vertex in a graph, in such a way that if the player traverses an edge, it is destroyed. The aim of the game is to destroy as many edges as possible. For example, the graph below had the player move \( A \to B \to C \).

   ![Game Image]
Thereafter the edges \( \{A, B\} \) and \( \{B, C\} \) are destroyed:

Consider the below graph. Can a player destroy all the edges? Justify your answer.

2. Winter 2019 Define what is meant by an Euler circuit, and find one for the following graph known as Muhammad’s Scimitars:

3. Determine which of graphs 1-10 on the exercise sheet have Eulerian Trails/Circuits. For those which have such a trail/circuit, find it.
4. **Autumn 2020** For the following graph:

   i. Give the degree of each vertex.

   ii. Is the graph a tree? Explain.

5. **Winter 2018** Consider a computer network given by the following (undirected) graph:

   i. Give the degree of each vertex.

   ii. Is the graph connected? Give a reason for your answer.

   iii. Is the graph a tree? Give a reason for your answer.

   iv. Does the graph have an Euler Cycle? If yes, find one. If not, explain your answer.
6. Is this graph a tree? Give a reason for your answer.

7. Autumn 2020 A tournament consists of five teams, where each team plays every other team exactly once.
   
i. Model this using a graph. Identify the associated sets $V$ and $E$.
   
ii. Give the degree of each vertex and hence find the sum of all the degrees in the graph.
   
iii. Hence determine the number of games.
Chapter Summary

1. A digraph (directed graph) is a set of vertices $V$ together with a set of directed edges $E$ given by a relation on $V$ (a subset of $V \times V$). Nodes $v_i, v_j \in V$ are joined by a directed edge, if and only if $(v_i, v_j) \in E$.

2. Loops are edges of the form $(v_i, v_i) \in E$.

3. (Undirected) Graphs or Networks are equivalent to digraphs given by symmetric relations. The elements $(v_i, v_j), (v_j, v_i)$ are replaced by a single element $\{v_i, v_j\}$ (and an edge drawn between them), and the element $(v_i, v_i)$ is replaced by a single element $\{v_i\}$ (with a loop $v_i \rightarrow v_i$). Therefore a graph is a set of nodes $V$ together with a set of edges $E$ given as a set of one- and two-element subsets of $V$.

The remaining points concern (undirected) graphs:

4. A simple graph is an (undirected) graph with no loops.

5. A walk on a (simple) graph is a sequence of edges:

$$\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\},$$

briefly

$$v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \cdots \rightarrow v_n.$$  

$v_0$ is known as the initial node and $v_n$ the final node.

6. A graph is connected if there exists a walk between any two nodes. To show that a graph is connected write down a walk containing all vertices of $V$.

7. The connected subgraphs of an unconnected graph are called components.

8. The degree or valency of a node is the number of edges incident to the node (loops count twice).

9. The sum of the degrees of the nodes is even. As a corollary (consequence), the Handshaking Lemma says that the number of nodes of odd degree is even.

10. A trail is a walk in which all the edges are distinct.

11. A path is a trail in which all the nodes are distinct.

12. A cycle is a path $v_0 \rightarrow \cdots \rightarrow v_n$ such that $v_0 = v_n$.

13. A tree is a simple, connected graph with no cycles.

14. An Eulerian Trail is a trail which contains every edge of the graph (exists if exists zero or two nodes of odd degree). If not a circuit, the trail starts and ends at the nodes of odd degree.

15. An Eulerian Circuit is an Eulerian Trail $v_0 \rightarrow \cdots \rightarrow v_n$ with $v_0 = v_n$ (exists if zero nodes of odd degree).
Chapter 3

Functions I - Theory

*Mathematics is the art of giving the same name to different things.*

Henri Poincaré

Figure 3.1: Category Theory is more abstract than this stuff again and becoming more popular in computer science.
3.1 Functions as Relations

The abstract definition of a function is as follows:

Definition

A function \( f : X \rightarrow Y \) is a relation between \( X \) (\( \sim \) inputs) and \( Y \) (\( \sim \) outputs) such that each element \( x \in X \) appears in exactly one ordered pair \( (x, y) \in X \times Y \). Therefore a function \( f \in \mathcal{P}(X \times Y) = \{ (x, y) \} = \{ \text{(input,output)} \} \).

The crucial features are

- every element of \( X \) appears in an ordered pair
- every element of \( X \) appears in only one ordered pair.

If \( (x, y) \in f \), then we can write \( x \mapsto y \): \( x \) is mapped to \( y \) by \( f \). We can also write \( y = f(x) \): \( y \) is the image of \( x \) under \( f \).

The set \( X \) is known as the domain/source (inputs), the set \( Y \) is known as the codomain/target (outputs), and the range is the subset of \( Y \) that is contained in at least one ordered pair: the set of things that are ‘hit’. As functions are relations we can represent them using arrow diagrams:

Sometimes, a function \( f : X \rightarrow Y \) can be specified by a ‘rule’/formula \( f(x) \), and the set of ordered pairs given by:

Given an element of the domain \( X \), say \( x \), the ‘rule’ gives a way of finding the image of \( x \) under \( f \), \( f(x) \).

Example and Non-Example

1. Let \( P = \{ \text{people in this room} \} \). Consider the relation \( R \subseteq P \times \mathbb{N} \):

This is a function because every person in the room has an age and no person has two ages (each person only occurs in one ordered pair). It is absolutely fine that \( 19 \in P \) occurs in more than one ordered pair. The domain is \( P \), the people in this room. The codomain is \( \mathbb{N} \). The range is not all of \( \mathbb{N} \) but the subset:
Represented as an arrow diagram, each element $p \in P$ is mapped to a single $a \in \mathbb{N}$. It is fine that different people $p \in P$ can be mapped to the same age $a \in \mathbb{N}$:

2. Let $S = \{1, 2, 3\}$ and define a relation $R$ on $S$ by:

$$R = \{(1, 1), (1, 2), (2, 3)\}$$

This is not a function (why)? Let us look at its arrow diagram:

3. Let $X = \{1, 2, 3, 4\}$. Define a function, $s : X \to \mathbb{N}$ by $s(x) = x^2$. Then the ordered pairs of $s$ are given by:

What is the domain? The codomain? The range?
Further Remark: Dependent & Independent Variables

The concept of a function began with Galileo to describe variables that depend on other variables. For the moment we can think of variables as properties that can be measured. Suppose we have variables $x$ and $y$ with perhaps some of the following properties:

A1. we can control the value of $x$; e.g. the word length $x \sim L$ of an essay.

A2. the value of $x$ doesn’t seem to depend on other variables; e.g. $x \sim t$, time.

B1. given a measurement of $x$, we know the value of $y$; e.g. if we know the length of a square, $x \sim s$, we know its area, $y \sim A$. In this example, $s$ seems to have a causal relationship with $A$: $A$ depends on $s$, $A$ is a function of $s$.

B2. for a fixed value of $x$, there is a single value of $y$; e.g. for $x \sim t = 10$ days after an app is launched, the total sales are given by the figure $y \sim S = 1,238$. In this example, $t$ does not seem to have a causal relationship with $S$: $S$ does not depend on $t$.

If you have a relationship between variables $x$ and $y$ that satisfy A1 and B1 you can control and record the value of $x$ and measure and record the corresponding value of $y$. For example, $x \sim s$, side-length; and $y \sim A$, area, satisfy properties A1 and B1. If you record $s$ and its corresponding value of $A$ and record them as ordered pairs:

$$\{(1, 1), (2, 4), (3, 9), (4, 16), \ldots \}.$$ 

Now, being a bit fast and loose with the domain — let them be the real numbers — we see that $s \mapsto s^2$, that is $A = s^2 = f(s)$, so that $f : \mathbb{R} \to \mathbb{R}$, $s \mapsto s^2$ is a function. We can say that area is a function of side-length and we can write stuff like:

$$A = A(s) = s^2.$$ 

Here side-length, $s$, is independent, $s$ can take on any value, but once $s$ is chosen, the area, $A$, is determined: $A$ is dependent on $s$. We can say that $s$ is the independent variable with domain $\mathbb{R}^+$, $A$ is the dependent variable with codomain $\mathbb{R}$, and area is a function of side-length, $A = f(s)$.

What about variables that satisfy A2 and B2? For example, the total sales of an app, $S$, $t$ days after launch. Now $t$ doesn’t cause $S$, and knowing $t$ doesn’t automatically give you the value of $S$, but if you measure and record $t$ and $S$ at the same time, the ordered pairs look very much like that of a function:

$$\{(1, 10), (2, 23), (3, 46), (4, 78), (5, 100), \ldots \}.$$ 

Indeed each day has only one corresponding cumulative sales figure (so no day $t$ occurs in two different ordered pairs), and if the domain of $t$ is given by $\mathbb{N}$ (or perhaps a cut of $\mathbb{N}$, $\{1, 2, 3, \ldots, n\}$), then every day has a cumulative sales figure. These ordered pairs are that of a function. Can we identify the independent and dependent variables? The domain and codomain?

$$S : \mathbb{N} \to \mathbb{N}; \quad t \mapsto S(t).$$ 

So, in context — when we want to think of one variable depending on another — or when for each value of a variable $x$ there is a single value of a variable $y$ — when we have a function $f : X \to Y$, then $x \in X$ is the independent variable, $y \in Y$ is the dependent variable, and we can write $y = f(x)$: $y$ is a function of $x$. We can also, in this picture, think of $x \in X$ as an input, and $f(x) = y \in Y$ as the corresponding output.
Exercises

1. **Winter 2017** Let $X = \{0, 1, 2, 3, 4\}$ and $Y = \{8, 9, 10, \ldots, 16\}$. Define $f : X \to Y$ as $f(x) = 2x + 8$. List the ordered pairs of the relation that define this function.

2. Let $X = \{0, 1, 2, 3\}$. There are 256 functions $X \to X$. Choose one of the 256 and answer the following:
   (a) Write down the ordered pairs defining the function.
   (b) Represent it using a digraph.
   HINT: Choosing a function $f : X \to X$ amounts to choosing four values: for $f(0), f(1), f(2), f(3)$, so that you choose an $f$ by filling in:
   
   $$f = \{(0, \_), (1, \_), (2, \_), (3, \_}\}.$$  

3. Let $X = \{1, 2, \ldots, 10\}$. Define the successor “function” $S : X \to X$ by $S(n) = n + 1$.
   (a) Why is $S : X \to X$ not a function?
   (b) * Give an example of a domain, $D$, such that $S : D \to D$ is a function.

4. A relation $R$ on $X = \{1, 2, 3, 4\}$ is defined by $xRy$ if and only if $x > y$.
   (a) Write down the ordered pairs that define this relation.
   (b) Graphically represent the relation using a digraph.
   (c) Is this relation a function? Justify your answer.

5. * Let the “function” $f : \mathbb{R} \to \mathbb{R}$ be defined by the formula $f(x) = \frac{1}{x}$.
   (a) Why is $f$ not a function?
   (b) * Give an example of a domain $D$ such that $f : D \to \mathbb{R}$ is a function.

6. * Is the function $f : \mathbb{Q} \to \mathbb{Z}$ defined by $f\left(\frac{m}{n}\right) = m$, a function?

### 3.2 Graphs of Functions

The graph of a function $f : A \to B$ consists of all the ordered pairs of $f$. Note that every function satisfies the *vertical line test*. A relation on $\mathbb{R}$ is a relation from $\mathbb{R}$ to $\mathbb{R}$ — a subset of $\mathbb{R} \times \mathbb{R}$, i.e. the plane. A 2D curve/region/shape is a subset of the plane; i.e. a relation on $\mathbb{R}$. A function $\mathbb{R} \to \mathbb{R}$ is a relation — a subset of the plane — such that each element in the first $\mathbb{R}$ appears once and only once.
**Vertical Line Test**

A graph $G \subseteq A \times B$ is *not* the graph of a function $f : A \to B$ if a vertical line $x = a$ (for $a \in A$) intersects $G$ more or less than exactly once.

*Proof.* Suppose that such vertical line intersects the graph zero times... or twice:

![Figure 3.2: Suppose that the line $x = a_1$ does not intersect the graph and the line $x = a_2$ intersects the graph twice.](image)

What can we say about ordered pairs of the form $(a_1, y)$ and $(a_2, y)$?
Examples

1. Let \( S = \{1, 2, 3\} \) and define a relation \( R \) on \( S \) (subset of \( S \times S \); some ordered pairs) by:

\[
R = \{(1, 3), (1, 2), (2, 2)\}.
\]

We can represent this relation graphically as a network, graph, or arrow diagram:

![Graph of \( R \)](image)

Figure 3.3: When you graph a relation \( R \subseteq X \times Y \) in the conventional way — source on \( x \)-axis and target on \( y \)-axis — any graph which fails the vertical line test (a vertical line contains two elements of \( R \)) has at least one element \( x \in X \) that appears in two ordered pairs and so is not a function (or indeed the graph of any function).

2. Is the following the graph of a function?

![Graph](image)

Figure 3.4: This is the graph of the relation \( 2x^3 - y = 3y^2 \) \( ((x, y) \in R \) if and only if \( x, y \), satisfies the equation). Is it a function?
Exercises

1. The graph of \( f : \mathbb{R} \to \mathbb{R} \) is shown below. Write down the values of \( f(0) \), \( f(2) \), and \( f(5) \).

\[
\begin{array}{c}
\text{Exercises} \\
1. \text{The graph of } f : \mathbb{R} \to \mathbb{R} \text{ is shown below. Write down the values of } f(0), f(2), \text{ and } f(5).
\end{array}
\]

2. Winter 2017 Let \( B = \{0, 1\} \). There are four functions \( B \to B \). For each of the four functions:

(a) List the ordered pairs of the relation that define the function,
(b) Represent the function using an arrow diagram,
(c) Plot the graph using the following axes:

\[
\begin{array}{c}
\text{Winter 2017 Let } B = \{0, 1\}. \text{ There are four functions } B \to B. \text{ For each of the four functions:} \\
\text{(a) List the ordered pairs of the relation that define the function,} \\
\text{(b) Represent the function using an arrow diagram,} \\
\text{(c) Plot the graph using the following axes:}
\end{array}
\]
3. The following are graphs of eight relations on $X = \{0, 1, 2, \ldots, 10\}$. Which are graphs of functions?

4. Determine in each case whether the graph is that of a function:

3.3 1-1, Onto, Bijective

Recall that for relations: $R \cap S \cap T = E$. We are going to have that 1-1 and Onto = Bijective = Invertible.

3.3.1 Definitions

A function is 1-1 if each element of the range of $f$ occurs in one and only one ordered pair. You may also come across the term injective for 1-1.

A function $f : X \to Y$ is onto if the range of $f$ is the whole of $Y$. You may also come across the term surjective for onto.

A function $f : X \to Y$ that is both 1-1 and onto is called bijective: each element of $Y$ occurs in one and only one ordered pair. This means we have what might be called a perfect matching between $X$ and $Y$: each $x \in X$ is associated to a unique $y \in Y$, and vice versa. In the case of finite sets this means that $|A| = |B|$. When you move to the world of infinite sets, things get a little strange. For more, read about the Hilbert Hotel.
Using arrow diagrams, here are examples of not onto but 1-1, not 1-1 but onto, neither onto nor 1-1, and bijective:

We can tell if a function is 1-1 or onto by looking at the horizontal line test.

**Horizontal Line Test**

A function \( f: A \to B \) is

1. not 1-1 if any horizontal line \( y = b \) intersects the graph of \( f \) more than once,

2. not onto if any horizontal line \( y = b \) fails to intersect the graph of \( f \).

*Proof.* Suppose that such vertical line intersects the graph zero times... or twice:

![Diagram](image)

Figure 3.5: Suppose that the line \( y = b_1 \) does not intersect the graph and the line \( y = b_2 \) intersects the graph twice.

What can we say about ordered pairs of the form \((x, b_1)\) and \((x, b_2)\)?
Let us look at these in terms of arrow diagrams:

For bijections $f : A \rightarrow B$, every element of $B$ must be mapped to from a unique element of $A$.

Examples

1. Let $X = \{a, b, c\}$ and $Y = \{d, e, f\}$. Define $F : X \rightarrow Y$ by

$$\{(a, d), (b, e), (c, d)\}.$$ 

Is $F$ 1-1? Is $F$ onto? Is $F$ bijective? Look at an arrow diagram:

$$\text{Figure 3.6: In general, while digraphs are the way to analyse relations, arrow diagrams (above) are the best way to analyse functions. Here we use the horizontal and vertical line tests.}$$
2. Let \( X = \{a, b, c\} \) and \( Y = \{d, e, f, g\} \). Define \( F : X \to Y \) by
\[
\{(a, g), (b, f), (c, d)\}.
\]
Is \( F \) 1-1? Is \( F \) onto? Is \( F \) bijective?

3. Let \( X = \{a, b, c\} \) and \( Y = \{d, e\} \). Define \( f : X \to Y \) by
\[
\{(a, d), (b, e), (c, d)\}.
\]
Is \( f \) 1-1? Is \( f \) onto? Is \( f \) bijective?

4. Autumn 2019 Let \( X = \{-1, 0, 1, 2\} \) and \( Y = \{-2, -1, 1, 2\} \). Define the function \( f : X \to Y \) as \( f(x) = x^2 - 2 \).
   (a) List the ordered pairs of the relation that defines this function.
   (b) Is \( f \) one-to-one? Give a reason for your answer.
   (c) Is \( f \) onto? Give a reason for your answer.

Solution: Recall the ordered pairs are of the form \((x, f(x))\). We calculate:

Thus the ordered pairs are:
To study a function we can draw its arrow diagram:

5. Let $P$ be the set of students I had in this class in 2017/18. Define a function $Q : P \to \mathbb{R}$ by:

$$Q(p) = \begin{cases} 
\text{score of } p \text{ on Quick Test} & \text{if } p \text{ sat the QT}, \\
0 & \text{otherwise}.
\end{cases}$$

Note without the otherwise clause $Q$ is not a function on $P$. Why? The scores, ranked from top to bottom were:

$$22, 21, 18.5, 18.5, 18, 16.5, 15.5, 14.5, 14.5, 14, 14, 14, \ldots$$

If we call by $p_i$ the person who got the $i$th highest score (ties split alphabetically), what do the ordered pairs look like:

Is $Q$ 1-1? Is $Q$ onto? Is $Q$ bijective?
6. Consider the following graph of a relation on \( \mathbb{R} \):

\[
\begin{array}{c}
\text{Is it the graph of a function? Is it 1-1? Is it onto? Is it bijective?}
\end{array}
\]

What does the graph suggest is the value \( f(0) \) (what is zero sent to; \( 0 \mapsto f(0) \)), and \( f(3) \) (what is three sent to; \( 3 \mapsto f(3) \))?

For what value(s) of \( x \) is \( f(x) = 0 \) (what is mapped to zero; \( x \mapsto 0 \))?

What does the graph suggest are the value(s) of \( x \) such that \( f(x) = 4 \)?
7. Let $P$ the set of students in this class and define a function $ID : P \rightarrow N$, the set of student numbers, by:

Here, where $D = \{0, 1, 2, \ldots, 9\}$,

$$N = \{\mathbb{R}\} \times D^8,$$

where $D^8 = D \times D \times \cdots \times D$, eight times. Is ID 1-1? Is ID onto? Is ID bijective?

8. Define a function $I : \mathbb{N} \rightarrow \mathbb{R}$ by $I(n) = n$. $I$ is not onto because, for example, the element $\frac{1}{2} \in \mathbb{R}$ is not in the range. Is $I$ 1-1? Is $I$ bijective?

What if we changed the codomain from $\mathbb{R}$ to $\mathbb{N}$ to get a new function $I' : \mathbb{N} \rightarrow \mathbb{N}$. Is $I'$ onto? Is $I'$ 1-1? Is $I'$ bijective?

9. Consider the function $s : \mathbb{R} \rightarrow \mathbb{R}$ given by $s(x) = x^2$. Is $s$ 1-1? Is $s$ onto? Is $s$ bijective?

What about $s' : \mathbb{R} \rightarrow \mathbb{R}_+$ (positive real numbers). Is $s'$ 1-1? Is $s'$ onto? Is $s'$ bijective?

What about $s'' : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Is $s''$ onto? Is $s''$ 1-1... eh... for this we will need algebra. We state the problem and return to it later.
Exercises

1. Let $X = \{a, b, c\}$ and $Y = \{d, e, f\}$. Define $F : X \rightarrow Y$ by
   $$\{(a, f), (b, d), (c, e)\}.$$ 
   Is $f$ 1-1? Is $f$ onto? Is $f$ bijective?

2. Winter 2019 Let $X = \{1, 0, 1\}$ and $Y = \{2, 1, 0, 1, 2\}$. Define the function $f : X \rightarrow Y$ by $f(x) = -x^2 - x$.
   (a) List the ordered pairs of the relation that defines this function.
   (b) Is $f$ one-to-one? Give a reason for your answer.
   (c) Is $f$ onto? Give a reason for your answer.

3. Autumn 2018 Let $X = \{-1, 0, 1, 2\}$ and $Y = \{-4, -2, 0, 2\}$. Define the function $f : X \rightarrow Y$ as $f(x) = x^2 - x$. Is the function $f$ invertible?

4. The question(s) is in the captions: circle the correct answer(s).

![Figure 3.7: Function: Yes or No?](image-url)
5. Winter 2017 Show that the function \( f : \mathbb{N} \rightarrow \mathbb{N} \) given by \( f(n) = n + 2 \) in not onto.

6. * Let \( A \) and \( B \) be finite sets with \( |A| = m \) and \( |B| = n \).

   (a) If there is an onto function from \( A \) to \( B \), what can you say about the relationship between \( m \) and \( n \)?

   (b) If there is a 1-1 function from \( A \) to \( B \), what can you say about the relationship between \( m \) and \( n \)?
3.4 Composition and Inverses

3.4.1 Definitions

If you have functions \( g : X \to Y \) and \( f : Y \to Z \) you can form their composition \( (f \circ g) \) (\( f \) after \( g \)). The ordered pair definition is a little tricky:

However the formula definition, \( (f \circ g)(x) \), and arrow diagram, is more straightforward:

A composition can be represented by a chain of arrow diagrams. For example, the composition of \( g : \mathbb{R} \to \mathbb{R}, x \mapsto x^2 \), and \( f : \mathbb{R} \to \mathbb{R}, x \mapsto 3x \), denoted \( g \circ f \) (\( g \) after \( f \)) may be represented as:

If you have a bijective function \( f : X \to Y \) then it has an inverse, \( f^{-1} : Y \to X \). The ordered pairs of \( f^{-1} \) are just the ‘flips’ of the ordered pairs of \( f \). For example for \( X = Y = \{1, 2, 3, 4\} \) consider the bijective function:

An inverse may be represented as a backwards arrow:

If \( f \) is not onto then a \( y \in Y \) not in the range of \( f \) is not in the set of ordered pairs and so \( f^{-1}(y) \) is undefined: therefore \( f^{-1} \) is not a function. Similarly if \( f \) is not 1-1 then some \( y \) appears in at
least two ordered pairs of \( f \) and flipping these:

also leads to a non-function. Why?

Any set \( X \) has on it the \textit{Identity Function} \( I_X : X \to X \), \( I_X(x) = x \), consisting of the ordered pairs:

Equivalently given by the formula \( I_X(x) = x \) for all \( x \in X \). If \( f : X \to Y \) is invertible, then we have:

What is the inverse of \( f^{-1} : Y \to X \)? That is inverses come in pairs, and for any inverse pair we have:

By the horizontal line test a function \( f : A \to B \) is a bijection, and thus invertible, if every horizontal line \( y = b \) cuts the graph exactly once. In which case \( f \) is invertible, and the graph of \( f^{-1} \) is found by reflecting the graph of \( f \) through the line \( y = x \).

\textbf{Remark}

Functions that are 1-1 have a \textit{left} inverse. For example, recall the Student ID function: let \( P \) the set of students in this class and define a function \( \text{ID} : P \to N \), the set of student numbers, by:
Figure 3.10: Here we see the graph of $e^x$ and its inverse $\ln x$ — the graph of an inverse is found by reflecting through the line $y = x$. The graph of $e^x$ intersects (positive) horizontal lines only once: therefore $e^x$ is bijective $\mathbb{R} \to \mathbb{R}_+$. Therefore it has an inverse $\mathbb{R}_+ \to \mathbb{R}$: this is the natural logarithm.

The function ID is 1-1? The function $f : \text{range(ID)} \to P$, identifying to a student number the student is what is called a left-inverse:

$$f \circ \text{ID} = I_P.$$  

In terms of an arrow diagram:


Sometimes we might say that a 1-1 function is invertible on its range. Note that we do have

$$\text{ID} \circ f = I_{\text{range(ID)}} \neq I_N,$$

as $I_N : N \to N$. A 1-1 function is enough to identify objects uniquely.
Examples

1. Autumn 2019 Let $f = \{(0,8), (1,10), (2,12), (3,14), (4,16)\}$ and let $g(x) = x^2$.
   (a) Specify the domain and range of $f$.
   (b) Is $f$ invertible? If yes, list the elements of $f^{-1}$. Otherwise explain your answer.
   (c) Calculate $(f \circ g)(2)$ and $(g \circ f)(2)$.

Solution: The domain and range of $f$?

Best to analyse functions using an arrow diagram:

The compositions:
2. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^2 \), and \( S_4 : \mathbb{R} \to \mathbb{R} \) defined by \( S_4(x) = x \) rounded to four significant figures. Here we see the arrow diagram for \( (S_4 \circ f) \):

Note that this is different to \( (f \circ S_4(x)) \) as \( f(S_4(\sqrt{2})) = f(1.414) = 1.999396 \neq 2 = S_4(2) = S_4(f(\sqrt{2})) \).

3. Which of the following functions \( \{1, 2, 3, 4\} \to \{1, 2, 3, 4\} \) are invertible and why (arrow diagrams left and right):

\[ \{(1, 2), (2, 3), (3, 2), (4, 4)\} \]

\[ \{(1, 1), (2, 3), (3, 4), (4, 2)\} \]

4. Let \( a \in \mathbb{R} \) and define a function \( +_a : \mathbb{R} \to \mathbb{R} \):

What is the inverse of \( +_a \):

That is \( (+_a, -_a) \) is an inverse pair. Note \( +_a = -a \):

\^1some part of it anyway: some ordered pairs: \( (\sqrt{2}, 2), (1.414214, 2.000001 \ldots), (1.52, 2.3104), (-3.3, 10.89), (0.111, 0.012321) \)
5. Let $a \in \mathbb{R}\setminus\{0\}$ and define a function $\times_a : \mathbb{R} \to \mathbb{R}$ by:

What is the inverse of $\times_a$:

That is $(\times_a, \div_a)$ is an inverse pair. Note that $\times_{1/a} = \div_a$:

Why did we exclude $a = 0$?

6. We think that $(\sqrt{2}, \sqrt{\cdot})$ is an inverse pair. However if we look at the domains and codomains:

They are in fact not an inverse pair. However if we restrict $\sqrt{2} : \mathbb{R}_+ \to \mathbb{R}_+$ then they are an inverse pair:

7. Consider for $p \in \mathbb{N}$ the power function $\Box^p : \mathbb{R}_+ \to \mathbb{R}_+$:

What is the inverse of $\Box^p$:

That is $(\Box^p, \sqrt[1/p]{\cdot})$ are an inverse pair.

8. Let $\Box^x : \mathbb{R} \to \mathbb{R}_+$ be defined by:

What is the inverse of $\Box^x$:

That is $(\Box^x, \log_b(x))$ is an inverse pair:

In particular, $(e^x, \ln x)$ is an inverse pair:
Exercises

1. Let \( X = \{1, 2, 3, 4\} \). Let \( f : X \rightarrow \mathbb{R} \) be a function defined as the set of ordered pairs \( \{(1, 2), (2, 3), (3, 4), (4, 5)\} \). Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be the function defined as \( g(x) = x^2 \). Represent \( g \circ f \) using arrow diagrams.

2. Let \( X = \{1, 2, 3, 4\} \). Functions \( g : X \rightarrow \mathbb{R} \) and \( f : \mathbb{R} \rightarrow \mathbb{R} \) are defined by:
\[
g(x) = x^2 - 1 \quad \& \quad f(x) = 3x - 2.
\]
Represent \( f \circ g \) using an arrow diagram. Label all sets.

3. The following are arrow diagrams of relations from \( A \) to \( B \). For each relation answer the following:

   (a) Is the relation a function? If it is not explain why, and then go onto the next relation.

   (b) If the arrow diagram is of a function, is the function:
      i. 1-1?
      ii. onto?
      iii. invertible?

4. Consider the function \( f : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\} \) given by
\[
f = \{(1, 3), (2, 2), (3, 4), (4, 1)\}.
\]
   (a) Graphically represent the function using an arrow diagram. Label all sets.
   (b) Draw the graph of the function (using the axes on the next page).
   (c) Is the function invertible? If yes, write down the ordered pairs that define \( f^{-1} \). Otherwise, explain why the function is not invertible.

5. Winter 2017 Let \( f \) be the function that maps strings of characters and blank spaces onto strings of characters by removing all blank spaces and vowels. For example, \( f(\text{“dog cat”}) = \text{“dgct”} \). Let \( g \) be the function that maps strings of characters onto integers such that the value of a string is simply the number of characters (including blanks) in the string. What is \( f(\text{“Michael D Higgins”}) \)? What is \( g(\text{“Michael D Higgins”}) \)? What is \( (g \circ f)(\text{“Michael D Higgins”}) \)?

6. Let \( X = \{1, 2, 3, 4, 5\} \) and let \( f : X \rightarrow X \) be defined by:
\[
f(1) = 2, \quad f(2) = 2, \quad f(3) = 4, \quad f(4) = 4, \quad f(5) = 4.
\]
Determine whether or not \( f \circ f = f \).
7. Let \( f : \mathbb{Z} \to \mathbb{Z} \) and \( g : \mathbb{Z} \to \mathbb{Z} \) be defined by:
\[
 f(n) = n + 1, \quad g(n) = 2n.
\]
Determine whether or not \( f \circ g = g \circ f \).

8. Autumn 2018 Let \( X = \{1, 2, 3, 4\} \). Let \( f : X \to \mathbb{R} \) be a function defined as the set of ordered pairs \( \{(1, 2), (2, 3), (3, 4), (4, 5)\} \). Let \( g : \mathbb{R} \to \mathbb{R} \) be the function defined as \( g(x) = x^2 \).
List the ordered pairs of \( g \circ f \).

9. Let \( X = \{1, 2, 3, 4\} \) and \( Y = \{5, 6, 7, 8, 9\} \). Let \( f = \{(1, 5), (2, 7), (4, 9), (3, 5)\} \).
   (a) Is \( f \) a function?
   (b) Is \( f \) invertible? Explain your answer.

10. Let \( f = \{(0, 8), (1, 10), (2, 12), (3, 14), (4, 16)\} \).
    (a) Is \( f \) a function? Specify its domain and range.
    (b) Is \( f \) invertible? If yes, list the elements of \( f^{-1} \). Otherwise explain your answer.

11. Let \( f \) and \( g \) be the functions \( \mathbb{R} \to \mathbb{R} \) given by \( f(x) = 2x \) and \( g(x) = x^2 \). Find formulae for the functions
   (i) \( g \circ f \), (ii) \( f \circ g \), (iii) \( f \circ f \), (iv) \( f \circ g \circ f \), (v) \( (f \circ g) \circ f \), (vi) \( f^{-1} \circ g \), (vii) \( g \circ f^{-1} \).

12. For each of the following graphs of functions \( f : A \to B \), what does the graph suggest about the invertibility (or otherwise) of \( f \)?

\[
\begin{array}{c}
\text{Figure 3.11: A constant function}
\end{array}
\]
Figure 3.12: A line or affine function

Figure 3.13: The exponential function, from the real numbers to positive real numbers, $\mathbb{R} \rightarrow \mathbb{R}_+$. 
Figure 3.14: A vertically shifted $x^3$ function.

Figure 3.15: A quadratic function.
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Figure 3.16: The natural logarithm, from positive real number to real numbers, $\mathbb{R}_+ \to \mathbb{R}$.

Figure 3.17: The floor function, from positive real numbers to integers, $\mathbb{R} \to \mathbb{Z}$. It certainly isn’t invertible as a function $\mathbb{R} \to \mathbb{R}$ (what is floor$^{-1}(1/2)$ equal to?).

13. * Let $f : \mathbb{N} \to \mathbb{N}$ and define $f^i : \mathbb{N} \to \mathbb{N}$ by

$$f^i(n) = (f \circ f \circ \cdots \circ f)(n).$$

Explain why it is not possible that $f(1) = 2$, $f^2(1) = 4$, $f^3(1) = 9$, $f^4(1) = 2$ and $f^5(1) = 5$. 
Chapter Summary

1. A function, \( f : X \to Y \), is a subset of \( X \times Y \) (so a set of ordered pairs \((x, y)\)), such that each element \( x \in X \) occurs in one and only one ordered pair. The ordered pairs are denoted \((x, f(x)) \sim \text{(in, out)}\). The set \( X \) is called the domain, the set \( Y \) is called the codomain and the set
\[
\{ f(x) \mid x \in X \} \sim \text{all things that are ‘hit’},
\]
is called the range.

2. Given a function \( f : X \to Y \), the elements of \( X \) are called independent variables, and the elements of \( Y \) are called dependent variables.

3. A function \( f : X \to Y \) is 1-1 if each element \( y \in Y \) occurs in at most one ordered pair.

4. A function \( f : X \to Y \) is onto if each element \( y \in Y \) occurs is at least one ordered pair; that is everything is ‘hit’.

5. A function 1-1 and onto is bijective.

6. The composition of functions \( g : X \to Y \) and \( f : Y \to Z \) is a function \( (f \circ g) : X \to Z \) given by:
\[
(f \circ g)(x) = f(g(x)).
\]

7. The Identity function on \( X \), \( I_X : X \to X \), is given by (do nothing)
\[
I_X(x) = x.
\]

8. A bijective function \( f : X \to Y \) has an inverse function \( f^{-1} : Y \to X \): reversed ordered pairs and so with the property that:
\[
(f \circ f^{-1}) = I_Y \quad \text{and} \quad (f^{-1} \circ f) = I_X.
\]

9. The Vertical Line Test is used to see if a curve is the graph of a function.

10. The Horizontal Line Test is used to see if the graph of a function is that of a 1-1 and/or onto function.
Chapter 4

Algebra

Algebra is the metaphysics of arithmetic.

John Ray

These are not letters, they are numbers

Me, just there

Recall that we were unable to prove that $sq : \mathbb{R}^+ \to \mathbb{R}^+$ was 1-to-1. To do this, we would have to show that no output is hit twice: that the only way that two inputs could hit the same output, was if the two inputs were equal.

Let $P_{(x,y)}$ be the statement that for fixed $x, y \in \mathbb{R}^+$:

$$sq(x) = sq(y) \Rightarrow x = y.$$  

It is quite easy to find pairs $(x, y)$ such that $P_{(x,y)}$ is true. For example, $P_{(6,6)}$ is true because $6^2 = 6^2$ and $6 = 6$. In mathematics, when we say $P_{(x,y)}$ is true, as in

$$sq(x) = sq(y) \Rightarrow x = y,$$

is true, we actually mean the statement $P$:

for any possible pair $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$, $P_{(x,y)}$ is true.

To demonstrate the truth of this, one has to demonstrate the truth of (uncountably) many statements.
Call this statement $P$: the power of algebra is that it can achieve this, algebra can prove the (countably) infinite conjunction:

$$P = \bigwedge_{x,y \in \mathbb{R}^+} (P(x,y)).$$

**Further Remark: De Morgan’s Law and Algebra**

On the other hand, let $Q_n$ be the statements that for $n \in \mathbb{N}$:

$$n^2 + n + 41 \text{ is a prime number.}$$

Again, it is quite easy to find numbers $n$ such that $Q_n$ is true. For example, $Q_{20}$ is true because:

$$20^2 + 20 + 41 = 461 \text{ is a prime number.}$$

Consider the statement $Q$ that

for any natural number, $Q_n$ is true.

That is $Q_1$ is true and $Q_2$ is true and $Q_3$ is true... and we have infinitely many statements to prove again, because

$$Q = Q_1 \land Q_2 \land Q_3 \land \cdots.$$  

We might again employ algebra to help us, but we struggle. Perhaps we believe the statement is in fact FALSE. De Morgan’s Theorem can help us with this. If $Q$ is FALSE $\neg Q$ is true and by De Morgan

$$\neg Q = (\neg Q_1) \lor (\neg Q_2) \lor (\neg Q_3) \lor \cdots.$$  

is true. What this means is that at least one of these statements is true. That is, there exists an $m$ such that $\neg Q_m$ is true aka $Q_m$ is false. So we search for a *counterexample* to the truth of $Q_m$; that is an $m$ such that $m^2 + m + 41$ is NOT a prime. As it happens, $Q_{40}$ says that $40^2 + 40 + 41 = 1681$ is a prime number, which is false, because $1681 = 41^2$. Therefore $Q$ is false. There exists no counterexample to $P$.

**Autumn 2019**

Consider $A = \{0, 1\}^4$ = set of bit-strings of length 4, and define the relation $R$ on $A$ by:

$$(a_1a_2a_3a_4, b_1b_2b_3b_4) \in R \iff a_1a_2 = b_1b_2.$$  

For example $(1001, 1011) \in R$. This relation forms an equivalence relation.

1. Find $|A|$.
2. Prove that the relation $R$ is transitive.
3. Describe $E(1000)$, the equivalence class containing the bit-string 1000.

**Solution:** $|A|$ is the number of bit-strings of length 16:
If $|A|$ were smaller, we could draw the digraph, show that it forms a partition, therefore $R$ is an equivalence relation, and therefore transitive. The set $A$ is a bit large for that so we will do an algebraic type proof. Suppose that $a = a_1a_2a_3a_4$, $b = b_1b_2b_3b_4$, and $c = c_1c_2c_3c_4$ are bit-strings. Suppose that $(a, b) \in R$ and $(b, c) \in R$. This implies that:

The equivalence class $E(1000)$ consists of all elements related to 1000:

---

**Appendix: The Axioms of the Real Numbers**

**Closure**
For any $x, y \in \mathbb{R}$; $x + y \in \mathbb{R}$ and $x \times y \in \mathbb{R}$.

**Commutativity**
For any $x, y \in \mathbb{R}$; $x + y = y + x$ and $xy = yx$.

**Associativity**
For any $x, y, z \in \mathbb{R}$;

$$
\begin{align*}
x + (y + z) &= (x + y) + z, \\
x \times (y \times z) &= (x \times y) \times z.
\end{align*}
$$

**Identity**
There is a special real number $0 \in \mathbb{R}$ such that:

$$
0 + x = x,
$$
for every $x \in \mathbb{R}$. Also there is a special number $1 \in \mathbb{R}$ ($1 \neq 0$) such that for every $x \in \mathbb{R}$:

$$
1 \times x = x,
$$

**Subtraction and Division**
For every number $x$ there corresponds a number $-x \in \mathbb{R}$ such that:

$$
x + (-x) = 0.
$$
Also if $x \neq 0$ there is a number $x^{-1} \in \mathbb{R}$ ($x^{-1} = 1/x$) such that:

$$
x \times x^{-1} = 1.
$$

**Distributive Law**
For $x, y, z \in \mathbb{R}$,

$$
x \times (y + z) = (x \times y) + (x \times z).
$$
4.1 Basics of Algebra

4.1.1 Evaluation Convention

The following question was asked on an Irish television quiz programme:

*Fill in the missing number:* \(2 + 4 \times \square = 30\).

A contestant said the answer was 5. “Correct” said the quizmaster. But they were both wrong. The answer should have been 7.

In Mathematics, by convention alone, multiplications are done before additions, so that

\[
2 + 4 \times 5 = 2 + 20 = 22, \quad \text{but} \quad 2 + 4 \times 7 = 2 + 28 = 30.
\]

So the correct answer is 7.

More generally, the order in which mathematical operations should be done — the *Hierarchy of Mathematics* — is as follows:

Hence if we want to write add two and four, then multiply this by five we write:

**Example**

Evaluate \((3 + 5)^2 \times 2 \div 4 - 2\).

*Solution:* We do the brackets first to get \(3 + 5 = 8\). Now do the exponential to get \((3 + 5)^2 = 64\). Now multiply by 2 to get 128. Now divide by 4 to get 32 and finally take away two to get 30.

*Exercise:*

1. What is the output of the following Python code:

```python
>>> a=6-4/2+5*2
>>> print a
```

2. Calculate each of the following without the use of a calculator:

\[
(i) 6 \times 2 + 3 \quad (ii) 3 + 12 \div 6 \quad (iii) 2(5) + 4
\]

\[
(iv) 2(2 + 3) \quad (v) 4 \times 5 - 12 \quad (vi) 36 \div 9 - 1
\]

\[
(vii) 5 + 2 \times 3 + 12 \div 4 \quad (viii) 4 - 18 \div 6 \quad (ix) 30 \div (6 - 1) + 2
\]

**Answers:** (i) 15 (ii) 5 (iii) 14, (iv) 10 (v) 8 (vi) 3 (vii) 14 (viii) 1 (ix) 8
4.2 Equations

Motivation

The relationship between temperature measured in degrees Fahrenheit (°F) and temperature in degrees Celsius (°C) is described by

\[ F = \frac{9}{5} C + 32. \]

where \( F \) represents the temperature in degrees Fahrenheit and \( C \) represents the temperature in degrees Celsius. If we have measured a temperature in °F but need to convert it to °C then we will need to rearrange the equation to make \( C \) the subject (note that a variable is the subject of the equation if it appears once, by itself, on one side of the equation).

Task: Rearrange the equation below to make \( L \) the subject:

\[ R = \sqrt{\frac{VL + PR}{L}}. \]

4.2.1 Equality, Balancing Scales, and Equivalent Equations

An equation says that a pair of quantities are equal. It helps to think of a weighing scales in balance and of both sides of the equation ‘weighing’ the same amount.
We must **preserve the equality** thus keep the scales in **balance** at all times. If we **do the same thing** (apply the same function) **to both sides** of the equation we keep the scales in balance, and create another (equivalent) equation.

---

**Key Concept:**

Recall that **Inverse** functions are functions that **undo** each other.
<table>
<thead>
<tr>
<th>Function</th>
<th>Inverse</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adding, $+_{a}$</td>
<td></td>
<td>$a + 2 = b$</td>
</tr>
<tr>
<td>Subtracting, $-<em>{a}$, $+</em>{-a}$</td>
<td></td>
<td>$a - 2 = b$</td>
</tr>
<tr>
<td>Multiplying, $\times_{a}$</td>
<td></td>
<td>$5 \times a = b$</td>
</tr>
<tr>
<td>Dividing, $\div_{a}$</td>
<td></td>
<td>$\frac{a}{4} = b$</td>
</tr>
<tr>
<td>$\times_{-1}$</td>
<td></td>
<td>$-a = b$</td>
</tr>
<tr>
<td>Squaring, sq</td>
<td></td>
<td>$a^2 = b$</td>
</tr>
<tr>
<td>Square root, $\sqrt{}$</td>
<td></td>
<td>$\sqrt{a} = b$</td>
</tr>
</tbody>
</table>
### Exercise

<table>
<thead>
<tr>
<th>Equation</th>
<th>Function Applied</th>
<th>Inverse</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x = 3$</td>
<td>$\times 2$</td>
<td>$\div 2, \times 1/2$</td>
<td>$x = \frac{3}{2}$</td>
</tr>
<tr>
<td>$x + 2 = 3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^2 = 9$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^2 = 5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{x}{b} = b$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sqrt{x} = 3$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### 4.2.2 Solving Equations: Know Right from Wrong

<table>
<thead>
<tr>
<th>True Laws</th>
<th>Why?</th>
<th>Exercise</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ax + ay = a(x + y)$</td>
<td></td>
<td>$mt + ms =$</td>
</tr>
<tr>
<td>$\frac{a \cdot x}{b \cdot x} = \frac{a}{b}$</td>
<td></td>
<td>$\frac{2k}{7k} =$</td>
</tr>
<tr>
<td>$\frac{a \cdot p + b \cdot p}{cp} = \frac{a + b}{c}$</td>
<td></td>
<td>$\frac{2 + k}{7k} =$</td>
</tr>
<tr>
<td>$(c + d)^2 = c^2 + 2cd + d^2$</td>
<td></td>
<td>$\frac{a \cdot p + b \cdot p}{5} =$</td>
</tr>
<tr>
<td>$(5z)^2 = 5^2 z^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(3n)^2 =$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Some Definitions

**Terms** are the elements of what is called a *linear combination*:

**Factors** are the elements of what is called a *product*:

If the *same factor* is present in *every term* we can *factor it out/take it out*:

**Key Concepts:**

- When multiplying both sides of an equation by something, because \( a(b + c) = ab + ac \), you must multiply *each term* on both sides by that thing.

- The square of a sum is not equal to the sum of the squares

  \[(q + 4)^2 = (q + 4)(q + 4) \neq q^2 + 16.\]

- In general, \( \sqrt{a + b} \neq \sqrt{a} + \sqrt{b} \).

*Main Principle of Solving Equations*

1. we can apply any function to an equation as long as we *apply the same function to both sides*

2. we simplify/rearrange equations *step-by-step* by applying *inverse functions*
More General Guidelines

• Perhaps start by removing things that are furthest away from your subject. Work in small steps, removing one thing at a time, thus getting closer and closer to the subject.

• Think of peeling an onion, removing outer layers before getting to the core, as an analogy of getting to the subject in your equation:

- It is OK to do many small (but correct) steps. It is also OK to only think of the next small step instead of ‘having to plan the entire route in detail from the start’.

Examples

1. Solve for $v$:

$$T = \frac{2v}{g} + 5$$

<table>
<thead>
<tr>
<th>Want to get rid of</th>
<th>Apply to Both Sides</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = \frac{2v}{g} + 5$</td>
</tr>
</tbody>
</table>
2. Solve for $t$

$$q = \frac{4t}{5} - 2$$

<table>
<thead>
<tr>
<th>Want to get rid of</th>
<th>Apply to Both Sides</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Exercises:**

1. What's the first step towards solving for the variable?

<table>
<thead>
<tr>
<th>Equation</th>
<th>Want to Get Rid Of</th>
<th>Inverse Function to Apply</th>
<th>New Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4s - 7 = 10$</td>
<td>$-7$</td>
<td>$+7$</td>
<td>$4s = 17$</td>
</tr>
<tr>
<td>$3 \cdot \sqrt{z^2} - 4 = 6$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{2p + 5}{3} = 4$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sqrt{x} - 6 = 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Solve each of the following equations:

(a) Solve for $c$:

$$bc - d = a.$$  

(b) Solve for $t$:

$$v = u + at.$$  

(c) Solve for $q$:

$$p + q = r.$$  

(d) Solve for $d$:

$$I = \frac{db^3}{12}.$$  

(e) Solve for $s$:

$$u^2 + 2as = v^2.$$
(f) Solve for \( R \):

\[
A = P \left( 1 + \frac{R}{100} \right).
\]

3. **“Spot the Error”** In the table below, identify and circle the error (on the left column, will be one of the \( \Rightarrow \)s) then write down the correct solution in the right column:

<table>
<thead>
<tr>
<th>Wrong Solution</th>
<th>Correct Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ax = 2b )</td>
<td>( ax = 2b )</td>
</tr>
<tr>
<td>( \Rightarrow x = 2b - a )</td>
<td>( \Rightarrow x = \frac{2b}{a} )</td>
</tr>
<tr>
<td>( 3s = 7 - 2s )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow 3s - 2s = 7 )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow s = 7 )</td>
<td></td>
</tr>
<tr>
<td>( R = \frac{p}{a} + q )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow R - q = \frac{p}{a} )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow aR - q = p )</td>
<td></td>
</tr>
<tr>
<td>( v^2 = u^2 + 2as )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow \frac{v^2}{2a} = u^2 + s )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow s = \frac{v^2}{2a} - u^2 )</td>
<td></td>
</tr>
<tr>
<td>( \sqrt{m - n} = p + q )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow m - n = p^2 + q^2 )</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow m = p^2 + q^2 + n )</td>
<td></td>
</tr>
</tbody>
</table>

Recall:

1. we can apply any function to an equation as long as we **apply the same function to both sides**

2. we simply/rearrange equations **step-by-step** by applying **inverse functions**

Variables represent physical quantities and the symbols are agreed by scientists within a specific discipline. These symbols are not necessarily just one letter but often are a combination of symbols, sometimes also involving subscripts. For example, the symbol for a change in temperature is usually \( \Delta T \), and the symbol for the concentration of chemical ‘1’ is \( C_1 \).

**Examples**

1. Solve for \( m \):

\[
Q = m \cdot c \cdot \Delta T.
\]

**Solution:**

2. Solve for \( V_1 \):

\[
C_1 V_1 = C_2 V_2.
\]

**Solution:**
3. Solve for $m$:

$$\rho = \frac{m}{v}$$

_Solution:

4. Solve for $I$:

$$P = RI^2.$$  

_Solution:

5. Solve for $g$:

$$\omega = \sqrt{\frac{g}{L}}.$$  

_Solution:

When solving equations that involve fractions, it can be helpful to ‘get rid’ of the fractions by multiplying both sides of the equation by the (product of the) bottom(s)/denominator(s).

**Examples**

1. Solve for $x$:

$$\frac{1}{2 + x} - 4 = 5.$$  

_Solution:
2. Solve for $a$: 
\[ \frac{2}{a - 2} = 5. \]

*MY Solution:*

3. Solve for $R$: 
\[ \frac{1}{R} + \frac{1}{R_1} = \frac{1}{R_2}. \]

*Solution:*

4. Solve for $a$: 
\[ \frac{1}{a} + \frac{1}{b} = 5. \]

*MY Solution:*

*Exercises:*

1. Solve for $R$: 
\[ \frac{M}{T} = \frac{E}{R} \quad \text{Ans: } R = \frac{EI}{M}. \]
2. Solve for \( L \):
\[
y = \frac{n\lambda L}{d} \quad \text{Ans: } L = \frac{yd}{n\lambda}
\]

3. Solve for \( v \):
\[
\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \quad \text{Ans: } v = \frac{uf}{u - f}
\]

**Revision Exercises:**

1. Solve each of the following equations:
   
   \((i)\) \(-2x = -20\) \quad \((ii)\) \(3x - 7 = 8\)

   \text{Ans: } (i) \ x = 10 \ (ii) \ x = 5

2. Solve each of the following equations:
   
   \((i)\) \(3(x - 1) + 5(x + 1) = 18\) \quad \((ii)\) \(3(2x + 1) - 3(x + 4) = 0\)

   \text{Ans: } (i) \ x = 2 \ (ii) \ x = 3

3. **Winter 2019** You are writing some code to convert Fahrenheit, \( F \), to Celsius, \( C \). You Google ‘Fahrenheit to Celsius’ and find the following equation:

   \[ F = \frac{9}{5}C + 32. \]

   This formula converts Celsius to Fahrenheit. You want to make the inverse conversion, and therefore need to find \( C \) written in terms of \( F \):

   (a) Solve this equation for \( C \) in terms of \( F \).
   
   (b) Hence convert \( 100^\circ F \) to Celsius. \text{Ans: } 37.78^\circ C.

4. You are writing some code to that takes as input the volume of a sphere, \( V \), and outputs its radius, \( r \). You Google ‘volume of sphere’ and find the following equation:

   \[ V = \frac{4}{3}\pi r^3. \]

   This formula takes \( r \) as input and outputs the volume. You want to do the opposite, and therefore need to find \( r \) written in terms of \( V \):

   (a) Solve this equation for \( r \) in terms of \( V \).
   
   (b) Hence find, correct to three significant figures, the radius of a sphere of volume \( 100 \text{ cm}^3 \). \text{Ans: } 2.88 \text{ cm}.

5. Solve each of the following equations:
   
   \((i)\) \( \frac{2x}{5} = \frac{3}{2} + \frac{x}{4} \)
   
   \((ii)\) \( \frac{x - 2}{2} = 5 - \frac{x + 10}{9} \)

   \text{Ans: } (i) \ x = 10 \ (ii) \ x = 8
6. Solve for $x$: $\sqrt{x + 7} = 5$.

7. **Winter 2018** Solve for $x$: 
\[
\frac{2x}{x - 3} + 4 = \frac{6}{x - 3} \quad \text{Ans: No solutions.}
\]

8. **Autumn 2018** Solve for $a$: 
\[
\frac{a + x}{b - x} = c.
\]

9. In the following we have “Lhs = Rhs; variable”. Solve for “variable”.

   (i) $E = P + k$; $k$.  
   (ii) $F = ma$; $m$.  

   (iii) $y = mx + c$; $m$.  
   (iv) $E = mc^2$; $c$.  

   (v) $E = V/R$; $R$.  
   (vi) $v^2 = u^2 + 2as$; $u$.  

   (vii) $t = (3 + v)k + c$; $v$.  
   (viii) $s = ut + at^2/2$; $a$.  

   (ix) $z = p/(2s + q)$; $s$.  
   (x) $s = a/(1 - r)$; $r$.  

   (xi) $A = 4\pi r^2$; $r$.  
   (xii) $x = \sqrt{y + z}$; $y$.  

   (xiii) $A = \frac{XY}{X + Y}$; $X$.  
   (xiv) $\frac{1}{A} = \frac{1}{B} + \frac{1}{C}$; $C$.  

   (xv) $A = \sqrt{\frac{M+X}{Y}}$; $Y$.  
   (xvi) $T = 2\pi \sqrt{\frac{L}{G}}$; $L$.  

   (xvii) $V = \frac{KAB}{B-A}$; $A$.  
   (xviii) $d = \sqrt{b^2 - 4ac}$; $b$.  

   (xiv) $v = c\sqrt{2gh}$; $h$.  
   (xix) $z = \sqrt{R^2 + \omega^2 L^2}$; $L$.  

   (xx) $E = mgh + mv^2/2$; $m$.  
   (xxi) $A = P(1 + r)^3$; $r$.  

4.3 Exponents/Powers

4.3.1 Definition

Let $a \in \mathbb{R} — a$ is a ‘real number’ — and $n \in \mathbb{N} — n$ is a ‘natural number’. Then

$$a^n = a \times a \times \cdots \times a$$

(4.1)

We can refer to $a^n$ as “$a$ to the power of $n$” and say $a^n$ is “a power of $a$”. Finally $a^2$ is a square ($a$-squared), $a^3$ is a cube ($a$-cubed) and $a^n$ is an $n$-th power. For example, $3^7$ is an example of a power of 3.

4.3.2 Basic Properties of Powers

Now that we have a shorthand way of writing repeated multiplication we must investigate how they combine together? What happens when we multiply together two powers? If we divide one power by another? What about a power of a power? What if we took two numbers, multiplied them together and took a power of that? We will see that answering these questions will raise more questions.

Before we start our investigation I want you to be aware of the following:

Definition

Let $a \in \mathbb{R}$ and $m,n \in \mathbb{N}$ (both $m$ and $n$ are natural numbers). Then $a^m$ and $a^n$ are like powers (same base, “base $a$”). Now nobody apart from me in a second calls this the Golden Rule of Powers, but it’ll become clear very soon why it is indeed the case:

(The Golden Rule of Powers)

Multiplication of Like Powers

Consider $7^2$ and $7^3$. What is really happening when we multiply them together?

There is nothing special about $7^2$ and $7^3$ here. Indeed any real number $a$ and natural numbers $m, n$ could replace the roles of 7, 2 and 3:

Try to get any kind of similar rule for unlike powers and you’ll see what I mean by the Golden Rule.

Example

Write $4 \times 4 \times 16$ as a power of 4.

Solution:
Division of Like Powers

Consider $10^5$ and $10^2$. What is really happening when we divide, say, $10^5$ by $10^2$:

Again there is nothing special about $10^5$ and $10^2$ here. Again let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$ such that $m > n$:

Examples

1. Write $\frac{x^{11}}{x^5}$ as a single power.

   Solution

2. Write $\frac{10^2 \times 10^5}{100}$ in the form $10^n$ for $n \in \mathbb{N}$.

   Solution:

Repeated Powers?

What is $(9^3)^4$?

Again this is a general result. Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$:

Examples

1. Write $(4^3)^5$ in the form $2^n$ for $n \in \mathbb{N}$.

   Solution: First write 4 as a power of 2:

   Now we can write $(4^3)^5$ as a power of two:
2. Express as a power of 4: \( \frac{4^4(16^x - 3)}{64} \).

*Solution:* First note that \( 16 = 4^2 \) and \( 64 = 4 \times 4^2 = 4^3 \):

Now we have a repeated power and a product of powers:

Finally we have a division of powers:
Powers of a Product

What about \((2 \times 5)^3\)?

Again this is a general result. Let \(a, b \in \mathbb{R}\) and \(n \in \mathbb{N}\):

Examples

1. Let \(x \in \mathbb{R}\). Write \(4x^2\) as a square.

   Solution:

2. Winter 2017 Write \(\left(\frac{a^7 a^3}{a^2}\right)^3\) in the form \(a^p\), where \(p\) is a rational number (i.e. a fraction, an element of \(\mathbb{Q}\)).

   Solution: Taking our time

Another similar case is that of a power of a fraction. Now our work from the first chapter should pay dividends. We know how to multiply fractions together so we have

4.3.3 Zero Powers, Negative Powers & Fractional Powers

You may have noticed that our definition for indices/ powers has only defined powers when the exponent is a natural number. What about the following:

(P1) \(2^0\)?

(P2) \(3^{-8}\)??

(P3) \(4^{1/2}\)???

(P4) \(5\sqrt{2}\)???

(P5) \(6^{1+i}\)???
Note that every natural number is an integer and that every integer is a fraction (e.g. $-11 = -11/1$). What we will do is construct definitions for zero, negative numbers and fractions that *extend the definition for natural numbers* and *make the laws of indices consistent*.

In other words we will choose a definition for fractional powers\(^1\) such that for all $a \in \mathbb{R}$, $m, n \in \mathbb{Q}$ — $m$ and $n$ are fractions — the following make sense:

If it makes sense for fractions it will also make sense for integers and natural numbers as these are fractions also.

**Zero Powers**

Consider multiplying by, say, $7^0$:

Once again this is a result independent of 7. Let $a \in \mathbb{R}$:

**Example**

Suppose that $a, b \in \mathbb{R}$ such that $a = b^3$. Simplify

$$
\frac{a^2}{(b^2)^3}
$$

*Solution:* First we replace $a = b^3$:

**Negative Powers**

What is $3^{-8}$? Using the first law of indices and the definition of $3^0$:

We know the story by now. Let $a \in \mathbb{R}$, $a \neq 0$. Let $n \in \mathbb{N}$:

So positive powers are multiplying and negative powers are dividing.

**Example**

Evaluate $10^{-1} \times 100$. What does this tell you about $x^{-1}$?

*Solution:*

\(^1\) will be for $a > 0$ only
This tells us a few things. Firstly if you want to divide by \(x\) you can multiply by \(x^{-1}\). We already saw that if you want to divide by \(x\) multiply by \(\frac{1}{x}\)... but then of course \(x^{-1} = \frac{1}{x^1} = \frac{1}{x}\).

There is actually another way of explaining why zero powers are one that might also sheds light on how we do negative powers. We start with the fact that multiplying by one doesn’t change anything. So given any number you want, say 8 we can write

In fact, when multiplying, we can think of always starting at one. Now we can make products or powers of eight as follows:

So to increase powers you multiply by 8. Now coming back in the other direction from \(8^4\).

To go decrease powers you divide by 8. Hence if we want to go from \(8^1 \rightarrow 8^0\) we divide by eight a final time:

If we go back to the idea that we should always start at one we see that \(8^0\) starts at 1 and... well we don’t do anything there are no eights to multiply it and we are stuck at one! This is called an empty product. Utterly bizarre examples include

\[\text{Let } S \text{ be the set of all students who are at least ten foot tall. Now take the ages of all these students and multiply them together. The answer is one.}\]

Such is the mystery of the empty-set!

If we continue this policy of dividing reduces powers we can see why the negative powers make sense:

**Fractional Powers**

What is \(9^{1/2}\)? Now using the first law of indices:

So we see that \(9^{1/2}\) is the number that when squared gives 9... Now to keep things easy for us, we will only consider fractional powers of positive real numbers (i.e. numbers bigger than zero. We write \(a > 0\) for “a is bigger than zero”). Try and figure out what \((-1)^{1/2}\) if you want to see what I mean. Hence let \(a \in \mathbb{R}, a > 0:\)
What about $5^{1/6}$? Similar story, using the third law of indices:

So we see that $5^{1/6}$ is the number when brought to the power of 6 gives you 5. This is known as the sixth root of 5 which we denote by $\sqrt[6]{5}$, and extends naturally from “the” square root (e.g. Let $x \in \mathbb{R}$, $x > 0$. Then $\sqrt{x} = \sqrt[2]{x}$). Again, let $a \in \mathbb{R}$, $a > 0$ and $n \in \mathbb{N}$:

What about $6^{3/5}$? If we agree that whatever it is, it has to agree with, say the third law of indices, so that:

A subtlety of this is that we have $3 \times (1/5) = (1/5) \times 3$:

Finally, let $a \in \mathbb{R}$, $a > 0$ and $m/n \in \mathbb{Q}$:

**Warning**

Students frequently have trouble with terms of the form $(ab)^n$, $ab^n$ and $(ax^n)^m$. These problems are helped if we remember BEDMAS — most importantly exponents happen before multiplication (but after brackets). Let us maybe start with $(ab)^n$ which simply means $ab$ multiplied by itself $n$ times:

$$(ab)^n = (ab) \cdots (ab) = a \cdots a \cdot b \cdots b = a^n b^n.$$  

If $n$ is not a natural number then we can show that $(ab)^n = a^n b^n$ still holds.

If you write $(2x)^2 = 2x^2$ however you are wrong! When we write $2x^2$ we do the exponent/power first and so

$$2x^2 = 2 \cdot x \cdot x \neq (2 \times 2) \times x^2 = (2x) \cdot (2x) = (2x)^2.$$  

Finally the other thing that people have problems with are terms of the form $(ax^n)^m$, for example $(5x^2)^3$. This is not equal to $5x^6$ (which is not $(5x)^6$ either) but

$$(5x^2)^3 = (5x^2)(5x^2)(5x^2) = (5 \times 5 \times 5) \times (x^2 \cdot x^2 \cdot x^2) = 125x^6.$$  

If you have a term of the form $(ax^n)^m$ you need to use $(ab)^n$ and then your ‘rules’:

$$(ax^n)^m = a^m(x^n)^m = a^m x^{nm}.$$  

These facts about powers are collated at the back of this manual (and will appear on your exam paper also).
Examples

1. **Autumn 2019** Write the following in the form $a^m b^n$, where $m$ and $n$ are rational numbers:

$$\left( \frac{a^3 b^{-2}}{a^{-1} b^2} \right)^{-1}.$$

**Solution:** Taking our time:

2. **Winter 2019** Suppose that $a > 0$ is a real number. Simplify as much as possible:

$$\sqrt{\frac{a}{a^2 \cdot a^{-3}}}.$$

**Solution:** Taking our time, and recalling $\sqrt{a} = a^{1/2}$:

4.3.4 Exponential Equations

An exponential equation is one of the form

Sometimes we can solve these via the following fact:

**Fact/Theorem**

If $b > 0$ but $b \neq 1$ and

$$b^x = b^y,$$

then $x = y$. 
Proof. All functions of the form \( p(x) = b^x \) are increasing for \( b > 1 \):

![Graph showing exponential functions]

Figure 4.1: If \( b^x = b^y \) then in fact \( x = y \). Note that for large negative values of \( x \), say \( x = -10 \), that \( b^x \approx 0 \). For example, \( b^{-10} = \frac{1}{b^{10}} \approx 0 \). Note that \( x \mapsto b^x \) is 1-to-1 and so invertible on it’s range. If \( b < 1 \) then the function is decreasing. If \( b = 1 \) we have the constant function \( x \mapsto 1^x = 1 \).

Therefore if we can write an exponential equation in the form

we can then write \( f(x) = g(x) \).

Exercises These questions should be done, with the possible exception of calculating roots, WITHOUT the aid of a calculator. While a calculator can give you the answers to some of the questions it cannot help you with all of the questions. Struggle without the calculator so you understand the concepts involved. Later on you will be either to use the calculator or your head and certainly your head when the calculator is not an option.

1. Evaluate:
   (i) \( 36^{1/2} \)  
   (ii) \( 125^{1/3} \)  
   (iii) \( 16^{1/4} \)  
   (iv) \( 1000^{1/3} \)  
   (v) \( 1000^{2/3} \)  
   (vi) \( 2^{-5} \)  
   (vii) \( 5^{-2} \)  
   (viii) \( 8^{2/3} \)  
   (ix) \( 4^{-1/2} \)  
   (x) \( 4^{-1} \)  

2. Write these as \( a/b \), where \( a, b \in \mathbb{Z} \):
   (i) \( 2^{-2} \)  
   (ii) \( (1/4)^{1/2} \)  
   (iii) \( 32^{-3/5} \)  
   (iv) \( 16^{-1/4} \)  
   (v) \( 27^{-2/3} \)  

3. Write \( 8^{3/4} \) and \( 256^{-3/4} \) as powers of two.

4. Write \( a^{3/2} \) in terms of roots and whole number powers.

5. Write \( p^{-1/2} \) in terms of roots and whole number powers.

6. **Winter 2018** Write the following in the form \( q^p \), where \( p \) is a rational number:

\[
q^{-4} \cdot \frac{q}{(q^{-2})^{1/4}}
\]
7. **Autumn 2020** Write the following in the form $x^p$:

$$\left(\frac{x^3x^5}{x\sqrt{x}}\right)^{-4} \quad \text{Ans: } x^{-26}.$$ 

8. Which of each pair is greater?:

(i) $2^5$ or $5^2$  
(ii) $4^{1/2}$ or $(1/2)^4$  
(iii) $2^{-1/2}$ or $(-1/2)^2$  
(iv) $(1/2)^6$ or $(1/2)^7$  
(v) $7^2 \times 7^3$ or $(7^2)^3$

9. Solve for $k$. If we have started looking at logs in lectures you may use them here:

(i) $2^k = 4$  
(ii) $4^k = 64$  
(iii) $8^k = 64$  
(iv) $2^k = 128$  
(v) $4^k = 2$  
(vi) $25^k = 5$  
(vii) $8^k = 4$  
(viii) $1000^k = 100$  
(ix) $32^k = 16$  
(x) $8^k = 1/2$

10. Write each of these in the form $a^p$, where $p \in \mathbb{Q}$:

(i) $a^7 \div a^2$  
(ii) $a^7 \times a^2$  
(iii) $(a^7)^2$  
(iv) $\sqrt{a}$  
(v) $\sqrt[3]{a}$  
(vi) $1/a^3$  
(vii) $1/\sqrt{a}$  
(ix) $(\sqrt{a})^3$  
(x) $1/(a\sqrt{a})$

11. State if these are true or false:

(i) $2^3 \cdot 5^3 = 10^3$  
(ii) $(x^6)^7 = x^{13}$  
(iii) $(3\sqrt{3})^3 = 3^{4.5}$  
(iv) $(10\sqrt{10})^4 = 10^6$  
(v) $2^7 \times 3^7 = 5^7$  
(vi) $(-3)^4 = -3^4$

12. **Autumn 2018** Write $a^a \left(\frac{1}{a^2}\right)^3$ in the form $a^p$, where $p$ is a rational number.

13. Simplify the following

(a) $x^3 \times x^7 \times (x^3)^5$  
(b) $(x^{-3})(x^h)^2(x)$  
(c) $\frac{4q}{q^{-2}}$  
(d) $\frac{3y}{y^2 y^{-8}}$  
(e) $z^2 \cdot 0.5$  
(g) $x^{\frac{1}{7}} x^{\frac{1}{4}}$  
(h) $(x\sqrt{2})^{\sqrt{2}}$  
(i) $\frac{x(x+2)(x-3)}{(x+2)^{-2}(x-3)^{1/2}}$  
(j) $(x+1)^{1-\pi}(x+1)^{1+\pi}$  
(k) $\left(\frac{x^2 x^3}{x-2}\right)^3$

14. Simplify

(a) $\left(\frac{2y}{x}\right)^2$  
(b) $\frac{a}{a^4} \left(\frac{3}{a^2}\right)^{-3}$

15. Write $\sqrt[3]{\sqrt{2}}$ as $2^p$ for $p \in \mathbb{Q}$. 
Further Remarks for Computing: Units of Data

The fundamental unit of data is a *bit*. At the most basic level a bit is like a switch with two states 1 or 0. The *byte* then is eight bits. Note that

\[ \text{byte} = 8 \text{ bits} = 2^3 \text{ bits}. \]

You can compare this with

\[ \text{km} = 10^3 \text{ m}. \]

As computers are built on binary AKA powers of two rather than decimal AKA powers of ten, if you want to talk about large magnitudes of data you use powers of two rather than powers of ten. There is a good news and a bad news. The good news is that \( 2^{10} = 1024 \approx 10^3 \). The bad news is *you computer people* still call this kilo even though it isn’t a thousand! We use the symbol \( B \) for a byte.

We have

\[
\begin{align*}
\text{kB} &= 2^{10} \text{ B} \approx 1000 \text{ B} = 10^3 \text{ B} \\
\text{mB} &= 2^{10} \text{ kB} = 2^{20} \text{ B} \approx 10^6 \text{ B} \\
\text{GB} &= 2^{10} \text{ MB} = 2^{30} \text{ B} \approx 10^9 \text{ B} \\
\text{TB} &= 2^{10} \text{ TB} = 2^{40} \text{ B} \approx 10^{12} \text{ B}
\end{align*}
\]

These are the kilobyte, megabyte, gigabyte and terabyte. Note that the errors in calling \( 2^{10} = 1024 = 10^3 \) mean that a kilobyte is 2.4% bigger than one might think, a megabyte is 4.86% bigger than one might think, a gigabyte is 7.37% bigger than one might think and a terabyte is 9.95% bigger than one might think. Surely someone has made money out of this.

A download or upload speed then is a rate of change of data so we have, where \([ \cdot ]\) denotes *dimension*,

\[ \text{[download speed]} = \left[ \frac{\Delta \text{ Data}}{\Delta t} \right] = \left[ \frac{\text{Data}}{t} \right], \]

so for example bytes per second.

**Examples**

1. Convert 8 GB to MB.

   *Solution:* We have that
   
   \[ \text{GB} = 2^{10} \text{ MB}. \]

   Therefore
   
   \[ 8 \text{ GB} = 8(2^{10} \text{ MB}) = 8192 \text{ MB}. \]

2. A page from a novel is approximately 4 kB of data. How many pages could a 650 MB CD-ROM theoretically hold?

   *Solution:* First of all let us convert 650 MB to kB:
   
   \[ 650 \text{ MB} = 650(2^{10} \text{ kB}) = 665,600 \text{ kB}. \]

   Now
   
   \[ \text{page} \equiv 4 \text{ kB} \Rightarrow \text{kB} \equiv \frac{\text{page}}{4}, \]

   so we have
   
   \[ 665,600 \text{ kB} = 665,600 \left( \frac{\text{page}}{4} \right) = 166,400 \text{ pages}. \]
3. How long does it take to download 1 GB of data with a download speed of 1.4 Megabits per second?

Solution: We have the relationship

\[ \text{download speed} = \frac{\text{data}}{\text{time}}. \]

Carefully (by multiplying by time and dividing by download speed), we can write this as

\[ \text{time} = \frac{\text{data}}{\text{download speed}}. \]

Off we go

\[ \text{time} = \frac{1 \text{ GB}}{1.4 \frac{\text{Mbit}}{s}} \]

What we need to do is note that there are 8 bits in a byte:

\[ 8 \text{ bit} = B \Rightarrow \text{bit} = \frac{B}{8} \Rightarrow \text{Mbit} = \frac{MB}{8} = \frac{MB}{2^5}, \]

and use

\[ \text{GB} = 2^{10} \text{ MB}. \]

Now put it back together

\[ \text{time} = \frac{2^{10} \text{ MB}}{1.4 \frac{\text{Mbit}}{s}} \]

To tidy this up multiply above and below by second, s and also by \(2^3\):

\[ \text{time} = \frac{2^{10} \text{ MB}}{1.4 \frac{\text{Mbit}}{s}} = \frac{2^{10} \text{ MB} \cdot s}{1.4 \frac{\text{MB}}{2^5}} = \frac{2^{10}2^3 \text{ MB} \cdot s}{1.4 \text{ MB}} \]

and of course \( \frac{\text{MB}}{\text{MB}} = 1 \) and we are just left with

\[ \text{time} = \frac{2^{10}2^3 \cdot s}{1.4} = 5851.43 \text{s} \approx 100 \text{ mins}. \]

4. Someone in the class is coming in early in the morning and downloading \textit{inappropriate material} in the computer lab. The download speed is 1.5 MB per second and the inappropriate material is 622 MB. If the cleaners come into the lab at 8.30 am, what time should the student come in to get the download done?

Solution: How long will this take? We use the same relationship we derived last time:

\[ \text{time} = \frac{\text{data}}{\text{download speed}} \]

\[ = \frac{622 \text{ MB}}{1.5 \frac{\text{MB}}{s}} \]

\[ = \frac{622 \text{ MB}}{1.5 \text{ MB}} \cdot s = \frac{1244}{3} \text{s} \approx 415 \text{s} \approx 7 \text{ mins} \]

Well they’d probably want to be in earlier than 8.23 am to get those machines going on time.
4.4 Logarithms

Fix a \( b > 1 \) (usually in CS this will be \( b = 2 \)) and consider the function \( p : \mathbb{R} \to \mathbb{R}, p = b^\circ, p(x) = b^x \) (\( b \) for base). For example, for \( b = 2 \), we can tabulate some of the values that \( p(x) \) takes:

<table>
<thead>
<tr>
<th>Input</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>-1/2</th>
<th>0</th>
<th>1/2</th>
<th>1</th>
<th>4/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>0.0625</td>
<td>0.125</td>
<td>0.25</td>
<td>0.7071...</td>
<td>1.414...</td>
<td>2</td>
<td>2.520...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We could plot the points of \( p(x) = 2^x \) on a graph:

Figure 4.2: The graph of \( p(x) = 2^x \). Note that \( p(x) \neq 0 \) and the function is strictly increasing. A graph is all the points of the form (input,output). By the Horizontal Line Test, \( p \) is 1-to-1 and onto \( \mathbb{R}^+ \), and therefore bijective and invertible. That is \( p^{-1} = (2^\circ)^{-1} \) exists.

This suggests that we could define a function going in the opposite direction, where \( y \) would be the input. Now this other function, which, for a fixed \( b \), we call ‘\( l(x) \)’ only takes positive numbers as input. So

We define \( \log(x) \) as:

The terminology here is that \( \log_b(x) \) is the inverse of \( p(x) = b^x \).

4.4.1 Definition

Let \( b > 1 \) and let \( p(x) = b^x \). The logarithm to base \( b \) is a function \( \log_b : \mathbb{R}_+ \to \mathbb{R} \) that is the inverse function of \( p(x) \) so that

That is when you apply \( b^\circ \) and then \( \log_b \) you get back to where you started, and also when you
apply $b^x$ and then $\log_b x$ you also get back to where you started. These functions are inverses of each other and the action of one undoes the action of the other.

Figure 4.3: $\log_b n = p \iff b^p = n$. This graph shows that $\log_b$ is an increasing, unbounded function.

Another way to remember this is to say:

$\text{LOGS ARE POWERS.}$

**Examples**

Evaluate
1. $\log_2 8$

   *Solution*

2. $\log_3 81$

   *Solution*

3. $\log_{25} 5$

   *Solution*

4. $\log_2 0.125$

   *Solution*

---

²don’t use the ‘c’ word in this class... c****
5. $\log_8 4$

Solution

6. **Winter 2018** Solve for $x$:

$$8 + 2^{5x} = 16.$$ 

Solution:

7. * Solve $5e^{3x} = 90$.

Solution: First of all we will need log-base-‘$e$’ in this question. We will learn more about this number $e \approx 2.718$ later but for now note we have $\log_e(x) = \ln(x)$ so if you want say $\log_e(18)$ on the calculator you type

$$\ln(18)$$

so you don’t have to input

$$\log_\bigg(18\bigg).$$

Back to the question: we must note that this is $5(e^{3x})$ and not $(5e)^{3x}$ (BEMDAS):

4.4.2 Properties of Logarithms

Now that we have defined what a logarithm is we must investigate common properties of them. What happens when we take the log of a product? If we take the log of a fraction? What about the log of a power?
The Log of a Product?
What is the log of $8 \times 16$ to the base 2:

$$\log_2(8 \times 16) = X$$

As we’ve seen time and again, a process like this can be abstracted to prove a general rule for all numbers:

Example
Evaluate $\log_4 2 + \log_4 32$.

Solution:

The Log of a Power
What is $\log_2 (8^4)$?

So...

Example
Find $\log_5 (25^{50})$.

Solution:
The Log of a Fraction

What about $\log_3(9/27)$?

So

Example

Evaluate $\log_2 80 - \log_2 5$.

Solution:

Remark

The best way to remember these formulae/facts/rules —

$$\log(xy) = \log x + \log y, \tag{4.2}$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y, \tag{4.3}$$

$$\log(x^n) = n \log x. \tag{4.4}$$

is to think of the log $x$ function as a transform from the positive real numbers to the real numbers that simplifies operations: multiplication $\to$ addition, division $\to$ subtraction, indices $\to$ multiplication:

These facts about logs are collated at the back of this manual (and will appear on your exam paper also).

Examples

1. What is $\log_3 3^x$?

Solution: Arrow Diagram:
2. What is $4^{\log_4 42}$?

Solution: Arrow Diagram:

3. What is $\log_b 1$?

Solution:

4. What is $\log_b \sqrt{b}$?

Solution:

5. Solve for $x$:

$$\log_2(x^2) = 3.$$  

Solution:

6. Solve the equation $4^x = 100$.

Solution:
7. **Winter 2017** Solve for $x$: $4^{1-2x} = 3^{4x+1}$.

*Solution:* First identify the difficulty:

---

### 4.4.3 The Natural Exponential & Logarithm Functions*

#### Two Distinguished Bases

For good reasons the two bases that ‘turn-up’ most frequently in computer science are 2 and the special real number $e \approx 2.718$.

I have robbed the following directly from a user ‘templatetypedef’ on Stack Overflow ([https://stackoverflow.com/a/42568759/2548552](https://stackoverflow.com/a/42568759/2548552)). It is an answer to the question

> I saw the following in a Computer Science text book,

> In computer science, all logarithms are to the base 2 unless specified otherwise

> ... so, I was wondering if someone could explain to me why.

The answer is comprehensive:

*One of the most common ways that logarithms arise in computer science is by repeatedly dividing some array in half, which often occurs with divide-and-conquer algorithms like binary search, mergesort, etc. In those cases, the number of times you can divide an array of length $n$ in half before you get down to single-element arrays is $\log_2 n$."

*Another very common way logarithms arise is by looking at the bits of a number. Writing out a number in binary uses $\lceil \log_2 n \rceil$ bits for the number $n$. Algorithms like radix sort that sometimes work one bit at a time often give rise to logs like these. Other algorithms like the binary GCD algorithm work by dividing out powers of two and therefore end up with log factors floating around."

*Logarithms in physics, math, and other sciences often arise because you’re working with continuous processes that grow as a function of time. The natural logarithm comes up in those contexts because the “natural” growth rate of some process over time is modeled by*
\(e^x\) (for some definition of “natural” growth rate). But in computer science, exponential growth usually occurs as a consequence of discrete processes like the divide-and-conquer algorithms described above or in manipulation of binary values. Consequently, we typically use \(\log_2 n\) as a logarithmic function, since it just arises so frequently.

This isn’t to say that we always use base-two logarithms in CS. For example, the analysis of AVL trees often involves logarithms whose base is the golden ratio \(\varphi\) due to the presence of Fibonacci numbers. Many randomized algorithms do involve \(e\) in some way, such as the standard analysis of quicksort, which involves harmonic numbers and thus connects back to natural logarithms. Those are examples of processes where the growth rate is modeled by something else — Fibonacci numbers or the exponential function — and so we opt for different log bases there. It’s just that it’s sufficiently common to work with binary numbers or to divide things in half that base-two logarithms end up being the default.

In many cases, it doesn’t even matter what base you choose. For example, in big-O notation, all logarithms are asymptotically equivalent (they only differ by a multiplicative constant factor), so we usually don’t even specify the base when writing something like \(O(n \log n)\) or \(O(\log n)\).

The importance of the base \(e\) comes from following observation:

Figure 4.4: The slope of the curve of \(b^x\) is similar to \(b^x\) itself — when the function gets bigger the slope gets bigger — aka exponential growth... ye aren’t smooth though so...

As it turns out, \(e^x\) is the unique function of the form \(f(x) = b^x\) for which the slope of the curve of \(b^x\) is equal to \(b^x\). We will write \(\ln x\) for \(\log_e x\), where ‘\(\ln\)’ stands for log naturelle (French). \(e\) is a number, \(e \approx 2.718\).

Further Remark: Base-10

I would argue that:

the pre-eminence of base ten logarithms is a relic from pre-calculator days.

Finding the (base-10) logarithm of positive real numbers without a calculator can be reduced to finding the (base-10) logarithm of numbers (strictly) between 1 and 10 via scientific notation

\[
\log_{10}(x) = \log_{10}(a \times 10^n) = \log_{10}(a) + \log_{10}(10^n) = \log_{10}(a) + n, \quad (*)
\]
It is possible to compile (approximate) logarithm tables\(^3\) for \(1 < a < 10\) and hence we can calculate logs base 10. However this can now be done with a calculator... and why would you want to calculate \(\log_{10}(x)\) in the first place?

The next reason that we might need \(\log_{10}(x)\) is to solve equations like

\[
 b^x = n. 
\]

Now we know that \(x = \log_b n\) but we can use the change of base “formula” to express this in terms of log base 10. Of course the change of base “formula” comes from a calculation like

\[
 \log_{10}(b^x) = \log_{10} n \\
 \Rightarrow x \log_{10}(b) = \log_{10} n \\
 \Rightarrow x = \frac{\log_{10} n}{\log_{10} b}. 
\]

However the new modern calculators can calculate \(\log_b n\) in the first place.

Then you could say what about solving

\[
 b^x = c^{g(x)}. 
\]

Well you don’t need to take a base-10 log: we have the perfectly good base \(e\) natural log!

it seems to me that it is only stuff like the Richter Scale and sound intensity and similar derived quantities and scales that really use base-10 logs and that while base \(e\) logs are clearly useful, that the pre-eminence of base 10 logs is due only to the by-hand-calculation (*).\(^4\)

### 4.4.4 Logarithmic Equations

An exponential equation is one of the form

We can usually takes an appropriate logarithm of both sides to simplify these.

A logarithm equation is of the form

Similarly we can apply a suitable \((\log_b)^{-1} = b^{\log_b}\) to both sides:

---

\(^3\)does this phrase sound familiar?
Examples

1. Solve
\[ \log_2(x^4) - \log_2(x^3) = \log_2(5x) - \log_2(2x). \]

*Solution:* One option is to write this as \( \log(f(x)) = \log(g(x)) \)? We will use \( \log\left(\frac{x}{y}\right) = \log x - \log y \):

Another option is to ‘attack’ the logs straightaway:

2. **Winter 2017** Solve for \( x \)
\[ \log_5(x + 7) + \log_5(x - 3) = 2 \log_5(x). \]

*Solution:* It is a good idea to write down the ‘transform identities’:

Now we proceed as follows:
Alternatively we can ‘attack’ the logs straightaway:

**Computer Science Example: How Many Bits to Represent an Integer?**  
As you may (or may not) know by now, computer memory is based on the concept of a bit or a binary digit. A bit stores just a 0 or a 1. Therefore a single bit can represent two numbers. Now two bits can represent four numbers:

Three bits can represent eight integers:

If we associate e.g. 110 with the ordered triple (1, 1, 0), then we can see that the set of length three binary strings is the same as:

\[
\{0, 1\} \times \{0, 1\} \times \{0, 1\} =: \{0, 1\}^3.
\]

Therefore the number of length three binary strings is:

Length \( n \) binary strings are represented by:

and so, in general, \( n \) bits can represent \( 2^n \) integers. Therefore, the question is, if you want to represent \( N \) integers, how many bits do you need? Suppose for example, you want to represent \( N = 500 \) integers. Suppose that \( n \) bits can do the job. What do you want:

We can solve this equation/inequality for \( n \):
Autumn 2020 How many (binary) bits are required to encode 10,000 numbers?

Solution:

Computer Science Example: Password Length

Suppose that you have an application that users must choose a password for. Suppose further that
the application is poorly designed and accounts are susceptible to random attacks (a bot keeps
trying random assortments of letters). If you can force users to have a long password then the bot
is less likely to guess the password.

Suppose you want there to be at least \( N \) different passwords and users must choose passwords from
an alphabet of size \(|A|\). Then the question is how long should the minimum length of a password
be? Well if the length of the password is \( L \) then there are

\[
|A|^L \quad \text{passwords},
\]

and we want this to be greater than \( N \):

\[
|A|^L \geq N
\]

\[
\Rightarrow L = \log_{|A|}(N)
\]

For example, if you have an alphabet of size 62 (letters, capitals and digits) and want one billion
passwords then you need a password length of six. If you just use 26 lower case letters... seven will
do... this idea is at the heart of the following xkcd comic:

Examples

1. Suppose you have an alphabet of size \(|A| = 52\) and you want at least \(10^8 = 100000000\) possible
   passwords of the same length. What is the minimum length of password?

   Solution: How many passwords of length \(L\)?

Therefore we want:
2. **Winter 2018** Suppose you have an alphabet of size 36 and you want at least 4,000,000,000 distinct passwords. What is the minimum length restriction on your passwords?

**Solution:** How many passwords of length $L$:

Therefore we want:

**Exercises**

1. If you can do these without a calculator do: if completely necessary use your calculator. Solve for $x$:

   (i) $\log_2 8 = x$  
   (ii) $\log_2 x = 3$  
   (iii) $\log_3 81 = x$

   (iv) $\log_5 x = 3$  
   (v) $\log_2 x = 10$  
   (vi) $\log_2 64 = 3$

   (vii) $\log_2 2 = x$  
   (viii) $\log_2 1 = x$  
   (ix) $\log_{25} x = 1/2$
2. Solve for \( n \) (to three significant figures): \( 2^n = 20 \) \( \text{Ans: } x \approx 4.322 \).

3. Solve for \( x \):
   
   (a) Autumn 2018 \( 5 = 3e^{2x} \) \( \text{Ans: } x = \frac{\ln(5/3)}{2} \approx 0.2554 \).
   
   (b) \( 36 = 72 \left(1 - e^{-\frac{x}{3}}\right) \) \( \text{Ans: } x = 3 \ln 2 \approx 2.079 \).
   
   (c) \( \log_{10} \left(\frac{x^2}{\pi}\right) = 2.5 \) \( \text{Ans: } x \approx 53.35 \).
   
   (d) \( 4^x = 512 \) \( \text{Ans: } x = 9/2 \).
   
   (e) \( 3^{2+x} = 5^{x-1} \) \( \text{Ans: } x \approx 7.452 \).
   
   (f) Autumn 2019 \( \text{Solve for } x: \ 3^{x+1} = e^{x+2} \) \( \text{Ans: } x \approx 9.141 \).
   
   (g) Autumn 2020 \( \text{Solve for } y: \ \frac{1}{\log_2 y} = 3 \) \( \text{Ans: } \sqrt{2} \).
   
   (h) Winter 2019 \( \text{Solve, correct to four significant figures, for } x: \)
   
   \[
   \frac{1}{2} \log_2 x + \log_2 9 = 6. \quad \text{Ans: } x \approx 50.57
   \]
   
   (i) Autumn 2019 \( \text{Solve for } x: \)
   
   \[
   \log_3(x) = 3 - \log_3(x + 6) \quad \text{Ans: } x = 3
   \]
   
   (j) \( \log_3(x + 1) - \log_3(x - 1) = 3 \log_3(2) \) \( \text{Ans: } x = 9/7 \).

4. Winter 2019 A computer virus is spreading rapidly through a large network, such that the number of infected machines, \( I \), \( n \) days after the first machine was affected is given by:
   
   \( I = 3^n \).

   After how many days, \( n \), is the number of infected machines greater than 10,000 \( \text{for the first time}: \)
   
   \( 3^n > 10,000 \) \( \text{Ans: } n = 9 \).

5. Find the number of bits required to represent \( N = 1,000,000 \).

6. How many times can we divide 1048576 by two before reaching one?

7. How many times can you divide an array of length 256 in half before you get down to a single-element array?

8. Winter 2019 Consider passwords for an app made from the alphabet \( A = \{a, b, c, \ldots, z\} \) of size 26.
   
   (a) How many passwords, \( N \), of length five can be formed using the alphabet \( A \)? \( \text{Ans: } 11,881,376 \)
   
   (b) Suppose that the app now requires twice as many, \( 2N \), passwords. What should be the minimum length restriction on the passwords? \( \text{Ans: six} \).

9. Winter 2017 Suppose you have an alphabet of size 26 and you want at least 2,000,000 possible passwords. How long should the minimum length of a password be?

10. Autumn 2019 Suppose you have an alphabet of size 26 and you want at least 1,000,000,000 distinct passwords. What is the minimum length restriction on your passwords? \( \text{Ans: seven} \)
11. **Autumn 2020** Solve, giving any solutions correct to four significant figures, for $x$:

$$2^x = 3^{x^2} \quad \text{Ans: } x \approx 0, 0.6309.$$

12. In the following we have “Lhs = Rhs; variable”. Solve for “variable”.

(a) $I = R \cdot t^b; \quad b \quad \text{Ans: } b = \log_t \left( \frac{I}{R} \right)$. 

(b) $y = a \cdot x^b; \quad x \quad \text{Ans: } x = \sqrt[2b]{\frac{y}{a}}$. 

**Chapter Summary**

1. What is $2 + 4 \times 5$ equal to?
   
   $2 + 20 = 22$. 

2. When we talk about ‘$x$’, what are we talking about, roughly?
   
   a number 

3. What is special about the number 1?
   
   $1 \times x = x$. 

4. What is $\frac{1}{0}$ equal to?
   
   it is undefined. 

5. What is $(x - y)(x + y)$ equal to? Put a name on it.
   
   $(x - y) \cdot (x + y) = x^2 - y^2$, the difference of two squares. 

6. What is three ‘sheeps’ plus four ‘sheeps’?
   
   Seven sheep.
   
   Just like $4x + 3x!$ 

7. If you have two fractions and want to add them together what do you need?
   
   A common denominator/bottom. 

8. What is *factorising*?
   
   Writing a sum as a product. 

9. Explain why we can say that

   $$\frac{ac}{bc} = \frac{a}{b}$$. 

   
   $$\frac{ac}{bc} = \frac{ac}{bc} \cdot \frac{\frac{1}{c}}{\frac{1}{c}} = \frac{a}{b}.$$ 

10. What does ‘$=$’ mean? What, when I write it, does $\Rightarrow$ mean?

    Equals. Therefore/implies. 

11. Name one thing that is going to make me go mad if you say it during algebra.

    c****] 

12. What is $a^n$ if $n$ is a natural number?

    $$\underbrace{a \times a \times \cdots \times a}_{\text{n copies of } a}$$ 

13. Do you understand why $a^m a^n = a^{m+n}$?

    $$\underbrace{a^m}_{\text{m copies}} \cdot \underbrace{a^n}_{\text{n copies}} = \underbrace{a^{m+n}}_{\text{m+n copies}}$$
14. What does \( r^0 \) equal?  
   One

15. Can you write \( \frac{1}{y} \) in the form \( y^n \)?  
   \( y^{-1} \)

16. \( 7^{1/2} \times 7^{1/2} = 7^{1+1/2} = 7^1 = 7 \) therefore \( 7^{1/2} = \sqrt{7} \)... does this make sense?  
   Yes as \( \sqrt{7} \) is a number you multiply by itself and get 7.

17. When we write, in algebra, \( ab \) what does this usually mean?  
   \( a \times b \)

18. Which is bigger... \( 0.1^4 \) or \( 0.1^{-4} \)?  
   \( 0.1^{-4} \), dividing by a small number is a big number

19. Explain why logarithms & roots are difficult.  
   They are defined as inverses.
Chapter 5

Functions II - Examples

5.1 Examples

5.1.1 Lines

To talk about lines we must first understand the concept of \textit{slope}.

![Figure 5.1: The slope of a line is a measure of its steepness. We use the symbol \(m\), which is named after the French word, \textit{monter}, to climb. A vertical line is not the graph of a function (why?)](image)

\textbf{Proposition}

Let \(m, c \in \mathbb{R}\) be constants. The graph of function \(\ell : \mathbb{R} \rightarrow \mathbb{R}\)

is a line of slope \(m\) and \(y\)-intercept \(c\).

\textit{Proof.} Suppose we have two points \((x_1, y_1)\) and \((x_2, y_2)\) that satisfy the equation of the curve aka are on the graph:

Take one away from the other:
That is any two points on the graph have the same slope between them. Therefore the graph of $\ell$ is a line. Let $x = 0$:

![Figure 5.2: The graph of $y = mx + c$ cuts the y-axis at $y = c$.](image)

This allows us to easily *roughly* sketch lines:

![Figure 5.3: Some lines. When is a line $\ell(x) = mx + c$ invertible?](image)

Lines are functions of the form $\ell(x) = mx + c$ while *constants* are functions of the form $c(x) = c$.

### 5.1.2 Quadratics

A *quadratic function* $q : \mathbb{R} \to \mathbb{R}$ is a function of the form:

so that

$$q = \{(x, ax^2 + bx + c) : x \in \mathbb{R}\}$$

Here $a, b, c \in \mathbb{R}$ are constants.
It can be shown that, the graph of \( q(x) \) is similar to the graph of \( f(x) = x^2 \) (it is got by translating and stretching the axes).

\[
\begin{align*}
\text{Figure 5.4: Quadratic functions look like } x^2 \quad \text{— hence they can only cut the } x\text{-axis at most twice.}
\end{align*}
\]

### 5.1.3 Polynomials

Any function \( p : \mathbb{R} \to \mathbb{R} \) of the form

\[
p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,
\]

for constants \( a_i \in \mathbb{R} \), is a *polynomial of degree* \( n \); in other words a sum of powers of \( x \). Examples include:

Note that lines are degree one polynomials and quadratics are degree two polynomial. In general, a polynomial of degree \( n \) has \( n \) roots, some of which may be repeated and some of which may be unreal.

The growth rate of polynomials is dominated by the highest power of \( x \) so that for example:
5.1.4 Exponentials

We can define $e^x : \mathbb{R} \to \mathbb{R}_+$ as the unique function such that the slope at each point is equal to the $y$-value, and $e^0 = 1$:

![Figure 5.5: The graph of the exponential function $y = e^x$. $e$ is an actual number, $\approx 2.718$. Note that $e^{-N} = \frac{1}{e^N} \to 0$ (as $e^N \to \infty$). $e^x$ is 1-1, onto $\mathbb{R}_+$, and so bijective with inverse $(\ln x)$.](image)

We are also interested in, for $a \in \mathbb{R}$ a constant, $f(x) = e^{ax}$. These functions have the property that their slope is equal to $a$ times the $y$-value. Thus $e^{a_1x} > e^{a_2x}$ if $a_1 > a_2 > 0$:

![Figure 5.6: The growth rate of exponential functions. $e^{a_1x} \gg e^{a_2x}$ if $a_1 > a_2$](image)
Exponentials eventually dominate — are much larger than — any polynomials:

For example, eventually $e^{0.001x}$ dominates $x^{10000}$. It takes a long time but eventually:

In contrast, for $a < 0$, $a = -b$, exponentials of the form $e^{ax} = e^{-bx}$ decay to zero as $x$ gets large:

Figure 5.7: A sketch of a polynomial and an exponential.

Figure 5.8: For functions of the form $f(x) = e^{-bx}$, the larger the value of $b$, the faster the decay rate.
Examples

1. The graph in the diagram matches which of the functions (a) to (d) below?

\[ f(t) = 5e^{-0.5t} \]  \( (a) \)  
\[ f(t) = 10(1 - e^{-0.5t}) \]  \( (b) \)  
\[ f(t) = 10e^{-0.5t} \]  \( (c) \)  
\[ f(t) = 0.5e^{-10t} \]  \( (d) \)

Solution: The first obvious thing to do is to look at the value of \( f(t) \) at \( t = 0 \)... note that we have \( e^{k(0)} = e^0 = 1 \):

\[
\begin{align*}
(a) & \quad f(0) = 5e^{-0.5(0)} = 5 \\
(b) & \quad f(0) = 10(1 - e^{-0.5t}) = 10(1 - 1) = 0 \\
(c) & \quad f(0) = 10e^{-0.5(0)} = 10 \\
(d) & \quad f(0) = 0.5e^{-10(0)} = 0.5.
\end{align*}
\]

Note now that the graph has \( y = 10 \) at \( t = 0 \). Hence the diagram matches (c).

2. Here is the graph of \( f(t) = 5 + 3e^{-2t} \). Find the values of \( a \) and \( b \).

Solution: We have that \( a \) is the value of \( f(t) \) at \( t = 0 \)... Now as \( t \) gets larger and larger, \( f(t) \) gets closer and closer to \( y = b \). What happens to \( 5 + 3e^{-2t} \) as \( t \) gets large?
5.1.5 Logs

The natural logarithm, \( \log_e \) or \( \ln : \mathbb{R}_+ \to \mathbb{R} \) is the inverse of the function \( e^x : \mathbb{R} \to \mathbb{R}_+ \). Therefore the graph looks as follows:

![Graph of \( y = \ln x \)](image)

Figure 5.9: A rough sketch of \( y = \ln x \). Note \( e^0 = 1 \) and so \( \ln(1) = 0 \). \( \ln x \) grows very slowly: \( \ln(100) \approx 4.6 \) while \( \ln(1000) \approx 6.9 \).

The log function grows slower than any polynomial of degree greater than one (or indeed any power \( f(x) = x^p \) with \( p > 0 \)). This means that even a line eventually dominates \( \ln x \):

![Graph of even lines dominating \( \ln x \)](image)

Figure 5.10: Even lines eventually dominate \( \ln x \).

Examples

1. **Winter 2018** List the following functions according to how fast they grow from slow to fast.

   - \( f(x) = 3^x + 4x \)
   - \( g(x) = 2x^3 - 5x^2 + 3x \)
   - \( h(x) = 6 \log_2(x) \)
   - \( k(x) = 7x + 4 \)
Solution: The dominant terms are:

Therefore, from slow to fast:

2. Autumn 2019 Sketch $f(x) = \log_3(x)$ and $g(x) = 3^x$ in the same plane

- Ensure you label each function.
- Clearly indicate the points at which the graphs intercepts the $x$ and $y$ axes.
- Clearly mark two other points on the graph of each function.

Solution: We need to know that roughly these functions look like:

From here we need to make sure we have answered the question that is asked. $g(x) = 3^x > 0$ and so never hits the $x$-axis. It hits the $y$-axis at $x = 0$ and so $(0,1)$ is in the graph of $g$. Note that as $g^{-1} = f$, $(1,0)$ is in the graph of $f$, and so $f$ hits the $x$ axis there (and never the $y$-axis. Two other points can be found by looking at $g(1)$ and $g(2)$:

Inverting we get $(3,1)$ and $(9,2)$ in the graph of $f$:

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{5.11.png}
\caption{A rough sketch of $f$ and $g$.}
\end{figure}
Further Remark: \( \mathcal{O} \)-Notation for Algorithmic Efficiency

If you pick up any textbook which analyses the speed of algorithms, you will see stuff like “the algorithm is \( \mathcal{O}(n^2) \)”, or the “algorithm is \( \mathcal{O}(\log_2(n)) \)”. For \( n \) the size of the input, these statements refer to the complexity or speed of the algorithms. \( \mathcal{O}(f(n)) \) is said “order \( f(n) \)”, and the faster the function \( f \) grows, the more complex the algorithm, and the longer it will take to run. Roughly, to say that an algorithm is \( \mathcal{O}(f(n)) \) is to say that there is a constant \( C \) such that when the size of the input gets large, \( n \to \), the complexity of the algorithm satisfies:

\[
\text{complexity as a function of } n \leq C \cdot f(n).
\]

This means that if an algorithm has complexity, say,

\[
\text{complexity as a function of } n = 6n^2 + 100n,
\]

we consider what happens when \( n \) is large, so that:

\[
\text{complexity as a function of } n \approx 6n^2 \leq 6 \cdot n^2,
\]

so we would merely say this algorithm is \( \mathcal{O}(n^2) \). Similarly, from above, where \( f, g, h, k \) are algorithm complexity functions:

- \( f(n) = 3^n + 4n \) is \( \mathcal{O}(3^n) \)
- \( g(n) = 2n^3 - 5n^2 + 3n \) is \( \mathcal{O}(n^3) \)
- \( h(n) = 6 \log_2(n) \) is \( \mathcal{O}(\log_2(n)) \)
- \( k(n) = 7n + 4 \) is \( \mathcal{O}(n) \)

and therefore, in terms of algorithmic speed, where \( \leq \) means faster:

\[
\mathcal{O}(1) \leq \mathcal{O}(\log_2 n) \leq \mathcal{O}(n) \leq \mathcal{O}(n^2) \leq \mathcal{O}(n^p) \leq \mathcal{O}(2^n) \quad (5.1)
\]

Some examples of well-known computer science algorithms:

- using a hash function — \( \mathcal{O}(\cdot) \)
- binary search — \( \mathcal{O}(\log_2 n) \)
- heapsort — \( \mathcal{O}(n \cdot \log_2 n) \)
- bubblesort — \( \mathcal{O}(n^2) \)
- travelling salesperson problem using dynamical programming — \( \mathcal{O}(2^n) \)

As exponential functions grow so fast, exponential-time algorithms — such as the last one here — are really slow. Such algorithms are said to be \( NP \) — non-polynomial. If you can prove that every problem that has an \( NP \) algorithm solution also has a \( P \) algorithm solution (\( \mathcal{O}(n^c) \) for finite \( c \)), then you will have solved the \( NP=P \) problem. You will achieve instant fame and instant wealth by way of a one million dollar prize. See [https://www.claymath.org/millennium-problems/p-vs-np-problem](https://www.claymath.org/millennium-problems/p-vs-np-problem) for more.

5.1.6 Absolute Value

The absolute value function \( | \cdot | : \mathbb{R} \to \mathbb{R} \) is defined by:

If you know the graph of \( f \), the graph of \( |f| = \text{abs} \circ f \) is easy to draw:
Figure 5.12: If $f(x_0) = -y_0 < 0$, then $|f(x_0)| = -(-y_0) = y_0 > 0$.

5.1.7 Floor & Ceiling

The *floor function* (round down), $\text{floor} : \mathbb{R} \to \mathbb{R}$ is given by:

The range of floor is $\mathbb{Z}$.

Similarly the *ceiling function* (round up), $\text{ceiling} : \mathbb{R} \to \mathbb{R}$ is given by:

Its range is also $\mathbb{Z}$.

August 2019

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x/2$. Calculate $\lfloor f(3) \rfloor$ and $\lceil f(3) \rceil$.

Solution: We calculate:
Exercises

1. **Autumn 2020** Let \( f : \mathbb{Z} \to \mathbb{Z} \) be defined by \( f(x) = 3x + 1 \).
   
   i. Prove that \( f \) is not onto.
   
   ii. Now let \( f : \mathbb{R} \to \mathbb{R} \). Is \( f \) onto or not? Give a reason for your answer.

2. Consider the function \( f : \{0, 1, 2, 3, 4\} \to \{0, 1, 2\} \) given by \( f(x) = \lfloor \frac{x}{2} \rfloor \).
   
   (a) Write down the ordered pairs that define this relation.
   
   (b) Graphically represent the function using an arrow diagram.
   
   (c) Is the function invertible? If yes, write down the ordered pairs that define \( f^{-1} \). Otherwise, explain why the function is not invertible.

3. The graph in the diagram matches which of the functions (a) to (d) below?

   \[
   (a) f(t) = 5e^{-0.5t} \quad (b) f(t) = 10(1 - e^{-0.5t}) \quad (c) f(t) = 10e^{-0.5t} \quad (d) f(t) = 0.5e^{-10t}.
   \]

4. Here is the graph of \( f(t) = 5 + 3e^{-2t} \). Find the values of \( a \) and \( b \).
5. Below there are seven lines plotted. Match the line functions to their graphs: \( \ell_1(x) = 20 \), \( \ell_2(x) = x - 10 \), \( \ell_3(x) = 2x \), \( \ell_4(x) = -4x \), \( \ell_5(x) = 5 - x \), \( \ell_6(x) = -2x - 5 \), \( \ell_7(x) = 2x + 5 \).

6. Winter 2017 Below see a plot of the graphs of the \( \mathbb{R} \to \mathbb{R} \) functions \( f(x) = 1 + e^x \) and \( g(x) = e^{2x} \). Label appropriately:
7. Below see a plot of the graphs of the $\mathbb{R} \to \mathbb{R}$ functions $f(x) = e^{-3x}$ and $g(x) = e^{-x}$. Label appropriately:

![Graph of $f(x) = e^{-3x}$ and $g(x) = e^{-x}$](image)

8. Below see a plot of the graphs of the $\mathbb{R}_+ \to \mathbb{R}$ functions $f(x) = \ln x$ and $g(x) = x^3 + 2$. Label appropriately:

![Graph of $f(x) = \ln x$ and $g(x) = x^3 + 2$](image)
9. Complete the following:

Figure 5.13: Circle the function that corresponds to the graph: \( x^2 + 2, \ |x^3|, \ 4x + 1, \ e^{2x}, \ e^{-x} + 2 \).

Figure 5.14: Circle the function that corresponds to the graph: \( x^2 + 2, \ |x^3|, \ 4x + 1, \ e^{2x}, \ e^{-x} + 2 \).

Figure 5.15: Circle the function that corresponds to the graph: \( x^2 + 2, \ |x^3|, \ 4x + 1, \ e^{2x}, \ e^{-x} + 2 \).
10. Let $m$ and $c$ be constants and define $\ell : \mathbb{R} \rightarrow \mathbb{R}$ by $\ell(x) = mx + c$.

(a) Use a rough sketch to determine for which value(s) of $m$ and $c$ is $\ell$ not a bijection?

(b) Under the assumption that $\ell$ is a bijection, find the inverse, $\ell^{-1} : \mathbb{R} \rightarrow \mathbb{R}$, by solving $y = \ell(x)$ for $x = \ell^{-1}(y)$.

(c) The algebra from the previous part appears to give a formula for $\ell^{-1}$ no matter what the value(s) of $m$ and $c$ are. Explain how this formula is not valid when $\ell$ is not a bijection.

11. **Autumn 2018** Produce rough sketches of the following lines (you may plot a number of lines on a single plane):

   (i) $\ell_1(x) = 2x$,  
   (ii) $\ell_2(x) = -3x$,  
   (iii) $\ell_3(x) = 2x + 3$,  
   (iv) $\ell_4(x) = -4x + 2$,  
   (v) $\ell_5(x) = -4x - 3$.

12. In each case, produce a rough sketch of the graphs of both functions on a single axis:

   (i) $e^x$ & $x^2$,  
   (ii) $e^{-x}$ & $e^{-2x}$,  
   (iii) $\ln x$ & $y = 3$,  
   (iv) $\ln x$ & $x + 1$. 
13. Consider the floor function, $\text{floor}(x) = \lfloor x \rfloor$.

(a) Is $\text{floor} : \mathbb{R} \to \mathbb{Z}$ onto?
(b) Is $\text{floor} : \mathbb{R} \to \mathbb{R}$ onto?

14. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x/2$ and $g(x) = x/3$. Calculate $\lfloor f(3) \rfloor$, $\lceil g(4) \rceil$, $\lceil f(g(23)) \rceil$.

15. Given the graph of $f : \mathbb{R} \to \mathbb{R}$ below, draw the graphs of $|f(x)|$, $\lfloor f(x) \rfloor$, and $\lceil f(x) \rceil$.

16. Autumn 2018 List the following list of functions $\mathbb{R}_+ \to \mathbb{R}$ from slowest growing to fastest growing:

\[ 2x^2, 4 \ln x, 2x, x^{10}, 3x^3, 2e^{3x}, e^{\frac{1}{2}x} \]

17. In each case, circle the fastest growing function of the three as $x \to \infty$:

(a) $4x^2, \ln x, 0.9x^3$
(b) $x^{100}, 100x, e^{-3x}$
(c) $10^9, e^{0.01x}, 0.01e^{-2x}, e^x$
(d) List the following list of functions $\mathbb{R} \to \mathbb{R}$ from slowest decay to fastest decay:

\[ e^{-x}, e^{-0.5x}, e^{-3x} \]
Chapter Summary

1. A constant function is of the form \( f(x) = k \) for a constant \( k \in \mathbb{R} \).
2. A line function is of the form \( f(x) = mx + c \) for constants \( m, c \in \mathbb{R} \). The slope of the line is \( m \), and the \( y \)-intercept is \( c \).
3. A quadratic function is of the form \( f(x) = ax^2 + bx + c \) for constants \( a, b, c \in \mathbb{R}, a \neq 0 \).
4. A polynomial of degree \( n \) is of the form \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) for constants \( a_i \in \mathbb{R}, a_n \neq 0 \).
5. The exponential function, \( e^x : \mathbb{R} \to \mathbb{R}_+ \), \( f(x) = e^x \), is the unique function \( f : \mathbb{R} \to \mathbb{R}_+ \) such that the slope of the graph as \( x \) is equal to the \( y \)-coordinate, and \( f(0) = 0 \).
6. An exponential function is of the form \( f(x) = e^{ax} \) for some non-zero constant \( a \in \mathbb{R} \). Growth for \( a > 0 \) and decay for \( a < 0 \).
7. The natural logarithm function, \( \ln : \mathbb{R}_+ \to \mathbb{R} \), is the inverse of the exponential function.
8. The absolute value function, \( f(x) = |x| \), is given by:
\[
|x| = \begin{cases} 
+x & \text{if } x \geq 0 \\
-x & \text{if } x < 0.
\end{cases}
\]
9. The floor function (round down), \( f(x) = \lfloor x \rfloor \), is given by:
\[
\lfloor x \rfloor = \max\{m \in \mathbb{Z} | m \leq x\}.
\]
10. The ceiling function (round up), \( f(x) = \lceil x \rceil \), is given by:
\[
\lceil x \rceil = \min\{m \in \mathbb{Z} | m \leq x\}.
\]
11. For sufficiently large \( x \), and constants \( a, k, m, a_n \in \mathbb{R} \), and \( n \in \mathbb{N}, n > 2 \):
\[
\left| e^{-(a+\delta)x} \right| \ll \left| e^{-ax} \right| \ll |k| \ll |\ln x| \ll |mx| \ll |ax^2| \ll |a_nx^n| \ll |e^{ax}| \ll |e^{+(a+\delta)x}| \quad \begin{array}{c} a,\delta>0 \\
\to0 \end{array} 
\quad \begin{array}{c} \to\infty \end{array}
\]
\[
\left| e^{-(a+\delta)x} \right| \ll \left| e^{-ax} \right| \ll |k| \ll |\ln x| \ll |mx| \ll |ax^2| \ll |a_nx^n| \ll |e^{ax}| \ll |e^{+(a+\delta)x}| \quad \begin{array}{c} a,\delta>0 \\
\to0 \end{array} 
\quad \begin{array}{c} \to\infty \end{array}
\]

Indices and Logarithms

\[ a^p a^q = a^{p+q} \]
\[ \frac{a^p}{a^q} = a^{p-q} \]
\[ (a^p)^q = a^{pq} \]
\[ a^0 = 1 \]
\[ a^{-p} = \frac{1}{a^p} \]
\[ \frac{1}{a^q} = \sqrt[q]{a} \]
\[ a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p \]
\[ (ab)^p = a^p b^p \]
\[ \left( \frac{a}{b} \right)^p = \frac{a^p}{b^p} \]

\[ \log_a (xy) = \log_a x + \log_a y \]
\[ \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \]
\[ \log_a (x^q) = q \log_a x \]
\[ \log_a 1 = 0 \]
\[ \log_a \left( \frac{1}{x} \right) = -\log_a x \]

\[ \log_a (a^x) = x \]
\[ a^{\log_a x} = x \]

Sets

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<tr>
<td>Identity Laws</td>
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<td>Annihilation Laws</td>
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