Quantum Permutations Re-Born

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⚠Warning ⚠

I told Teo I had a "safe" talk (about the ergodic theorem for random walks on finite quantum groups) and a "dangerous" talk.

This is the dangerous talk!

Hopefully your feeling at the end is something like:

> "I'm not sure was that even mathematics, but it was thought-provoking."

01/09/20

Motivation for Compact Quantum Groups

Gelfand Philosophy:

compact quantum space $\simeq$ (unital C^*-algebra)$^{\mathrm{op}}$,

$$\mathbb{X} := \Omega(C(\mathbb{X})) \quad \longleftrightarrow \quad C(\mathbb{X}) := A$$

Wang's Quantum Permutation Group:

- $C(S_N^+) := \mathrm{C}^*(u_{ij} :\ u$ a magic unitary),

$$\Delta(u_{ij}) := \sum_{k=1}^{N} u_{ik} \otimes u_{kj}; \quad \varepsilon(u_{ij}) = \delta_{i,j}; \quad S(u_{ij}) = u_{ji}, \text{ etc.}$$

- $[N] := \{1, 2, \ldots, N\} \curvearrowleft \mathrm{QAut}([N]) \quad \longleftrightarrow \quad \alpha : \mathbb{C}^N \to \mathbb{C}^N \otimes C(\mathrm{QAut}([N]))$.

Question about $C_r(G) := \pi_h(C(G))$.

01/09/20

Intuition for Quantum Permutations?

A quantum permutation matrix which will somehow map two "a little bit" to three and "a little bit" to four — which appears to be quite "quantum".

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ vs } \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$\nu = c\delta^{(1243)} + (1-c)\delta^{(23)}, \qquad c \in (0,1).$$

Non-commutativity/Non-simultaneous observation?

Can Gelfand Philosophy talk of a quantum permutation?

01/09/20

UTR TQM

Quantum state: $\psi \in P(\mathrm{H})$ measured by C^*-algebra $A \subset B(\mathrm{H})$.

⚠ $|\sigma(f)|$-finite $f \in B(\mathrm{H})_{\text{s.a.}}$ has o.n.b. and s.d.:

$$\Omega_f := \{e_\alpha\}_\alpha \quad \text{and} \quad f = \sum_i \lambda_i^f p_i^f : \quad \text{call} \quad p_i^f(\psi) =: \psi[f = \lambda_i^f].$$

$$f : \Omega_f \to \mathbb{R} : \qquad f = \lambda_i^f \text{ and } \psi \mapsto \psi[f = \lambda_i^f].$$

$$\mathbb{P}[f = \lambda_i^f] = \left|\left\langle \psi, \psi[f = \lambda_i^f] \right\rangle\right|^2 \text{ and } \mathbb{E}_\psi[f] = \langle \psi, f(\psi) \rangle. \quad ⚠$$

$$\mathbb{P}[(g = \lambda_j^g) \succ (f = \lambda_i^f) \mid \psi] = \underbrace{\left|\left\langle \psi, \psi[f = \lambda_i^f] \right\rangle\right|^2}_{\mathbb{P}[f = \lambda_i \mid \psi]} \underbrace{\left|\left\langle \psi[f = \lambda_i^f], \psi[(g = \lambda_j^g) \succ (f = \lambda_i^f)] \right\rangle\right|^2}_{\mathbb{P}[g = \lambda_j \mid \psi[f = \lambda_i^f]]}.$$

$$\mathbb{P}[(g = \lambda_j^g) \cap (f = \lambda_i^f) \mid \psi] = \mathbb{P}[(f = \lambda_i^f) \cap (g = \lambda_j^g) \mid \psi]$$

vs

$$\mathbb{P}[(g = \lambda_j^g) \succ (f = \lambda_i^f) \mid \psi] \neq \mathbb{P}[(f = \lambda_i^f) \succ (g = \lambda_j^g) \mid \psi]$$

01/09/20

Weaver Philosophy

Inspired by the Gelfand–Naimark Theorem rather than Gelfand's Theorem:

> "The fundamental idea of mathematical quantisation is that sets are replaced by Hilbert spaces... [and] the quantum version of a [real]-valued function on a set is a [self-adjoint] operator on a Hilbert space."
>
> Nik Weaver, Mathematical Quantisation

$|\sigma(f)|$-finite $f \in B(\mathsf{H})_{\text{s.a.}}$ is a random variable:

$$f : \Omega_f \to \mathbb{R}.$$

fg simultaneously observable $\Leftrightarrow$ see same set $\Leftrightarrow$ $fg = gf$

Quantum Space $\longleftrightarrow$ Hilbert Space

Quantum Permutation $\longleftrightarrow$ $\varsigma \in P(\mathsf{H})$.

Learn about ς with sequential measurements from $B(\mathsf{H})_{\text{s.a.}}$

01/09/20

The u_{ij}

$F(S_N)$ is the diagonal subalgebra of $B(\ell^2(S_N))$:

$$u_{ij}^{S_N}(e_\sigma) = \mathbb{1}_{j\to i}(\sigma)\, e_\sigma.$$

Projections are Bernoulli rv in UTRTQM; for $\varsigma \in P(\ell^2(S_N))$:

$$\mathbb{E}_\varsigma[u_{ij}^{S_N}] = \left\langle \varsigma, u_{ij}^{S_N}(\varsigma)\right\rangle = \mathbb{P}[u_{ij}^{S_N} = 1 \,|\, \varsigma] =: \mathbb{P}[\varsigma(j) = i].$$

$C(S_N^+) \subset B(\mathrm{H}_N^+)$, define the Birkhoff Slice $\Phi : P(\mathrm{H}_N^+) \to \mathscr{B}_N$:

$$\begin{aligned}\Phi(\varsigma) &= [\langle \varsigma, u_{ij}(\varsigma)\rangle]_{i,j=1}^N \\ &= \left[\frac{\langle \varsigma, u_{ij}(\varsigma)\rangle}{\|\varsigma\|^2}\right]_{i,j=1}^N \\ &=: [\mathbb{P}[\varsigma(j) = i]]_{i,j=1}^N\end{aligned}$$

01/09/20

Conditioning

Assuming $\mathbb{P}[\varsigma(k)=\ell]\neq 0$, condition ς to

$$\varsigma[\varsigma(k)=\ell]\equiv u_{\ell k}(\varsigma)\equiv \frac{u_{\ell k}(\varsigma)}{\|u_{\ell k}(\varsigma)\|}.$$

Now study, e.g.

$$\Phi(\varsigma[\varsigma(k)=\ell])=\Phi(u_{\ell k}(\varsigma))=\left[\frac{\langle \varsigma, u_{ij}(u_{\ell k}(\varsigma))\rangle}{\|u_{\ell k}(\varsigma)\|^2}\right]_{i,j=1}^{N}$$
$$=[\mathbb{P}[\varsigma(j)=i\,|\,\varsigma(k)=\ell]_{i,j=1}^{N}$$

Inductively, assuming $u_{i_n j_n}\cdots u_{i_1 j_1}(\varsigma)\neq 0$

$$\Phi(u_{i_n j_n}\cdots u_{i_1 j_1}(\varsigma))_{ij}=\mathbb{P}[\varsigma(j)=i\,|\,(\varsigma(j_n)=i_n)\succ\cdots\succ(\varsigma(j_1)=i_1)]$$

Assuming conditioned on non-null events:

$$\mathbb{P}[(\varsigma(j)=i)\succ(\varsigma(j_n)=i_n)\succ\cdots\succ(\varsigma(j_1)=i_1)]$$
$$=\Phi(u_{i_n j_n}\cdots u_{i_1 j_1}\varsigma)_{ij}\cdot\Phi(u_{i_{n-1} j_{n-1}}\cdots u_{i_1 j_1}\varsigma)_{i_n j_n}\cdots\Phi(\varsigma)_{i_1 j_1}$$

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Quantum Permutation Groups

CMQG $(C(K),u^K,\Delta_K)$: u^K is an $N\times N$ magic unitary:

$$\pi : C(S_N^+)\to C(K),\quad u_{ij}\mapsto u_{ij}^K \qquad \Rightarrow \qquad K\leq S_N^+.$$

Kac–Paljutkin Finite Quantum Group:

$$F(\mathfrak{G}_0)=\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus M_2(\mathbb{C}).$$

Where $f_1=(1,0,0,0,0)$, $f_2=(0,1,0,0,0)$, etc., $I_2\in M_2(\mathbb{C})$ the identity, and the projection

$$p:=\left(0,0,0,0,\begin{pmatrix}\frac{1}{2} & \frac{1}{2}e^{-i\pi/4}\\ \frac{1}{2}e^{+i\pi/4} & \frac{1}{2}\end{pmatrix}\right),$$

$\mathfrak{G}_0<S_4^+$ [BBN] via

$$u^{\mathfrak{G}_0}:=\begin{bmatrix} f_1+f_2 & f_3+f_4 & p & I_2-p\\ f_3+f_4 & f_1+f_2 & I_2-p & p\\ p^T & I_2-p^T & f_1+f_3 & f_2+f_4\\ I_2-p^T & p^T & f_2+f_4 & f_1+f_3\end{bmatrix}$$

01/09/20

Kac–Paljutkin Quantum Group

$F(\mathfrak{G}_0) \subset B(\mathbb{C}^6)$; $\quad e_1 = e_e,\, e_2 = e_{(34)},\, e_3 = e_{(12)},\, e_4 = e_{(12)(34)}.$

Consider $\varsigma = (0,0,0,0,e^{i\pi/4},1) \in P(\mathbb{C}^6)$ and

$$S := \{(13)(24),(14)(23),(1423),(1324)\}.$$

For all $\sigma \in S$

$$\mathbb{P}[(\varsigma(2)=\sigma(2)) \succ (\varsigma(1)=\sigma(1)) \succ (\varsigma(4)=\sigma(4)) \succ (\varsigma(3)=\sigma(3))] = \frac{1}{4}.$$

However: $\mathbb{P}[(\varsigma(2)=3) \succ (\varsigma(3)=1) \succ (\varsigma(1)=3)] = \dfrac{2+\sqrt{2}}{8}.$

Uncertainty phenomenon:

$$\Phi(u_{41}\varsigma) = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \to \Phi(u_{13}u_{41}\varsigma) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

01/09/20

An Infinite Example

Witness to $|S_N^+| \not< \infty$ for $N \geq 4$ via $u^{\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}}$:

$$\begin{bmatrix} \overline{p} & p & 0 & 0 \\ p & \overline{p} & 0 & 0 \\ 0 & 0 & \overline{q} & q \\ 0 & 0 & q & \overline{q} \end{bmatrix} \cong \begin{bmatrix} \frac{1}{2}\delta^e + \frac{1}{2}\delta^u & \frac{1}{2}\delta^e - \frac{1}{2}\delta^u & 0 & 0 \\ \frac{1}{2}\delta^e - \frac{1}{2}\delta^u & \frac{1}{2}\delta^e + \frac{1}{2}\delta^u & 0 & 0 \\ 0 & 0 & \frac{1}{2}\delta^e + \frac{1}{2}\delta^v & \frac{1}{2}\delta^e - \frac{1}{2}\delta^v \\ 0 & 0 & \frac{1}{2}\delta^e - \frac{1}{2}\delta^v & \frac{1}{2}\delta^e + \frac{1}{2}\delta^v \end{bmatrix}$$

Here u and v are respective generators of $\mathbb{Z}_2$ in $\mathbb{Z}_2 * \mathbb{Z}_2$:

$$\mathrm{C}^*(p,q) \cong \mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2) =: C_u(\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}).$$

$$\mathrm{C}^*(p,q) \cong \{f \in C([0,1], M_2(\mathbb{C})) \colon f(0), f(1) \text{ are diagonal}\}$$
$$\subset B(\mathsf{H}^{\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}}) := B\left(L^2\left([0,1], \mathbb{C}^2\right)\right) \qquad [Pederson]$$

$$\langle f, g \rangle_{\mathsf{H}^{\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}}} = \int_0^1 \langle f(x), g(x) \rangle_{\mathbb{C}^2}\, dx = \int_0^1 \left(\overline{f(x)_1}\, g(x)_1 + \overline{f(x)_2}\, g(x)_2 \right) dx$$

An Infinite Example

$p, q \in \mathrm{C}^*(p,q)$ are mapped to:

$$p(x) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } q(x) := \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}$$

As an example, consider

$$\varsigma(x) = \begin{pmatrix} ix \\ \sqrt{1-x} \end{pmatrix}$$

$$\Phi(\varsigma) = \begin{bmatrix} 3/5 & 2/5 & 0 & 0 \\ 2/5 & 3/5 & 0 & 0 \\ 0 & 0 & 3/10 & 7/10 \\ 0 & 0 & 7/10 & 3/10 \end{bmatrix} \to \Phi(\overline{p}\varsigma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$$

$$\Rightarrow \mathbb{P}[\varsigma(1) = 1] = \frac{3}{5} \text{ and } \mathbb{P}[(\varsigma(3) = 4) \succ (\varsigma(1) = 1)] = \frac{2}{5},$$

01/09/20

An Infinite Example

$$\Phi(q\overline{p}\varsigma) = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \Phi(pq\overline{p}\varsigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3/5 & 2/5 \\ 0 & 0 & 2/5 & 3/5 \end{bmatrix}$$

$$\Rightarrow \mathbb{P}[(\varsigma(3)=3) \succ (\varsigma(1)=2) \succ (\varsigma(3)=4) \succ (\varsigma(1)=1)] = \frac{3}{50}.$$

Recall $P(\mathbb{C}^6)$, measured by $F(\mathfrak{G}_0)$ had classical permutations, and:

$$\operatorname{span}(e_e, e_{(12)}, e_{(34)}, e_{(12)(34)}) = \ell^2(\mathbb{Z}_2 \times \mathbb{Z}_2),$$

the random permutations:

$$\text{Classical} \subset \text{Random} \subset \text{Quantum}\,.$$

01/09/20

Classical ⊂ Random ⊂ Quantum

Let $C(S_N^+) \subset B(\mathsf{H}_N^+)$ be fixed & faithful:

- Each $\sigma \in S_N$ gives (non-trivial?) subspace: $\mathsf{H}_\sigma := \bigcap_{j=1}^{N} \operatorname{ran} u_{\sigma(j),j}$.
- For all $f \in C(S_N^+)$, $f|_{\mathsf{H}_\sigma}$ is a scalar.
- Elements of different H_σ are orthogonal.
- For each $f \in C(S_N^+)$: $f|_{\bigoplus \mathsf{H}_\sigma} = \sum_{\sigma \in S_N} \lambda_\sigma I_{\mathsf{H}_\sigma}$.
- ⚠ The Hilbert spaces H_σ can be taken to be one-d. ⚠
- Choose unit e_σ from H_σ: $\ell^2(S_N) \cong \operatorname{span}(e_\sigma : \sigma \in S_N)$.

$P(\mathsf{H}^{\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}})$ measured by $\mathrm{C}^*(p,q)$ has no classical permutations.
Proof: Take a pair from $p, \overline{p}, q, \overline{q}$. Their ranges have trivial intersection •

⚠ Universal GNS: $\Phi(\xi_\varepsilon)_{i,j} = \langle \xi_\varepsilon, \pi_\varepsilon(u_{ij})(\xi_\varepsilon)\rangle = \varepsilon(u_{ij}) = \delta_{i,j}$ ⚠

01/09/20

Tweetable Mathematics

When playing poker with a quantum decks of cards, you can only look at one card at a time.

On first sight you might find three aces of hearts, and two of spades, but when you reveal your hand to claim the pot you suddenly have nothing but a pair of twos.

01/09/20

What else?

- $S_3^+ = S_3$: given "$\varsigma(1) = u_{11} + 2u_{21} + 3u_{31}$": $\varsigma(2)$ and $\varsigma(3)$ are "entangled".

- Quantum permutations in the lab?

- Thoughts on counit, antipode, and quantum group law:

$$\Delta(u_{ij})(\varsigma_2 \otimes \varsigma_1) = \sum_{k=1}^{N} u_{ik}(\varsigma_2) \otimes u_{kj}(\varsigma_1).$$

- Invariant Subspaces as Sub-Objects

$$F(\mathfrak{G}_0)(\mathbb{C}^4) = \mathbb{C}^4 \subset \mathbb{C}^6 \quad \longleftrightarrow \quad \mathbb{Z}_2 \times \mathbb{Z}_2 < \mathfrak{G}_0.$$

- Dual Finite Groups, believe $\widehat{S_3} < S_{36}^+$ [BBN].

01/09/20

What else?

- Hopf Images:

$$C(S_4^+) \to C(G_v) \to \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} =: v$$

- "Unobserved" random walks on S_N^+ behave classically? Random transposition random walk on S_N^+ converges at same rate as on S_N — $\frac{1}{2} N \ln N$ [FTW]

- Fast and slow strong convergence of $((p_k \cdots p_1)^n)_{n\geq 1}$ to projection onto $\bigcap_{i=1}^k \operatorname{ran} p_i$ [Badea, Grivaux, Müller]

- Beyond quantum symmetries of $\{1, 2, \ldots, N\} \sim \mathbb{C}^N$?

01/09/20

Some Moonshine

Let $\varsigma_0 \in P(\mathbb{H}_N^+)$ be a quantum permutation.

Consider (possibly 'random' is some sense) sequential measurements of ($|\sigma(f_i)|$-finite?) observables $(f_i)_{i\geq 1} \subset C(S_N^+)$.

These produce a (random) sequence of quantum permutations:

$$\varsigma_0 \longrightarrow \zeta_1 \longrightarrow \zeta_2 \longrightarrow \cdots \longrightarrow \zeta_k \longrightarrow \cdots$$

Consider (if it can be defined)

$$\mathbb{P}[\zeta_k \text{ classical}] = \sum_{\sigma \in S_N} \mathbb{P}[\zeta_k \in \mathbb{H}_\sigma].$$

How does this grow? (always zero on $P(\mathbb{H}^{\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}})$)

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Some Moonshine

Measured with $(u_{11}^{S_3}, u_{22}^{S_3}, u_{33}^{S_3}, u_{21}^{S_3})$, the random permutation

$$\varsigma_0 = \sum_{\sigma \in S_3} \frac{1}{\sqrt{6}} e_\sigma \in P(\ell^2(S_3));$$

k	1	2	3	4
$\mathbb{P}[\zeta_k$ classical$]$	0	1/2	2/3	1

Measured with $(u_{11}^{\mathfrak{G}_0}, u_{33}^{\mathfrak{G}_0}, u_{12}^{\mathfrak{G}_0}, u_{24}^{\mathfrak{G}_0})$, the quantum permutation

$$\varsigma_0 = \sum_{i=1}^{6} \frac{1}{\sqrt{6}} e_i \in P(\mathbb{C}^6),$$

k	1	2	3	4	∞
$\mathbb{P}[\zeta_k$ classical$]$	0	1/2	2/3	2/3	2/3

Is there meaningful restriction to (quantum not just random) ς_0 and measurements such that:

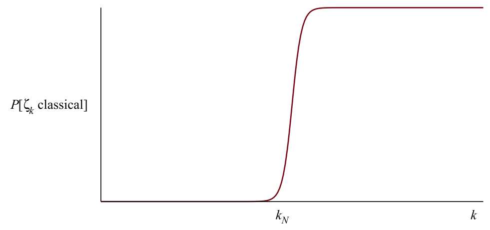
$$\lim_{k\to\infty} \mathbb{P}[\zeta_k \text{ classical}] = 1?$$

01/09/20

Really Reaching Moonshine

$\exists ?\ \varsigma_0^N \in \mathrm{H}_N^+$, measurements $(f_i^N)_{i\geq 1} \subset C(S_N^+)$, 'decoherence time' k_N, such that as $N\to\infty$ the 'window' $w_N/k_N \to 0$, and for $c>1$:

$$0 \underset{k<k_N-cw_N}{\approx} \mathbb{P}[\zeta_k \text{ classical}] \underset{k>k_N+cw_N}{\approx} 1.$$



"In macro", measurements "suddenly" reveal a classical object.

Inspired by [DS,F], might exist $\nu_N \in \mathscr{S}(C(S_N^+))$:

$$\mathbb{P}[\zeta_k \text{ classical}] \approx 1 - d(\nu_N^{\star k}, h)?$$

01/09/20

Fin

Thank you for listening/I am sorry

(Delete as appropriate)