THE ERGODIC THEOREM FOR RANDOM WALKS ON FINITE QUANTUM GROUPS

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Abstract. Necessary and sufficient conditions for a Markov chain to be ergodic are that the chain is irreducible and aperiodic. This result is manifest in the case of random walks on finite groups by a statement about the support of the driving probability: a random walk on a finite group is ergodic if and only if the support is not concentrated on a proper subgroup, nor on a coset of a proper normal subgroup. The study of random walks on finite groups extends naturally to the study of random walks on finite quantum groups, where a state on the algebra of functions plays the role of the driving probability. Necessary and sufficient conditions for ergodicity of a random walk on a finite quantum group are given on the support projection of the driving state.

Introduction

Let $\sigma_1, \sigma_2, \dots, \sigma_k$ be a sequence of shuffles of a deck of cards. If the deck starts in some known order, the order of the deck after these k shuffles is given by

$$\Sigma_k = \sigma_k \cdots \sigma_2 \cdot \sigma_1.$$

Suppose the shuffles are random variables independently and identically distributed as $\sigma_i \sim \nu \in M_p(S_{52})$, where $M_p(S_{52})$ is the set of probability distributions on S_{52}, then $\Sigma_k \sim \nu^{\star k}$ where $\nu^{\star k}$ is defined inductively by

$$\nu^{\star(k+1)}(\{\sigma\}) = \sum_{\varrho \in S_{52}} \nu(\{\sigma\varrho^{-1}\})\nu^{\star k}(\{\varrho\}).$$

This generalises to arbitrary finite groups. Given independent and identically distributed $s_i \sim \nu \in M_p(G)$, consider the random variable:

$$\xi_k = s_k \cdots s_2 \cdot s_1 \sim \nu^{\star k}. \tag{1}$$

If the convolution powers $(\nu^{\star k})_{k\geq 1}$ converge to the uniform distribution $\pi \in M_p(G)$, the random walk is said to be *ergodic*.

Although, Diaconis [8] references the appearance of the random transposition shuffle in an enumerative combinatorics problem in the study of Riemann Spheres, considered by

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Hurwitz in the 1890s, the study of random walks on finite groups probably has its roots in questions of Markov [25] and Borel (and coauthors) [5], who asked which card shuffles would mix up a deck of cards. This qualitative question, the inspiration for the current work, is answered by a folklore theorem, which gives conditions on the support of the *driving* probability that are equivalent to ergodicity:

Theorem. *Ergodic Theorem for Random Walks on Finite Groups* Let $\nu \in M_p(G)$ be a probability on a finite group G. The associated random walk is ergodic if and only if ν is not concentrated on a proper subgroup nor the coset of a proper normal subgroup •

One might remark that the detection of whether or not a subset is concentrated on a proper subgroup, or on a coset of a proper normal subgroup, is non-trivial in itself. The (rarely written down) proof may be found in the MSc thesis [28], and gives one an idea how to carry out the detection of this condition.

What follows this qualitative question is the quantitative: if a given random walk on a finite group is ergodic, how many transitions k before the distribution of ξ_k is 'close' to uniform? The distance to uniform is measured using *total variation distance*:

$$\|\nu^{\star k} - \pi\| := \sup_{S \subset G} |\nu^{\star k}(S) - \pi(S)|.$$

The representation-theoretic upper bound lemma of Diaconis and Shahshahani [9] proved a most useful tool in answering this question and coming up with estimates of convergence rates for many random walks.

The study of random walks on finite groups, under the programme of quantum probability, extends in a natural way to the study of random walks on *quantum* groups. Early work on quantum stochastic processes by various authors led to random walks on duals of compact groups (particularly Biane, see [13] for references), and other examples, but Franz and Gohm [13] defined with clarity a *random walk on a (finite) quantum group.*

A quantum group is a so-called *virtual object*; in general it does not exist as a mathematical object, but is instead defined via its algebra of functions, an object that is a noncommutative generalisation of some commutative algebra of functions on a group. Gelfand's Theorem [29] says that any unital commutative C*-algebra is the algebra of continuous functions, $C(X)$, on some compact Hausdorff space X. The *Gelfand Philosophy* says that a unital noncommutative C*-algebra A should be considered the algebra of functions on a *quantum space.* A quantum space X is a virtual object, but can be spoken about through its algebra of functions $C(X) := A$. The algebra of functions on a quantum group is an algebra which inherits, through a bialgebra structure, axioms equivalent to the group axioms whenever the algebra is commutative (see Section 1.1).

When, for example with the representation theory of compact quantum groups, the noncommutative theory generalises so nicely from the commutative theory, it can be useful to refer to a virtual object as if it exists: this approach helps point towards appropriate noncommutative definitions, and sometimes even towards results, such as the Peter-Weyl Theorem, that are true in this larger class of objects. Even when commutative results do not generalise to this larger class, the Gelfand Philosophy gives a pleasing notation, helping readers from the commutative world understand better what is going on in the noncommutative world. This current work employs the Gelfand Philosophy liberally. The algebra of functions on a finite quantum group G will be denoted $F(G)$; where references usually denote elements of a C*-algebra by $a \in A$, in this work $f \in F(G)$ is used to emphasise that such elements should be considered functions on a quantum space; instead of denoting the unit by 1_A, $\mathbb{1}_G$ is used; instead of denoting the states by $\mathcal{S}(A)$, the notation $M_p(G)$ (classically the probability measures on G) will be used; etc. This philosophical approach ramped up in the 2000s, and into the 2010s, and up to 2020 many authors denote an arbitrary quantum group with a blackboard $\mathbb{G}$. The current work will follow the more radical approach of some authors of just using 'G'.

As will be seen in Section 5.4, as the representation theory generalises so well from classical to quantum, the upper bound lemma of Diaconis and Shahshahani can also be used to analyse random walks on quantum groups. The upper bound lemma has been used to analyse random walks on the dual symmetric group, $\widehat{S_n}$ [27]; Sekine quantum groups, Y_n [2, 27]; the Kac–Paljutkin quantum group, $\mathfrak{G}_0$ [2]; free orthogonal quantum groups, O_N^+ [17]; free symmetric quantum groups, S_N^+ [17]; the quantum automorphism group of $(M_N(\mathbb{C}), \mathrm{tr})$ [17]; free unitary groups, U_N^+ [18]; free wreath products $\widehat{\Gamma} \wr_* S_N^+$, including quantum reflection groups H_N^{s+} [18]; duals of discrete groups, $\widehat{\Gamma}$, including for $\Gamma = \mathbb{F}_N$ the free group on N generators, [19].

However, the basic qualitative question: what are the conditions on the driving probability for a random walk on a quantum group to be ergodic; has remained open since at least 1996 when Pal [30] showed that the ergodic theorem for random walks on finite groups does not extend to the quantum setting, that there exist random walks on quantum groups that are not ergodic, but neither is $\nu \in M_p(G)$ concentrated on any proper quantum subgroup, nor does it have the periodicity associated with being concentrated on a coset of a proper normal subgroup.

The problem has been described as "clearly more complicated" (than the classical case) [14], and "open" [17]. The author has described not having this result a "deficiency" of their PhD thesis [26]. The irreducibility condition (see Section 3), however, has received a lot of attention through the study of idempotent states on quantum groups, initiated in [15] on compact quantum groups by Franz, Skalski (and coauthors). This programme of study, particularly [15], has been cited heavily in this work. To fully adapt the study of idempotent states to irreducible random walks was to prove Proposition 3.12 and Theorem 3.13, and these are mostly concerned with introducing to the study of idempotent states the concept of support projections (see Section 1.2.1). This programme continues to this day, with the focus now on locally compact quantum groups. A good history of this programme of study, with references, is summarised in the introduction of Kasprazak and Sołtan [22].

In contrast, the periodicity condition has seen little attention in the literature. After looking at Section 4.3.2, it could be speculated that because the presumably 'easy' direction of 'concentrated on a coset of a proper normal subgroup implies periodicity' does not hold in the quantum case, that easy progress was difficult to come by. The study of Fagnola and Pellicer [11], so crucial to this work, emerged after the intensive study of idempotent states began. Furthermore, those working in quantum groups were eager to work in the larger classes of compact, and locally compact quantum groups, and the study of these classes soon took precedence over the class of finite quantum groups, which as will be described shortly, comprise a fairly restricted class of quantum groups.

Perhaps one of the most surprising outcomes of the current work is that all the interesting phenomena, in terms of ergodicitity, that occur once commutativity is lost, already occur for dual groups, i.e. group algebras: quantum groups with a *cocommutative* algebra of functions. The important example of Pal: the same phenomenon (reducible but not concentrated on a subgroup) occurs for dual groups. There are irreducible random walks on dual groups that exhibit periodicity without being concentrated on the coset of a proper normal subgroup. Finally, there are irreducible random walks on dual groups that are concentrated on cosets of proper normal subgroups that do *not* exhibit periodicity. Indeed these phenomena can all be found in $\mathbb{C}S_3$, the very smallest quantum group whose algebra of functions is not commutative. It is worth mentioning that Freslon has proved the ergodic theorem for random walks on dual groups (see Section 5.3), but not in the language of supports projections (see Section 1.2.1).

The work leans most heavily on a paper of Fagnola and Pellicer [11], which itself follows a paper of Evans and Høegh-Krohn [10]. In this 2009 paper, the notions of irreducibility and periodicity of a stochastic matrix are extended to the case of a unital positive map on a finite dimensional C*-algebra, and a noncommutative version of the Perron–Frobenius theorem is given. This current work puts the results of Fagnola and Pellicer in the language of quantum groups, and in the language of support projections. The paper of Fagnola and Pellicer is cited so heavily in this work that it will be cited once and for all at this point, with further citations of "Fagnola and Pellicer" referring always to [11].

A number of partial results, stated for Sekine quantum groups; a sufficient condition of Zhang for aperiodicity [41], and an ergodic theorem of Baraquin for central states [2], have been shown to hold more generally. As remarked above, as the detection of whether or not a random walk satisfies the conditions for ergodicity is non-trivial, partial results such as these are most welcome for any study of random walks on quantum groups.

The current work liberally includes the commutative case in discussions. This is to improve readability for those from outside the field of quantum groups, and also to provide motivation for quantum generalisations of classical concepts. The current work is also unapologetically focussed on the problem for *finite* quantum groups, and no attempt is made to state results more generally, for example for compact quantum groups. Although the finiteness assumption is exploited many times, many, although certainly not all, of the results should be true in a more general, infinite setting (indeed, Fagnola and Rebolledo [12], and those who cite them, prove Perron–Frobenius-type results for infinite dimensional algebras). For example, if the convergence is defined with respect to a two-norm, a compact version of Baraquin's Ergodic Theorem 5.3, essentially due to Freslon, survives for a restricted class of random walks given by a central state with an appropriate $\mathcal{L}^2$ density.

As there is a generalisation of the classical finite symmetric group that is infinite dimensional for $n \geq 4$, it would be remiss not to point out that the restriction to *finite* quantum groups is more than a little unnatural, and also brushes many technical difficulties under the carpet. For example, natural examples of random walks on S_n, for example the random transposition shuffle, no longer have densities when generalised to quantum generalisation S_n^+. This means that Lemma 1.6 does not apply, and this takes away the upper bound lemma of Diaconis and Shahshahani from the toolkit. See Freslon [17], Section 4.2, for more. Just as Franz, Skalski, and Tomatsu [16] studied the problem of idempotent states for specific compact quantum groups after comprehensively understanding the finite case, the hope would be that this paper will inspire ergodic theorems for specific compact quantum groups. Unusually for a study of quantum groups, this paper uses little representation theory. A result of Hora states an ergodic theorem for random walks on finite groups using representation theoretic language (Th.1, [20]). If this result can be extended to compact groups, it almost certainly extends also to compact quantum groups.

The paper is organised as follows. In Section 1, the language of category theory is used to motivate the definition of a finite quantum group, and the dual quantum group of a finite quantum group defined. The important examples of the (commutative) algebra of functions on a finite group, and the (cocommutative) algebra of functions on a dual finite group are introduced. A number of properties of finite dimensional C*-algebras, particularly concerning projections, states, support projections, and densities, are included here also. Finally, the definition of a random walk on a finite quantum group is given. Section 2 takes a brief look at the stochastic operator associated to a random walk, and crucially states the relationship between the distribution of the random walk and powers of the stochastic operator. Results of Fagnola and Pellicer concerning the spectrum of a stochastic operator are stated. In Section 3, irreducible random walks are studied, and the example of Pal discussed in more detail. The programme of study of idempotent states, and their associated group-like projections, is introduced. The definition of irreducible by Fagnola and Pellicer, in the language of subharmonic projections, is shown to be equivalent to irreducible (in the sense of an irreducible random walk). Quasi-subgroups are introduced, and it is shown that a random walk concentrated on a proper subgroup is reducible, and it is shown that this is the only barrier to irreducibility. In Section 4, periodic random walks are studied. This section leans heavily on a result of Fagnola and Pellicer, which says that if an irreducible random walk is not ergodic, there exists a partition of unity that illustrates the periodicity of the walk. It is shown that these projections behave like indicator functions on cosets of proper normal subgroups, that the state defining the random walk is concentrated on one of these projections, and that one of the other projections gives a quasi-subgroup. This allows the Ergodic Theorem for Random Walks on Finite Quantum Groups to be written down. Some partial results are included in Section 5; 5.1 for random walks on Kac–Paljutkin and Sekine quantum groups; 5.2 for so-called Zhang Convergence; 5.3 for random walks on dual groups; 5.4 for random walks given by central states.

1. Preliminaries

1.1. **Finite Quantum Groups.** The following approach to introducing finite quantum groups is a very brief summary of the approach outlined in [27] (and covered in more detail in [26]). A finite group G together with its structure maps $(m,{}^{-1},e)$ can be considered an object together with some morphisms in the category of finite sets, with the associativity, inverse, and identity group axioms given by appropriate commutative diagrams. Apply, to the object G, the structure maps, and the group axiom commutative diagrams, the free functor $\mathbf{FinSet}\to\mathbf{FinVec}_{\mathbb{C}}$, and then compose with the contravariant dual endofunctor. Under this functor composition, $G\mapsto F(G)$, the algebra of complex-valued functions on G; $m\mapsto\Delta$ the *comultiplication*; ${}^{-1}\mapsto S$, the *antipode*; and the inclusion of the identity, $e\mapsto\varepsilon$, the *counit*. Using various isomorphisms (such as $F(G\times G)\cong F(G)\otimes F(G)$), the group axioms, under this functor composition, give *coassociativity*, the *counital property*, and the *antipodal property*:

$$
\begin{aligned}
(\Delta\otimes I_{F(G)})\circ\Delta &= (I_{F(G)}\otimes\Delta)\circ\Delta\\
(\varepsilon\otimes I_{F(G)})\circ\Delta &= I_{F(G)}=(I_{F(G)}\otimes\varepsilon)\circ\Delta\\
M\circ(S\otimes I_{F(G)})\circ\Delta &= \eta_{F(G)}\circ\varepsilon = M\circ(I_{F(G)}\otimes S)\circ\Delta
\end{aligned}
\tag{2}
$$

Here $M:F(G)\otimes F(G)\to F(G)$ is pointwise multiplication, and $\eta_{F(G)}$ is the inclusion of the unit, $\lambda\mapsto\lambda\mathbb{1}_G$. With the fact that $f^*f=0$ if and only if $f=0$, $F(G)$ can be given the structure of a finite dimensional C*-algebra. Note furthermore that Δ is a $*$-homomorphism, Δ satisfying $\Delta(f^*) = \Delta(f)^*$, where the involution in $F(G) \otimes F(G)$ is given by $(f \otimes g)^* = f^* \otimes g^*$.

A basis of $F(G)$ is given by the *delta functions*, $\{\delta_t\}_{t \in G}$, $\delta_{s_1}(s_2) = \delta_{s_1,s_2}$. *Indicator functions* of subsets $S \subseteq G$ are denoted and defined by

$$\mathbb{1}_S = \sum_{t \in S} \delta_t.$$

Concretely, the images of the group structure maps are linear maps, the comultiplication

$$\Delta : F(G) \to F(G) \otimes F(G)\,; \qquad \delta_s \mapsto \sum_{t \in G} \delta_{st^{-1}} \otimes \delta_t;$$

the antipode

$$S : F(G) \to F(G)\,; \qquad \delta_s \mapsto \delta_{s^{-1}};$$

and the counit

$$\varepsilon : F(G) \to \mathbb{C}\,; \qquad \delta_s \mapsto \delta_{s,e}.$$

However, there exist noncommutative finite dimensional C*-algebras A with a *-homomorphism $\Delta : A \to A \otimes A$, and maps $\varepsilon : A \to \mathbb{C}$, and $S : A \to A$, that satisfy the above relations. Such algebras (and indeed their commutative counterparts) are thus considered the *algebra of functions on a finite quantum group*. Such an algebra is called a C*-Hopf algebra.

Definition 1.1. *The* algebra of functions on a finite quantum group*, is a unital* C**-Hopf algebra A; that is a* C**-algebra A with a *-homomorphism Δ, a counit ε, and an antipode S, satisfying the relations (2).*

Denote the algebra of functions on a finite quantum group by $A =: F(G)$, with unit denoted by $\mathbb{1}_G$, and refer to G as a finite quantum group. Timmermann presents in Chapter 1 of his book [37] further properties of Hopf algebras, for example the fact that the counit is a *-homomorphism. Every commutative algebra of functions on a finite quantum group is the algebra of functions on some finite classical group. The simplest noncommutative example of an algebra of functions on a finite quantum group is $\mathbb{C}S_3$, where S_3 is the classical symmetric group on three elements, where the comultiplication is given by $\Delta_{\mathbb{C}S_3}(\delta^\sigma) = \delta^\sigma \otimes \delta^\sigma$, and is the dual of the pointwise multiplication in $F(S_3)$. Where τ is the flip map $a \otimes b \mapsto b \otimes a$, this comultiplication has the property that $\tau \circ \Delta_{\mathbb{C}S_3} = \Delta_{\mathbb{C}S_3}$. Algebras of functions on finite quantum groups, $F(C)$, whose comultiplications have this property, $\tau \circ \Delta = \Delta$, are said to be *cocommutative*, and are of the form $F(C) = \mathbb{C}G$ for G a finite (classical) group.

As will be explored in more depth in Section 1.5, for a finite (classical) group there is a duality:

$$F(G)' \cong \mathbb{C}G \text{ and } \mathbb{C}G' \cong F(G).$$

Let $\{\delta^t\}_{t\in G} \subset \mathbb{C}G \cong F(G)'$ be the basis dual to $\{\delta^t\}_{t\in G}$. Through this duality an element $\varphi \in \mathbb{C}G$ can be seen both as a discrete measure:

$$\varphi = \sum_{t\in G} \varphi(\{t\})\delta^t\,; \qquad \mathcal{P}(G) \ni S \mapsto \sum_{s\in S} \varphi(\{s\}),$$

and as a linear functional:

$$\varphi = \sum_{t\in G} \varphi(\delta_t)\delta^t\,; \qquad f = \sum_{t\in G} f(t)\delta_t \mapsto \sum_{t\in G} \varphi(\delta_t) f(t).$$

Similarly, through $\mathbb{C}G' = F(G)$, an element $f \in F(G)$ can be seen both as a function:

$$f = \sum_{t\in G} f(t)\delta_t,; \qquad G \ni s \mapsto f(s),$$

and as a linear functional:

$$f = \sum_{t\in G} f(\delta^t)\delta_t\,; \qquad \varphi = \sum_{t\in G} \varphi(\{t\})\delta^t \mapsto \sum_{t\in G} f(\delta^t)\varphi(\{t\}).$$

Where convenient, notation will toggle between these equivalent points of view.

A *projection* in a C*-algebra A is an element p such that $p = p^* = p^2$. For a finite (classical) group G, every function $G \to \{0,1\}$ is a projection in $F(G)$. Therefore denote by $2^G \subset F(G)$ the set of projections in the algebra of functions on a finite quantum group G.

As a finite dimensional C*-algebra, the algebra of functions on a finite quantum group G is a multi-matrix algebra:

$$F(G) \cong \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}).$$

Its left ideals are of the form $F(G)p$ for $p \in 2^G$. The central projections are sums of identity matrices:

$$Z(F(G)) \cap 2^G = \left\{ \sum_{i=1}^{N} \alpha_i I_{n_i} \,:\, \alpha_i = 0, 1 \right\}. \tag{3}$$

As the counit is a character, there is a one dimensional factor, whose basis element, a central projection $\eta \in 2^G$, is called the *Haar* element. By writing, for a general $p \in 2^G$,

$$p = \alpha\eta \oplus r,$$

and considering $p^2 = p$, it follows that $\alpha = 0$ or 1.

There are well established notions of compact and locally compact quantum groups. See, for example, Timmermann [37] for more. For the remainder of the current work, unless explicitly stated otherwise, G is a finite quantum group. As the term 'quantum group' is so variously defined, and any reader on the topic of quantum groups must carefully check which class/definition of quantum groups a study is using, it is safe in the current work to, unless explicitly stated otherwise, briefly say *quantum group* for *finite quantum group*. If talking about a finite (classical) group, with commutative algebra of functions, briefly say *classical group*.

1.2. **States.** A probability on a classical group $\mu:\mathcal{P}(G)\to[0,1]$, gives rise to an expectation, also denoted $\mu:F(G)\to\mathbb{C}$,

$$f\mapsto\sum_{t\in G}f(t)\mu(\{t\}).$$

The expectation is a positive linear functional on $F(G)$ such that $\mu(\mathbb{1}_G)=1$. Denoting the set of probabilities on G by $M_p(G)$, this motivates:

Definition 1.2. *A* state μ *on the algebra of functions on a quantum group G is a positive linear functional such that $\mu(\mathbb{1}_G)=1$. Denote the set of states on $F(G)$ by $M_p(G)$.*

The *convolution* of probabilities μ, ν on a classical group G is given by:

$$(\mu\star\nu)(\{s\})=\sum_{t\in G}\mu(\{st^{-1}\})\nu(\{t\})=(\mu\otimes\nu)\Delta(\delta_s).$$

Therefore define the *convolution of states* μ, $\nu\in M_p(G)$ on a quantum group:

$$\mu\star\nu:=(\mu\otimes\nu)\Delta. \tag{4}$$

The counit is a state that is an identity for this convolution:

$$\varepsilon\star\mu=\mu=\mu\star\varepsilon\,. \qquad (\mu\in M_p(G)) \tag{5}$$

Where π is the random/uniform probability on a classical group G, consider the state on $F(G)$:

$$f\mapsto\sum_{t\in G}f(t)\,\pi(\{t\})=\frac{1}{|G|}\sum_{t\in G}f(t).$$

This state is called the *Haar* state, and it is *invariant* in the sense that for all $\mu\in M_p(G)$,

$$\pi\star\mu=\pi=\mu\star\pi. \tag{6}$$

Still in the classical case, this invariance is equivalent to

$$\mathbb{1}_G\cdot\pi(f)=\left(I_{F(G)}\otimes\pi\right)\Delta(f)=\left(\pi\otimes I_{F(G)}\right)\Delta(f)\,. \qquad (f\in F(G)) \tag{7}$$

A quantum group also has a unique (tracial) Haar state (Theorem 1.3, [38]), denoted by $\int_G$, and whose invariance can be given by either of the equivalent conditions (6) or (7).

1.2.1. *The Support of a State.* Let $\nu\in M_p(G)$ be a state and consider the left ideal

$$N_\nu=\{g\in F(G)\,|\,\nu(|g|^2)=0\}.$$

As a left ideal of a finite dimensional C*-algebra, N_ν must be of the form $F(G)q_\nu$ for q_ν a projection such that $gq_\nu=g$ for all $g\in N_\nu$ [4]. It is the case that for all $f\in F(G)$,

$$\nu(q_\nu)=\nu(fq_\nu)=\nu(q_\nu f)=0.$$

This implies in particular that $\nu(N_\nu)=\{0\}$. Define $p_\nu:=\mathbb{1}_G-q_\nu$. It is the case that for all $f\in F(G)$,

$$\nu(f)=\nu(p_\nu f)=\nu(fp_\nu)=\nu(p_\nu fp_\nu), \tag{8}$$

and that $\nu(p_\nu)=1$. Suppose that p is another projection such that $\nu(p)=1$ and $p\leq p_\nu$. Then $p_\nu-p$ is a projection and

$$\nu((p_\nu-p)^*(p_\nu-p))=\nu(p_\nu-p)=\nu(p_\nu)-\nu(p)=0,$$

so that $p_\nu-p\in N_\nu$ and therefore $(p_\nu-p)q_\nu=p_\nu-p$. However

$$(p_\nu-p)q_\nu=(p_\nu-pp_\nu)q_\nu=(\mathbb{1}_G-p)p_\nu q_\nu=0,$$

and so $p=p_\nu$. Therefore p_ν is the smallest projection such that $\nu(p_\nu)=1$. Call p_ν by the *support projection* of ν.

1.3. **Random Walks on Quantum Groups.** To study random walks on classical groups, one can look at various objects. The random variables ξ_k (and ζ_i), their distributions $\nu^{\star k}$, or the stochastic operator that maps $\nu^{\star k}\to\nu^{\star(k+1)}$. Franz and Gohm [13] generalise this study to random walks on quantum groups, and find quantum generalisations of the random variables, their distributions, as well as the stochastic operators. Franz and Gohm, via Proposition 2.1, assert that the semigroup of stochastic operators, and the semigroup of distributions, are essentially the same thing, and so all of the data of a random walk on a quantum group is carried by the *driving probability* $\nu\in M_p(G)$. Indeed, Amaury Freslon [17] defines a random walk on a compact quantum group implicitly as a state on its algebra of continuous functions. In Section 3.2 of [26], this generalisation of Franz and Gohm from random walks on classical groups to random walks on quantum groups is explored in detail, but for the purposes of the current work the implicit approach of Freslon will be taken:

Definition 1.3. *A random walk on a quantum group is given by a state, $\nu\in M_p(G)$.*

To study a random walk on a quantum group therefore is to study its semigroup of convolution powers, $(\nu^{\star k})_{k\geq 1}$, defined inductively through

$$\nu^{\star(k+1)}=(\nu\otimes\nu^{\star k})\circ\Delta.$$

Of central interest are random walks that are *ergodic*:

Definition 1.4. *A random walk ν on a quantum group is said to be* ergodic *if the convolution powers $(\nu^{\star k})_{k\geq 1}$ converge to the Haar state. In this context, denote the Haar state by π, call the Haar state by the* random distribution, *and say the random walk* converges to random.

The random walk ν is associated with a stochastic operator $T_\nu : F(G) \to F(G)$ (see Section 2), and this object plays a key role in the current work.

1.4. **The Dual of a Quantum Group.** Consider the space, $\widehat{F(G)}$, of linear functionals on $F(G)$ of the form

$$g \mapsto \int_G gf \,. \qquad (f,\, g \in F(G))$$

As $F(G)$ is finite dimensional, the continuous and algebraic duals coincide. Furthermore, the Haar state is faithful and so

$$\langle f, g\rangle := \int_G f^* g$$

defines an inner product making $F(G)$ a Hilbert space. Via the Riesz Representation Theorem for Hilbert spaces, for every element $\varphi \in F(G)'$, there exists a density $f_\varphi^* \in F(G)$ such that:

$$\varphi(g) = \langle f_\varphi^*, g\rangle = \int_G f_\varphi g \,, \qquad (g \in F(G))$$

so that $F(G)' = \widehat{F(G)}$. This space can be given the structure of an algebra of functions on a quantum group by employing the contravariant dual functor to $F(G)$ and its structure maps. The quantum group formed in this way is called the *dual* of the quantum group G, and is denoted by $\widehat{G}$. Note that $M_p(G)$ is a subset of $F(\widehat{G})$. Indeed the convolution (4) defined on $M_p(G)$ is the image of the comultiplication in $F(G)$ under the dual functor, and thus defines the multiplication on $F(\widehat{G})$, and therefore, for φ_1, $\varphi_2 \in F(\widehat{G})$, briefly write $\varphi_1\varphi_2$ for $\varphi_1 \star \varphi_2$. By (5), $\mathbb{1}_{\widehat{G}} = \varepsilon$. The *-involution on $F(\widehat{G})$ is given by:

$$\varphi^*(f) = \overline{\varphi(S(f)^*)} \,. \qquad (\varphi \in F(\widehat{G}),\, f \in F(G))$$

A density $f_\nu \in F(G)$ defines a state $\nu \in M_p(G)$ if and only if f_ν is positive and $\int_G f_\nu = 1$. Denote the map $f_\nu \mapsto \nu$ by $\mathcal{F}$. The density of ε is $f_\varepsilon = \eta / \int_G \eta$, while the density of the Haar state is just $f_{\int_G} = \mathbb{1}_G$. In the sequel, unless specified otherwise, f_ν will denote the density of a state $\nu \in M_p(G)$.

The *convolution product* on $F(G)$ is given by:

$$f \circledast g := \left(\int_G \otimes I_{F(G)}\right) \left(\left((S\otimes I_{F(G)})\Delta(g)\right)(f\otimes \mathbb{1}_G)\right) . \tag{9}$$

There is a Convolution Theorem, well-presented in Section 1.1 of [6], relating this convolution to the convolution in $F(\widehat{G})$:

Theorem 1.5. *(Van Daele's Convolution Theorem) For $\varphi_1, \varphi_2 \in F(\widehat{G})$ with densities $f_{\varphi_1},\, f_{\varphi_2} \in F(G)$*

$$f_{\varphi_1\varphi_2} = f_{\varphi_1} \circledstar f_{\varphi_2} \qquad \bullet$$

Where $\pi := \int_G$ is the 'random distribution', the *distance to random* of a random walk on a quantum group G is measured using the total variation distance:

$$\|\nu^{\star k} - \pi\| = \sup_{p \in 2^G} |\nu^{\star k}(p) - \pi(p)|.$$

The Haar state is a normal, faithful trace, therefore non-commutative $\mathcal{L}^p$ machinery [32] can be used to put p-norms on $F(G)$:

$$\|f\|_p := \left(\int_G |f|^p\right)^{1/p}. \qquad (f \in F(G)) \tag{10}$$

Set the infinity norm equal to the operator norm.

Lemma 1.6. *[17, 27] Let G be a quantum group and $\nu,\, \mu \in M_p(G)$:*

$$\|\nu - \mu\| = \frac{1}{2}\|f_\nu - f_\mu\|_1 \quad \bullet$$

Random walks on quantum groups have the following ergodic property: while the distribution of a random walk may not converge, the distance to random does so monotonically:

Theorem 1.7. *The distance to random, $\|\nu^{\star k} - \pi\|$, is decreasing in k.*

Proof. Van Daele's Convolution Theorem implies that $f_\nu \circledstar f_{\nu^{\star k}} = f_{\nu^{\star(k+1)}}$. Note also that

$$f_\nu \circledstar \mathbb{1}_G = f_\nu \circledstar f_\pi = f_{\nu\pi} = f_\pi = \mathbb{1}_G,$$

as $\nu\pi = \pi$ for all $\nu \in M_p(G)$. Now consider

$$\begin{aligned}
\|\nu^{\star(k+1)} - \pi\| &= \frac{1}{2}\|f_{\nu^{\star(k+1)}} - \mathbb{1}_G\|_1 = \frac{1}{2}\|f_\nu \circledstar f_{\nu^{\star k}} - f_\nu \circledstar \mathbb{1}_G\|_1 \\
&= \frac{1}{2}\|f_\nu \circledstar (f_{\nu^{\star k}} - \mathbb{1}_G)\|_1 \leq \frac{1}{2}\|f_\nu\|_1\|f_{\nu^{\star k}} - \mathbb{1}_G\|_1 \\
&= \frac{1}{2}\|f_{\nu^{\star k}} - \mathbb{1}_G\|_1 = \|\nu^{\star k} - \pi\|.
\end{aligned}$$

The inequality is due to Simeng Wang ($\|f \circledstar g\|_1 \leq \|f\|_1\|g\|_1$, Prop. 2.2.1, [39]), while $\|f_\nu\|_1 = 1$ as $\nu \in M_p(G)$ if and only if f_ν is positive and $\displaystyle\int_G f_\nu = 1$ $\bullet$

1.5. **Group Algebras.** In the case of a classical group G, $F(\widehat{G})=\mathbb{C}G$, the group algebra of G. As remarked above, if the algebra of functions on a quantum group C is cocommutative, then $F(C)=\mathbb{C}G$, the group algebra of a classical group G. Fixing for this section G a classical group, and denoting $\mathbb{C}G=F(\widehat{G})$, the algebra structure of $F(\widehat{G})$ is the image of G, together with its structure maps, and group axiom commutative diagrams, under the free functor discussed in Section 1.1. Therefore, where its basis is given by $\{\delta^t\}_{t\in G}$, the multiplication is given by $\delta^s\otimes\delta^t\mapsto\delta^{st}$; and the unit map $\lambda\mapsto\lambda\,\delta^e$. The coalgebra structure is dual to the algebra structure of $F(G)$. This implies that the comultiplication is $\delta^s\mapsto\delta^s\otimes\delta^s$; the counit is, for all $s\in G$, $\delta^s\mapsto 1$; and the antipode is $\delta^s\mapsto\delta^{s^{-1}}$.

Each subgroup $H\leq G$ gives a non-zero projection denoted $\chi_H\in 2^{\widehat{G}}$:

$$\chi_H:=\frac{1}{|H|}\sum_{h\in H}\delta^h. \tag{11}$$

Note that $\chi_{\{e\}}=\delta^e=\mathbb{1}_{\widehat{G}}$, and that $\chi_H=\int_H\in M_p(G)\subset F(\widehat{G})$.

States on $F(\widehat{G})$ are given by positive definite functions $u\in M_p(\widehat{G})\subset F(G)$ (see Bekka, de la Harpe, and Valette (Proposition C.4.2, [3])). Furthermore, there is a bijective correspondence between positive definite functions and unitary representations on G together with a vector. In particular, for each positive definite function u there exists a unitary representation $\rho:G\to\mathrm{GL}(H)$ and a vector $\xi\in H$ such that

$$u(s)=\langle\rho(s)\xi,\xi\rangle, \tag{12}$$

and for each unitary representation ρ and vector ξ (12) defines a positive definite function on G. This inner product can be taken to be conjugate-linear on the right. For u to be a state, it is necessary that $u(e)=1$ and so $\langle\xi,\xi\rangle=1$; i.e. ξ is a unit vector. Therefore probabilities on $\widehat{G}$ can be chosen by selecting a given representation and unit vector.

The comultiplication being $\delta^s\mapsto\delta^s\otimes\delta^s$ implies that for a random walk on $\widehat{G}$ given by $u\in M_p(\widehat{G})$, the convolution powers are $(u^{\star k})_{k\geq 1}=(u^k)_{k\geq 1}$, the pointwise-multiplication powers. The Haar state is given by $\delta^e=:\int_{\widehat{G}}$, and so the random walk u is ergodic if and only if $|u(s)|=1$ for $s=e$ only. See Section 5.3 for more.

Using the $\{\delta_t\}_{t\in G}$ basis of $F(G)$, a state $u\in M_p(G)$ may be written as:

$$u=\sum_{t\in G}u(t)\delta_t$$

and so if $\mu=\sum_{t\in G}\mu(\delta_t)\delta^t\in F(\widehat{G})$:

$$u(\mu)=\sum_{t\in G}u(t)\mu(\delta_t).$$

To identify the density of the state $u \in M_p(G)$, denoted $\varphi_u \in F(\widehat{G})$, note that where $\widehat{\mathcal{F}} : F(\widehat{G}) \to F(G)$, $\varphi_u \mapsto u$ is given by, for $\mu \in F(\widehat{G})$:

$$u(\mu) = \widehat{\mathcal{F}}(\varphi_u)(\mu) = \int_{\widehat{G}} \mu\varphi_u = \sum_{t\in G} \mu(\delta_t)\varphi_u(\delta_{t^{-1}}),$$

shows that

$$\varphi_u = \sum_{t\in G} u(t^{-1})\delta^t. \tag{13}$$

2. Stochastic Operators

2.1. **Definition and Properties.** Considered as a Markov chain with finite state space $G = \{s_1, \dots, s_{|G|}\}$, a random walk on a classical group (1) *driven* by $\nu \in M_p(G)$ has stochastic operator $T_\nu \in M_{|G|}(\mathbb{C})$:

$$[T_\nu]_{ij} = \mathbb{P}[\xi_{k+1} = s_i \,|\, \xi_k = s_j] = \nu(s_i s_j^{-1}).$$

Then T_ν is an operator on $F(G)$ equal to

$$T_\nu = (\nu \otimes I_{F(G)}) \circ \Delta.$$

Thus given a random walk on a quantum group G driven by $\nu \in M_p(G)$, define its *stochastic operator* by the same formula. Sometimes the notation P_ν is used for $(\nu \otimes I_{F(G)}) \circ \Delta$, and, as in Franz and Gohm [13], T_ν reserved for $(I_{F(G)} \otimes \nu) \circ \Delta$. This boils down to a choice between generalising a right-invariant walk (1), or a left-invariant walk:

$$\xi_k = \xi_{k-1}\zeta_k.$$

This current work is using the generalisation of a right-invariant walk, and so the stochastic operator $(\nu \otimes I_{F(G)}) \circ \Delta$ is used, with the notation T_ν to avoid a clash in notation with p_ν, the support projection of a state $\nu \in M_p(G)$.

In the usual way, via its transpose, T_ν gives an operator on $F(\widehat{G})$, given by, for $\varphi \in F(\widehat{G})$ and $f \in F(G)$:

$$T_\nu^t(\varphi)(f) = \varphi(T_\nu(f)).$$

In the sequel write φT_ν for $T_\nu^t(\varphi)$.

Proposition 2.1. *Let G be a quantum group and $\mu,\,\nu\in M_p(G)$. Then the following hold:*

i. $\mu T_\nu=\nu\star\mu$.
ii. $T_\nu^k=T_{\nu^{\star k}}$.
iii. $\varepsilon T_\nu^k=\nu^{\star k}$.
iv. T_ν *is unital and positive.*
v. $M_p(G)$ *is invariant under* T_ν.
vi. $\displaystyle\int_G\circ T_\nu=\int_G$.
vii. $T_\nu(g)=S(f_\nu)\circledast g$ *for all* $g\in F(G)$.
viii. $\|T_\nu\|=1$.

Proof. Parts i-vi. can be checked easily. For vii., note that (using Sweedler notation [36])

$$\begin{aligned}T_\nu(g)&=(\nu\otimes I_{F(G)})\Delta(g)=(\nu\otimes I_{F(G)})\sum g_{(1)}\otimes g_{(2)}\\&=\sum g_{(2)}\int_G g_{(1)}f_\nu=\sum g_{(2)}\int_G S(g_{(1)})S(f_\nu),\end{aligned}$$

via $\int_G\circ S=\int_G$ (Th. 2.2.6, [37]) and the traciality of the Haar measure. Looking at (9), note this is nothing other than $S(f_\nu)\circledast g$.

For viii., note that $\|\cdot\|_\infty$ is the C*-norm on $F(G)$. Therefore as $T_\nu:F(G)\to F(G)$ is a positive map between unital C*-algebras, it satisfies the hypotheses of Corollary 2.9 of Paulsen [31]. This gives

$$\|T_\nu\|=\|T_\nu(\mathbb{1}_G)\|_\infty=\|\mathbb{1}_G\|_\infty=1\ \bullet$$

The most important of these will be Proposition 2.1 iii. That T_ν is unital and positive can be used to show that another distance to random is decreasing. Define a norm on $F(\widehat{G})$ by $\|\mu\|_{\max}=\|f_\mu\|_\infty$. Recall that in the classical case, commutativity of $F(G)$ means that $\|\cdot\|_\infty$, the operator norm, is nothing but the supremum norm. Furthermore, the density of $\nu\in M_p(G)$ is given by $f_\nu(s)=|G|\nu(\{s\})$. Let $\nu=\sum_{t\in G}\nu(\{t\})\delta^t\in M_p(G)$ and consider:

$$\begin{aligned}\|\nu-\pi\|_{\max}&=\|f_\nu-f_\pi\|_\infty\\&=\left\||G|\left(\sum_{t\in G}\nu(\{t\})\delta_t-\frac{1}{|G|}\mathbb{1}_G\right)\right\|_\infty\\&=|G|\max_{t\in G}\left|\nu(\{t\})-\frac{1}{|G|}\right|.\end{aligned}$$

This is related to the classical separation 'distance' used by e.g. Aldous and Diaconis [1]. Therefore, for a random walk on a quantum group, for a fixed $\nu\in M_p(G)$, call by the *quantum separation distance* the quantity $s(k):=\|\nu^{\star k}-\pi\|_{\text{QSD}}:=\|f_{\nu^{\star k}}-\mathbb{1}_G\|_\infty$.

Proposition 2.2. *The quantum separation distance is decreasing in k.*

Proof. Note by Proposition 2.1 vii., where $\mathcal{F}(f_\nu)=\nu$, and $T:=T_{\mathcal{F}(S(f_\nu))}$, that

$$T(f_\nu)=f_\nu\circledstar f_\nu,$$

so that $T(f_\nu^{\circledstar k})=f_\nu^{\circledstar(k+1)}$ (the fact that $S^2=I_{F(G)}$ was used). Note further, via the antipode $S:F(G)\to F(G)$ being an antimultiplicative $*$-linear (and thus positive) map (Prop. 1.3.12, Cor. 1.3.29, [37]), $S(f_\nu)$ is the density of a state, and so T is positive and unital (Prop. 2.1 iv.).

$$\begin{aligned}s(k+1)=\|\nu^{\star(k+1)}-\pi\|_{\text{QSD}}=\|f_{\nu^{\star(k+1)}}-\mathbb{1}_G\|_\infty&=\|T(f_{\nu^{\star k}}-\mathbb{1}_G)\|_\infty\\&\leq\|T\|\|f_{\nu^{\star k}}-\mathbb{1}_G\|_\infty=s(k)\quad\bullet\end{aligned}$$

To use the results of Fagnola and Pellicer the stochastic operator must be a *Schwarz Map*. All completely positive maps are Schwarz so the following suffices:

Proposition 2.3. *The stochastic operator T_ν of a random walk on a quantum group given by $\nu\in M_p(G)$ is* completely positive.

Proof. To show that T_ν is a completely positive map, for any positive $F=[f_{ij}]\in M_n(F(G))^+$, it must be shown that $T_\nu(F):=[T_\nu(f_{ij})]\in M_n(F(G))^+$. The comultiplication, as a *-homomorphism, is a completely positive map (Th. 1, [35]), and so $\Delta(F):=[\Delta(f_{ij})]\in M_n(F(G)\otimes F(G))^+$. As a *-homomorphism, the identity map $I_{F(G)}$ is a completely positive map. Linear functionals are completely positive and thus $\nu\in M_p(G)$ is also (Th. 3, [35]). It can be shown that the tensor product of completely positive maps is completely positive and thus $\nu\otimes I_{F(G)}$ is completely positive. Thus

$$(\nu\otimes I_{F(G)})\Delta(F)\in M_n(F(G))^+\quad\bullet$$

2.2. **Spectral Analysis.** Given a random walk on a quantum group, as T_ν is a linear operator on a finite dimensional C*-algebra $F(G)$, the convergence of $(T_\nu^k)_{k\geq1}$, and thus via Proposition 2.1 iii. of the convolution powers $(\nu^{\star k})_{k\geq1}$, is determined by its spectrum, $\sigma(T_\nu)$. Thus much of the standard spectral analysis of Markov chain stochastic operators (see, for example, [7]), applies in the quantum context. This analysis is often focussed on ergodic random walks, where $1\in\sigma(T_\nu)$ is multiplicity-free, and the only eigenvalue of modulus one. This same analysis is easier when the stochastic operator is symmetric, in which case T_ν is self-adjoint. In this case it is easy to demonstrate the classic Markov chain result, where λ_* is the second largest eigenvalue in magnitude, that the distance to random, $\|\nu^{\star k}-\pi\|$, is $\mathcal{O}(|\lambda_*|^k)$. This analysis is easy in the classical case precisely when the support of $\nu\in M_p(G)$ is symmetric in the sense that $\nu=\nu\circ S$.

Using the basis of matrix elements of irreducible representations, it can be shown that the stochastic operator of a random walk on a quantum group is also self-adjoint if ν is symmetric in the sense that $\nu=\nu\circ S$ (Th. 6.2.1, [26]). Elementary linear algebra shows that the second-largest-eigenvalue-in-magnitude analysis holds also for random walks on quantum groups (if $1\in\sigma(T_\nu)$ is multiplicity free and the only eigenvalue of magnitude one). Of course, in both the classical and quantum contexts, if T_ν is not self-adjoint, writing T_ν in Jordan normal form shows that if $1\in\sigma(T_\nu)$ is multiplicity free, and the only eigenvalue of magnitude one, that $(T_\nu^k)_{k\geq 1}$ converges and thus $(\nu^{\star k})_{k\geq 1}$ does too, by (7), to the map $f\mapsto \mathbb{1}_G\cdot\int_G f$.

Of course, for a random walk on a quantum group $1\in\sigma(T_\nu)$, however, following Evans and Høegh-Krohn [10], in a context more general than random walks on quantum groups, Fagnola and Pellicer say a number of things about $\sigma(T_\nu)$:

Proposition 2.4. *If T_ν is the stochastic operator of a random walk on a quantum group, then $\sigma(T_\nu)\subset\overline{\mathbb{D}}$. If $1\in\sigma(T_\nu)$ is multiplicity-free, then $\sigma(T_\nu)\cap\mathbb{T}\cong C_d$* •

3. Irreducibility

A random walk on a classical group G is said to be reducible if there are group elements that the random walk can not visit. If there are no such elements, the random walk is said to be irreducible. If $S\subset G$ is any subset of G not visited by the walk, the indicator function $\mathbb{1}_S$ has the property that $\nu^{\star k}(\mathbb{1}_S)=0$ for all $k\in\mathbb{N}$. This motivates the following definition:

Definition 3.1. *A random walk on a quantum group G given by $\nu\in M_p(G)$ is said to be* reducible *if there exists a non-zero $q\in 2^G$ such that $\nu^{\star k}(q)=0$ for all $k\in\mathbb{N}$. If there are no such non-zero projections, the random walk is said to be* irreducible.

The conditions for a random walk on a classical group to be irreducible are rather straightforward. The support of $\nu^{\star k}$,

$$\operatorname{supp}\nu^{\star k}=(\operatorname{supp}\nu)^k.$$

If $\operatorname{supp}\nu\subseteq H<G$, a proper subgroup, then $\operatorname{supp}\nu^{\star k}\subseteq H$, and so the indicator function $\mathbb{1}_{G\cap H^c}$ is such that $\nu^{\star k}(\mathbb{1}_{G\cap H^c})=0$ for all $k\in\mathbb{N}$. On the other hand, $\langle\operatorname{supp}\nu\rangle\leq G$ is a subgroup, and if there exists a non-zero projection $q\in 2^G$, given by a non-empty subset $S\subseteq G$, via $q=\mathbb{1}_S$, such that $\nu^{\star k}(\mathbb{1}_S)=0$ for all $k\in\mathbb{N}$, then $\langle\operatorname{supp}\nu\rangle\cap S=\emptyset$, and so $\langle\operatorname{supp}\nu\rangle<G$ is a proper subgroup. So a random walk on a classical group is irreducible if and only if the support is not concentrated on a proper subgroup.

A subgroup (H, m_H) of a classical group (G, m) is a classical group together with a monomorphism/injection $\iota : H \to G$ that satisfies:

$$\iota \circ m_H = m \circ (\iota \times \iota).$$

Via the functor composition mentioned in Section 1.1, this motivates the (standard) definition:

Definition 3.2. *If G and H are quantum groups and $\pi : F(G) \to F(H)$ is a surjective unital *-homomorphism such that*

$$\Delta_{F(H)} \circ \pi = (\pi \otimes \pi) \circ \Delta_{F(G)},$$

then H is called a subgroup *of G.*

As everything is in finite dimensions, the larger space can be decomposed as:

$$F(G) \cong F(H) \oplus \ker \pi,$$

and $F(H)$ is embedded via:

$$\imath : F(H) \hookrightarrow F(H) \oplus \ker \pi \subseteq F(G)\,; \;\; f \mapsto f \oplus 0.$$

Say that a state $\nu \in M_p(G)$ is *supported on H* if $p_\nu \leq \imath(\pi(\mathbb{1}_G)) =: \mathbb{1}_H$. From here it can be shown that if ν, $\mu \in M_p(G)$ are supported on $H \leq G$, then so is $\nu \star \mu$. As will be seen, this is only a special case of Proposition 3.12. In the classical case, *any* non-empty subset $\Sigma \subseteq G$ generates a subgroup $\langle \Sigma \rangle \leq G$. The quantum generalisation of this statement is not true.

3.1. **Idempotent States.** Consider a random walk on a quantum group G given by $\nu \in M_p(G)$. If the convolution powers $(\nu^{\star k})_{k\geq 1}$ converge they converge to an idempotent, a state ν_∞ such that $\nu_\infty = \nu_\infty \star \nu_\infty$. The Kawada-Itô Theorem implies that for classical groups, all idempotent states are integration against the uniform Haar measure on some subgroup [23].

3.1.1. *Group-Like Projections.* The notion of a group-like projection in the algebra of functions on a quantum group was first introduced by Lanstad and Van Daele [24].

Definition 3.3. *A non-zero $p \in 2^G$ is called a group-like projection if*

$$\Delta(p)(\mathbb{1}_G \otimes p) = p \otimes p.$$

It can be shown that $\varepsilon(p) = 1$ and $S(p) = p$ [24]. It is not difficult to show that if $H \leq G$ is a subgroup, $\mathbb{1}_H \in 2^G$ is a group-like projection. Franz and Skalski show that there is a one-to-one correspondence between idempotent states and group-like projections.

In particular, they prove:

Proposition 3.4. *(Cor. 4.2, [15]) Let $\phi \in M_p(G)$. The following are equivalent:*

i. *ϕ is idempotent*
ii. *there exists a group-like projection $p \in 2^G$ such that for all $f \in F(G)$:*

$$\phi(f) = \frac{1}{\int_G p} \int_G fp \qquad \bullet$$

In particular, if ϕ is an idempotent state, its density is $f_\phi = p/\int_G p$ and

$$\phi = \mathcal{F}(f_\phi) = \frac{1}{\int_G p}\mathcal{F}(p).$$

In the setting of locally compact quantum groups, Kasprzak and Sołtan [22] use the Gelfand philosophy to refer to the virtual object corresponding to a group-like projection as a *quasi-subgroup*. This paper will take the same approach, associating to a group-like projection p a quasi-subgroup $S \subseteq G$, and writing $p =: \mathbb{1}_S$, and the associated idempotent state by ϕ_S. Note also that a subgroup $H \leq G$ is a quasi-subgroup as $\mathbb{1}_H \in 2^G$ is a group-like projection.

Proposition 3.5. *If $S \subset G$ is a proper quasi-subgroup given by a group-like projection $\mathbb{1}_S$, the support of ϕ_S, $p_{\phi_S} = \mathbb{1}_S$.*

Proof. By Proposition 3.4,

$$\phi_S = \frac{1}{\int_G \mathbb{1}_S}\mathcal{F}(\mathbb{1}_S),$$

is an idempotent state such that

$$\phi_S(\mathbb{1}_S) = \int_G \mathbb{1}_S \frac{1}{\int_G \mathbb{1}_S}\mathbb{1}_S = 1,$$

as $\mathbb{1}_S \in 2^G$. Let p_{ϕ_S} be the support projection of ϕ_S, so that $p_{\phi_S} < \mathbb{1}_S$, and $p_{\phi_S}\mathbb{1}_S = p_{\phi_S}$. Consider

$$\int_G (\mathbb{1}_S - p_{\phi_S}) = \int_G (\mathbb{1}_S - p_{\phi_S}\mathbb{1}_S) = \int_G \mathbb{1}_S - \int_G p_{\phi_S}\mathbb{1}_S.$$

Note

$$\int_G p_{\phi_S}\mathbb{1}_S = \int_G \mathbb{1}_S \cdot \int_G p_{\phi_S}\frac{\mathbb{1}_S}{\int_G \mathbb{1}_S} = \int_G \mathbb{1}_S \cdot \phi_S(p_{\phi_S}) = \int_G \mathbb{1}_S \cdot 1 = \int_G \mathbb{1}_S,$$

so that $\int_G (\mathbb{1}_S - p_{\phi_S}) = 0$, and as $\int_G$ is faithful, $\mathbb{1}_S - p_{\phi_S} = 0$, and so the group-like projection of an idempotent state is also its support $\bullet$

This consideration, and Proposition 3.4, motivates:

Definition 3.6. *A state* $\nu \in M_p(G)$ *is* supported on a quasi-subgroup S *if, where* $\mathbb{1}_S$ *is the group-like projection associated with* S, $\nu(\mathbb{1}_S) = 1$.

It will be seen shortly that there are quasi-subgroups that are not subgroups. The easiest way to see this is through the following theorem:

Theorem 3.7. *(Th. 4.5, [15]) Let* G *be a quantum group and* $\phi_S \in M_p(G)$ *an idempotent state with group-like projection* $\mathbb{1}_S$*. The following are equivalent:*

i. $S \leq G$ *is a subgroup;*
ii. *the null space* N_{ϕ_S} *is a two-sided ideal of* $F(G)$*;*
iii. *the null space* N_{ϕ_S} *is a self-adjoint ideal of* $F(G)$*;*
iv. *the null space* N_{ϕ_S} *is an* S*-invariant ideal of* $F(G)$*;*
v. *the projection* $\mathbb{1}_S$ *is central* $\bullet$

Thus, by (3) and v., given a group-like projection $\mathbb{1}_S$ in a concrete algebra of functions $F(G) \cong \bigoplus_i M_{n_i}(\mathbb{C})$, $\mathbb{1}_S$ corresponds to a subgroup if and only if it is a sum of full identity matrices.

3.1.2. *Pal's Idempotents.* The Kac–Paljutkin quantum group $\mathfrak{G}_0$ has an algebra of functions structure

$$F(\mathfrak{G}_0) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}),$$

with basis elements η, e_2, e_3, e_4, and E_{ij} for $1 \leq i, j \leq 2$. Pal [30] determined that there are eight idempotent states $\{\phi_1, \dots, \phi_8\} \subset M_p(\mathfrak{G}_0)$ on the Kac–Paljutkin quantum group, six of these are non-trivial. Franz and Gohm [13] show that four of these six are algebras of functions on classical groups, but ϕ_6 and ϕ_7 are not. By looking at their supports:

$$p_{\phi_6} = \eta + e_4 + E_{11} + E_{21},$$
$$p_{\phi_7} = \eta + e_4 + E_{12} + E_{22};$$

it is easy to see that they correspond to quasi-subgroups that are not subgroups.

It might be tempting to think that perhaps a quasi-subgroup is always contained in a proper subgroup of G, however the only subgroup larger than the quasi-subgroup corresponding to ϕ_6 is the whole quantum group $\mathfrak{G}_0$.

Therefore, as will be seen with Proposition 3.12, given a random walk $\nu \in M_p(G)$ on a quantum group G, not being concentrated on a subgroup is a necessary but not sufficient condition for ergodicity.

3.1.3. *Cocommutative Idempotents.* Pal's counterexample showed, as Franz and Skalski remark [14], that the necessary and sufficient conditions for ergodicity of a random walk on a quantum group are "clearly more complicated" (than the classical situation). In fact, as was noted after Pal's counterexample, there exists an abundance of quasi-subgroups that are not subgroups as soon as cocommutative algebras of functions are considered.

Every subset $S\subseteq G$ gives an indicator function $\mathbb{1}_S$ that is an idempotent in $F(G)$. However not all of these are positive definite functions. Firstly if $\mathbb{1}_S$ is to be a state, the subset S must be a subgroup (Ex. 6.C.4, [3]). Consider a cocommutative algebra of functions $F(\widehat{G})$ and $H\leq G$. The indicator function on H, $\mathbb{1}_H$, is an idempotent state on $\widehat{G}$. By (13), its density is

$$\varphi_{\mathbb{1}_H}=\sum_{h\in H}\delta^h=\frac{\chi_H}{\int_{\widehat{G}}\chi_H}.$$

Therefore, by Proposition 3.4 its associated group-like projection is equal to χ_H, and by Proposition 3.5 the support projection $p_{\mathbb{1}_H}=\chi_H\in F(\widehat{G})$.

By Theorem 3.7 v., the quasi-subgroup given by χ_H is a subgroup if and only if χ_H is central, that is for all $s\in G$:

$$\chi_H\delta^s=\frac{1}{|H|}\sum_{h\in H}\delta^{hs}=\frac{1}{|H|}\sum_{h\in H}\delta^{sh}=\delta^s\chi_H,$$

which is the case if and only if H is a normal subgroup of G.

Therefore whenever $H\leq G$ is a non-normal subgroup, χ_H gives a quasi-subgroup of $\widehat{G}$ which is not a subgroup. It might be tempting to think that perhaps there is always a non-trivial subgroup contained in a quasi-subgroup, however if G is a simple classical group, the only subgroup of $\widehat{G}$ smaller than the quasi-subgroup corresponding to χ_H is the trivial subgroup of $\widehat{G}$ given by $\chi_{\{e\}}$.

3.2. **Subharmonic Projections.** Suppose that T_ν is the stochastic operator of a reducible random walk on a classical group G. Then the (proof of the) ergodic theorem for random walks on classical groups says that there exists a proper subgroup $H<G$ such that $p_\nu\leq\mathbb{1}_H$. Note

$$T_\nu(\mathbb{1}_H)=(\nu\otimes I_{F(G)})\Delta(\mathbb{1}_H)=(\nu\otimes I_{F(G)})\sum_{h\in H,t\in G}\delta_{ht^{-1}}\otimes\delta_t=\mathbb{1}_H.$$

Such a function, $\mathbb{1}_H\in F(G)$, will be called a T_ν-subharmonic. The ideal of functions equal to zero off H, and so *concentrated on H*, is given by:

$$F(H)=\mathbb{1}_HF(G)\mathbb{1}_H,$$

and it is a T_ν-invariant, *hereditary* C**-subalgebra* of $F(G)$. Fagnola and Pellicer identify such subalgebras as the appropriate quantum generalisation of functions concentrated on subsets, and this motivates their definition of irreducibility:

Definition 3.8. *A stochastic operator is* irreducible in the sense of Fagnola and Pellicer *if there exists no proper hereditary T_ν-invariant* C**-subalgebras of $F(G)$. A non-zero $p\in 2^G$ is called* T_ν-subharmonic *if $T_\nu(p)=p$.*

The definition that Fagnola and Pellicer use for T_ν-subharmonic is $T_\nu(p) \geq p$, but the operator norm $\|T_\nu\| = 1$. Fagnola and Pellicer prove:

Theorem 3.9. *A stochastic operator T_ν is irreducible in the sense of Fagnola and Pellicer if and only if the only subharmonic projections of T_ν are the trivial 0 or $\mathbb{1}_G$* •

As expected, irreducible in the sense of Fagnola and Pellicer coincides with the definition of irreducible for random walks.

Theorem 3.10. *A stochastic operator T_ν is irreducible in the sense of Fagnola and Pellicer if and only if the associated random walk is irreducible.*

Proof. Assume that T_ν is irreducible in the sense of Fagnola and Pellicer. As $F(G)$ is a finite dimensional C*-algebra, it may be concretely realised as

$$F(G) \cong \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \cong B\left(\bigoplus_{i=1}^{N} \mathbb{C}^{n_i}\right) =: B(H), \tag{14}$$

the bounded operators on a Hilbert space of dimension $\dim H = \displaystyle\sum_{i=1}^{N} n_i$. An inner product is given by:

$$\langle f, g\rangle = \int_G f^* g.$$

Let $q \in 2^G$ and suppose $T_\nu(q) = 0$. By Proposition 2.1 iv., this implies that $T_\nu(\mathbb{1}_G - q) = \mathbb{1}_G - q$, and so $p := \mathbb{1}_G - q$ is T_ν-subharmonic. This implies $q = 0$ or $\mathbb{1}_G$. If $q = 0$ there is nothing to say. If $q = \mathbb{1}_G$, then $T_\nu(q) = \mathbb{1}_G \neq 0$.

Therefore assume $T_\nu(q) \neq 0$. If $\nu(q) > 0$, there is nothing to say, so assume $\nu(q) = 0 \Rightarrow \varepsilon(T_\nu(q)) = 0$. This implies that, where $\eta = \eta^*$ is the Haar element:

$$\int_G \eta\, T_\nu(q) = 0 \Rightarrow \langle \eta, T_\nu(q)\rangle = 0.$$

As a positive linear map on $B(H)$, T_ν satisfies the hypothesis of Proposition 2.2 of Evans-Høegh-Krohn [10]. Therefore there exists an $k < \dim H$ such that

$$\langle \eta, (T_\nu)^k(T_\nu(q))\rangle > 0 \Rightarrow \int_G \eta\, T_{\nu^{\star(k+1)}}(q) > 0.$$

Therefore

$$\varepsilon(T_{\nu^{\star(k+1)}}(q)) > 0 \Rightarrow \nu^{\star(k+1)}(q) > 0,$$

and so the random walk given by ν is irreducible.

Suppose now that T_ν is reducible in the sense of Fagnola and Pellicer, so that there exists a non-trivial T_ν-subharmonic q such that $T_\nu(q) = q$ and indeed $T_\nu^k(q) = q$ for all $k \in \mathbb{N}$. This implies that for all $k \in \mathbb{N}$

$$\nu^{\star k}(q) = \varepsilon(T_\nu^k(q)) = \varepsilon(q).$$

Where $\eta \in 2^G$ is the Haar element, if

$$q = \alpha_q \eta \oplus r,$$

α_q is zero or one. If $\alpha_q = 0$ then

$$\nu^{\star k}(q) = \varepsilon(q) = 0$$

for all $k \in \mathbb{N}$, and so the random walk given by ν is reducible. If $\alpha_q = 1$, then $p := \mathbb{1}_G - q$ is a non-zero projection such that $\nu^{\star k}(p) = 0$ for all $k \in \mathbb{N}$, so that the random walk given by ν is reducible •

Let G be a classical group and $\Sigma \subseteq G$ a generating set. A trivial upper bound for the diameter of the Cayley graph is $|G|$. As in the classical case,

$$F(G) \cong B\left(\bigoplus_{i=1}^{|G|} \mathbb{C}\right),$$

the following corollary is a quantum generalisation of this fact. It can be seen in the proof of Theorem 3.10, via Evans-Høegh-Krohn, that the k_0 referenced below can be taken to be the dimension of the Hilbert space upon which $F(G)$ is the set of bounded operators:

Corollary 3.11. *Suppose that ν is an irreducible random walk on a quantum group. Then there exists $k_0 \in \mathbb{N}$ such that for all non-zero $q \in 2^G$, there exists $k \leq k_0$ such that $\nu^{\star k}(q) > 0$* •

3.3. **Irreducibility Criterion.**

Proposition 3.12. *Let ν, $\mu \in M_p(G)$ be supported on a quasi-subgroup S. Then $\nu \star \mu$ is also supported on S.*

Proof. That $\mathbb{1}_S$ is a group-like projection implies that (using Sweedler notation)

$$\Delta(\mathbb{1}_S)(\mathbb{1}_G \otimes \mathbb{1}_S) = \mathbb{1}_S \otimes \mathbb{1}_S$$
$$\Rightarrow \sum \mathbb{1}_{S(1)} \otimes (\mathbb{1}_{S(2)}\mathbb{1}_S) = \mathbb{1}_S \otimes \mathbb{1}_S.$$

Hit both sides with $\nu \otimes \mu$:

$$\sum \nu(\mathbb{1}_{S(1)})\mu(\mathbb{1}_{S(2)}\mathbb{1}_S) = \nu(\mathbb{1}_S)\mu(\mathbb{1}_S) = 1,$$

as ν, μ are supported on S. Note that $r_\mu := \mathbb{1}_S - p_\mu \in N_\mu$ as $\mu(r_\mu^* r_\mu) = 0$. Furthermore, as N_μ is a left ideal, $\mathbb{1}_{S(2)} r_\mu \in N_\mu$. Now consider

$$\begin{aligned}\mu(\mathbb{1}_{S(2)}\mathbb{1}_S) = \mu(\mathbb{1}_{S(2)}(p_\mu + r_\mu)) &= \mu(\mathbb{1}_{S(2)}p_\mu) + \mu(\mathbb{1}_{S(2)}r_\mu) \\ &= \mu(\mathbb{1}_{S(2)}),\end{aligned}$$

as $\mu(N_\mu) = \{0\}$ and by (8). This means that

$$\sum \nu(\mathbb{1}_{S(1)})\mu(\mathbb{1}_{S(2)}) = 1.$$

However this is the same as

$$(\nu \otimes \mu)\Delta(\mathbb{1}_S) = 1 \Rightarrow (\nu \star \mu)(\mathbb{1}_S) = 1 \qquad \bullet$$

Theorem 3.13. *A random walk ν is irreducible if and only if ν is not supported on a proper quasi-subgroup.*

Proof. Suppose that ν is supported on a proper quasi-subgroup, so that $p_\nu \leq \mathbb{1}_S < \mathbb{1}_G$. By Proposition 3.12, for all $k \in \mathbb{N}$, $\nu^{\star k}$ is supported on $\mathbb{1}_S$. Consider the projection $q_S := \mathbb{1}_G - \mathbb{1}_S > 0$. Then for all $k \in \mathbb{N}$

$$\nu^{\star k}(\mathbb{1}_S) = 1 \Rightarrow \nu^{\star k}(\mathbb{1}_G - q_S) = 1 \Rightarrow \nu(q_S) = 0,$$

that is the random walk given by ν is reducible.

Suppose now that the random walk given by ν is reducible so that there is a non-zero $q \in 2^G$ such that for all $k \in \mathbb{N}$, $\nu^{\star k}(q) = 0$. This implies that for all $n \in \mathbb{N}$, $\nu_n(q) = 0$, where

$$\nu_n := \frac{1}{n}\sum_{k=1}^{n} \nu^{\star k}.$$

Where

$$\nu_\infty := \lim_{n\to\infty} \nu_n,$$

ν_∞ is an idempotent state (this is well known, see e.g. Th. 7.1, [13]) such that $\nu_\infty(q) = 0$. Thus ν_∞ cannot be the Haar state as the Haar state is faithful.

Where p_{ν_n} is the support projection of ν_n,

$$\nu_n(p_{\nu_n}) = \frac{1}{n}\sum_{k=1}^{n} \nu^{\star k}(p_{\nu_n}) = 1 \Rightarrow \nu^{\star k}(p_{\nu_n}) = 1,$$

for each $1 \leq k \leq n$. Now consider ν_m with $m < n$. As $\nu^{\star k}(p_{\nu_n}) = 1$ for all $1 \leq k \leq m < n$, $\nu_m(p_{\nu_n}) = 1$ and so $p_{\nu_m} \leq p_{\nu_n}$. Note that $F(G) \cong B(H)$, and so $(p_{\nu_n})_{n\geq 1}$ is an increasing net of projections on some Hilbert space, and therefore (Th. 4.1.2, [29]) p_{ν_∞} is the projection onto the closed vector subspace:

$$\overline{\lim_{N\to\infty}\bigcup_{n=1}^{N} p_{\nu_n}(H)}.$$

As $p_{\nu_n}(H)\subseteq p_{\nu_\infty}(H)$, each $p_{\nu_n}\leq p_{\nu_\infty}$ (Th. 2.3.2, [29]), and this implies that:

$$p_\nu = p_{\nu_1}\leq p_{\nu_2}\leq p_{\nu_3}\leq\cdots\leq p_{\nu_n}\leq\cdots\leq p_{\nu_\infty} < p_{\int_G} = \mathbb{1}_G, \tag{15}$$

in particular ν is concentrated on the quasi-subgroup given by p_{ν_∞} •

4. Periodicity

If a random walk is irreducible, the other way it can fail to be ergodic is if periodic behavior occurs. In the classical case, if a random walk given by $\nu\in M_p(G)$ is irreducible, yet fails to be ergodic, one can construct a proper normal subgroup $N\lhd G$, and show that $\operatorname{supp}\nu\subseteq gN$ [28]. As the random walk is irreducible, it must be the case that $G/N\cong C_d$, where $d:=[G:N]$.

Suppose that T_ν is the stochastic operator of an irreducible but not ergodic random walk on a classical group G. Note that $p_\nu\leq\mathbb{1}_{gN}$, and $(\mathbb{1}_{g^iN})_{i=0}^{d-1}$ is a partition of unity. Furthermore it is straightforward to show that

$$T_\nu(\mathbb{1}_{g^jN}) = (\nu\otimes I_{F(G)})\Delta(\mathbb{1}_{g^jN}) = (\nu\otimes I_{F(G)})\sum_{i=0}^{d-1}\mathbb{1}_{g^iN}\otimes\mathbb{1}_{g^{j-i}N} = \mathbb{1}_{g^{j-1}N},$$

where the subtraction is understood mod d. Such a family, $\{\mathbb{1}_{g^iN}\}_{i=0}^{d-1}\subset F(G)$, will be called a T_ν-cyclic partition of unity. The ideal of functions equal to zero off g^iN, and so *concentrated on* g^iN, is given by:

$$F(g^iN) = \mathbb{1}_{g^iN}F(G)\mathbb{1}_{g^iN},$$

and indeed:

$$T_\nu(F(g^iN)) = F(g^{i-1}N).$$

This motivates the following definition by Fagnola and Pellicer for random walks on quantum groups:

Definition 4.1. *Let T_ν be the stochastic operator of an irreducible random walk. A partition of unity $\{p_i\}_{i=0}^{d-1}\subset F(G)$ is called T_ν-cyclic if (where the subtraction is understood mod d):*

$$T_\nu(p_iF(G)p_i) = p_{i-1}F(G)p_{i-1}.$$

The stochastic operator, and the associated random walk, is called periodic *if there exists a T_ν-cyclic partition of unity with $d\geq 2$. The biggest such d is called the* period *of the random walk.*

Proposition 4.1 of Fagnola and Pellicer states that $\{p_i\}_{i=0}^{d-1}$ is T_ν-cyclic if and only if $T_\nu(p_i) = p_{i-1}$. Furthermore, Theorems 3.7 and 4.3 of Fagnola and Pellicer imply:

Proposition 4.2. *If ν is an irreducible but periodic random walk, there exists a T_ν-cyclic partition of unity, $\{p_i\}_{i=0}^{d-1}$ such that*

$$T_\nu(p_i) = p_{i-1},$$

where the subtraction is understood mod d •

Clearly each p_i is subharmonic for $T_{\nu^{\star d}}$. These T_ν-cyclic partitions of unity behave very much like indicator functions of cosets of normal subgroups of classical groups, such that ν is concentrated on the coset, given by p_1, of the normal subgroup given by p_0.

Proposition 4.3. *Suppose that $\{p_i\}_{i=0}^{d-1}$ is a T_ν-cyclic partition of unity. Then the indexing $i = 0, 1, \ldots, d-1$ can be chosen such that*

i.
$$\varepsilon(p_i) = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

ii.
$$\nu(p_i) = \begin{cases} 1, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore $\displaystyle\int_G p_i = \frac{1}{d}$.

Proof. i. Where $\eta \in 2^G$ is the *Haar element*, writing

$$p_i = \alpha_i \eta \oplus r_i,$$

each α_i is zero or one. Note

$$\sum_{i=0}^{d-1} p_i = \left(\sum_{i=0}^{d-1} \alpha_i\right) \eta \oplus \left(\sum_{i=0}^{d-1} r_i\right) = \mathbb{1}_G = 1\eta \oplus \bigoplus_i I_{n_i},$$

and this implies that only one of the $\alpha_i = 1$. Choose it to be $i = 0$.

ii. From Proposition 2.1 iii., $\nu(p_i) = \varepsilon(p_{i-1})$.

By Proposition 2.1 v.,

$$\int_G \circ T_{\nu^{\star k}} = \int_G .$$

Let $i, j \in \{0, 1, \ldots, d-1\}$:

$$\int_G T_{\nu^{\star(i-j)}}(p_i) = \int_G p_i$$
$$\Rightarrow \int_G p_j = \int_G p_i =: c,$$

and so for any $j \in \{0, 1, \dots, d-1\}$,

$$dc = d \cdot \int_G p_j = \sum_{i=0}^{d-1} \int_G p_i = \int_G \sum_{i=0}^{d-1} p_i = \int_G \mathbb{1}_G = 1 \qquad \bullet$$

For the remainder of the current work, this indexing is understood.

Theorem 4.4. *Suppose that $\{p_i\}_{i=0}^{d-1}$ is a T_ν-cyclic partition of unity for an irreducible random walk. Then p_0 is a group-like projection.*

Proof. If $d = 1$, then $p_0 = \mathbb{1}_G$ is a group-like projection. Therefore assume $d > 1$. Using the Pierce decomposition with respect to p_0, where $q_0 = \mathbb{1}_G - p_0$,

$$F(G) = p_0 F(G) p_0 + p_0 F(G) q_0 + q_0 F(G) p_0 + q_0 F(G) q_0.$$

As ν is irreducible, by Corollary 3.11, there exists a $k_0 \in \mathbb{N}$, such that for all non-zero $q \in 2^G$, there exists $k_q \leq k_0 \in \mathbb{N}$ such that $\nu^{\star k_q}(q) > 0$.

Let $\phi := \nu^{\star d}$ so that, via $T_\phi = T_\nu^d$, $T_\phi(p_0) = p_0$ and $T_\phi(p_0 F(G) p_0) = p_0 F(G) p_0$. Define:

$$\phi_n = \frac{1}{n} \sum_{k=1}^{n} \phi^{\star k}.$$

Consider $\phi^{\star k}(p_0)$ for any $k \in \mathbb{N}$. Note that

$$\phi^{\star k}(p_0) = \varepsilon(T_{\phi^{\star k}}(p_0)) = \varepsilon(T_\phi^k(p_0)) = \varepsilon(T_{\nu^{\star d}}^k(p_0)) = \varepsilon(T_\nu^{kd}(p_0)) = \varepsilon(p_0) = 1, \tag{16}$$

that is each $\phi^{\star k}$ is supported on p_0. This means furthermore that $\phi_{k_0}(p_0) = 1$. The corner $p_0 F(G) p_0$ is a hereditary C*-subalgebra, such that $p_0 \in p_0 F(G) p_0$. Suppose that the support $p_{\phi_{k_0}} < p_0$. This implies that $p_{\phi_{k_0}} \in p_0 F(G) p_0$ (Sec. 3.2, [29]).

Consider the projection $r := p_0 - p_{\phi_{k_0}} \in p_0 F(G) p_0$. There exists a $k_r \leq k_0$ such that

$$0 < \nu^{\star k_r}(p_0 - p_{\phi_{k_0}}) \Rightarrow \nu^{\star k_r}(p_{\phi_{k_0}}) < \nu^{\star k_r}(p_0).$$

This implies that $\nu^{\star k_r}(p_0) > 0 \Rightarrow k_r \equiv 0 \mod d$, say $k_r = \ell_r \cdot d$ (note $\ell_r \leq k_0$):

$$\begin{aligned} &\nu^{\star \ell_r \cdot d}(p_{\phi_{k_0}}) < \nu^{\star \ell_r \cdot d}(p_0) \\ \Rightarrow\ &(\nu^{\star d})^{\star \ell_r}(p_{\phi_{k_0}}) < (\nu^{\star d})^{\star \ell_r}(p_0) \\ \Rightarrow\ &\phi^{\star \ell_r}(p_{\phi_{k_0}}) < \phi^{\star \ell_r}(p_0) \\ \Rightarrow\ &\phi^{\star \ell_r}(p_{\phi_{k_0}}) < 1 \end{aligned}$$

By assumption $\phi_{k_0}(p_{\phi_{k_0}})=1$. Consider

$$\phi_{k_0}(p_{\phi_{k_0}})=\frac{1}{k_0}\sum_{k=1}^{k_0}\phi^{\star k}(p_{\phi_{k_0}}).$$

For this to equal one every $\phi^{\star k}(p_{\phi_{k_0}})$ must equal one for $k\leq k_0$, but $\phi^{\star\ell_r}(p_{\phi_{k_0}})<1$. Therefore p_0 is the support of ϕ_{k_0}.

Define

$$\phi_\infty=\lim_{n\to\infty}\phi_n.$$

This is an idempotent state. Consider (15) for ϕ, but note by (16) that $p_{\phi_\infty}\leq p_0$:

$$p_\phi=p_{\phi_1}\leq\cdots\leq p_{\phi_{k_0}}\leq\cdots\leq p_{\phi_\infty}\leq p_0,$$

however $p_{\phi_{k_0}}=p_0$ which squeezes $p_0=p_{\phi_\infty}$, so p_0 is the support of a group-like projection, and therefore, by Proposition 3.5, p_0 is a group-like projection •

The possibility remains that p_0 might always correspond to a subgroup. The following example shows that this is not the case.

4.1. **Cocommutative Example.** Consider the algebra of functions on a dual group $\widehat{G}$. If $H\leq G$ is a subgroup,

$$\chi_H=\frac{1}{|H|}\sum_{h\in H}\delta^h$$

is a group-like projection, and so corresponds to a quasi-subgroup. The quasi-subgroup is a subgroup if and only if $H\lhd G$.

Consider the algebra of functions on the dual group $\widehat{S_3}$, and a state $u\in M_p(\widehat{S_3})$ given by:

$$u(\sigma)=\langle\rho(\sigma)\xi,\xi\rangle,$$

where ρ is the permutation representation $S_3\to\mathrm{GL}(\mathbb{C}^3)$, $\rho(\sigma)e_i=e_{\sigma(i)}$, and

$$\xi=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)\in\mathbb{C}^3.$$

Indeed

$$u(\delta^\sigma)=\begin{cases}1, & \text{if } \sigma=e\\ -1, & \text{if } \sigma=(12)\\ -\frac{1}{2}\operatorname{sgn}(\sigma), & \text{otherwise.}\end{cases}$$

Let $p=\sum_{\sigma\in S_3}\alpha_\sigma\delta^\sigma\in F(\widehat{S_3})$ be a fixed point of T_u:

$$T_u(p)=(u\otimes I_{F(\widehat{S_3})})\circ\Delta(p)=\sum_{\sigma\in S_3}\alpha_\sigma u(\delta^\sigma)\delta^\sigma.$$

This implies that either $p=0$ or $p=\mathbb{1}_{\widehat{S_3}}$. This implies that T_u is irreducible in the sense of Fagnola and Pellicer, and thus irreducible (Th. 3.9 and 3.10).

Define $p_0:=\chi_{\langle(12)\rangle}$ and $p_1=\mathbb{1}_{\widehat{S_3}}-p_0$. Note that (p_0,p_1) is a T_u-cyclic partition of unity, but p_0 does not correspond to subgroup of $\widehat{S}_3$ because $\langle(12)\rangle$ is not normal in S_3.

Definition 4.5. *Let G be a quantum group. A state $\nu\in M_p(G)$ is* supported on a cyclic coset of a proper quasi-subgroup *if there exists a pair of projections, $p_0\neq p_1$, such that $\nu(p_1)=1$, p_0 is a group-like projection, $T_\nu(p_1)=p_0$, and there exists $d\in\mathbb{N}$ such that $T_\nu^d(p_1)=p_1$.*

4.2. **Ergodic Theorem.** The main result may now be stated:

Theorem 4.6. *(The Ergodic Theorem for Random Walks on Finite Quantum Groups) A random walk on a quantum group G given by $\nu\in M_p(G)$ is ergodic if and only if the state is not supported on a proper quasi-subgroup, nor on a cyclic coset of a proper quasi-subgroup.*

Proof. Assume that the support of ν, $p_\nu\leq\mathbb{1}_S<\mathbb{1}_G$ for a proper quasi-subgroup $S\subset G$. By Proposition 3.12, $p_{\nu^{\star k}}\leq p_S$ for all $k\in\mathbb{N}$, and thus for $q_S:=\mathbb{1}_G-\mathbb{1}_S>0$, $\nu^{\star k}(q_S)=0$ for $k\in\mathbb{N}$ and so ν is not ergodic. Assume now that the support of ν is concentrated on a cyclic coset of a proper quasi-subgroup of index $d>1$. If ν were ergodic, by Proposition 4.3,

$$\lim_{k\to\infty}\nu^{\star(dk+1)}=\int_G\Rightarrow\lim_{k\to\infty}\nu^{\star(dk+1)}(p_1)=\int_G p_1=\frac{1}{d}.$$

However for all $k\in\mathbb{N}$:

$$\nu^{\star(dk+1)}(p_1)=\varepsilon(T_\nu^{dk+1}(p_1))=\varepsilon(T_\nu^{dk}(p_0))=1,$$

is constant not equal to $\int_G p_1$, and so ν is not ergodic.

Assume now that ν is not ergodic. If ν is reducible, by Proposition 3.13, ν is concentrated on the proper quasi-subgroup given by p_{ν_∞}. Assume therefore that ν is irreducible but periodic. Proposition 4.2 provides a T_ν-cyclic partition of unity $\{p_i\}_{i=0}^{d-1}$ such that $d>1$ (and so $p_0\neq p_1$), and $T_\nu(p_1)=p_0$. Note that

$$\nu(p_1)=\varepsilon(T_\nu(p_1))=\varepsilon(p_0)=1,$$

so that the support of ν, $p_\nu\leq p_1$. By Theorem 4.4, p_0 is a group-like projection. Finally $T_\nu^d(p_1)=p_1$ so that ν is supported on a cyclic coset of a proper quasi-subgroup •

As an easy corollary, a finite version of a result of Franz and Skalski:

Corollary 4.7. *(Prop. 2.4, [14]) A random walk on a quantum group given by a faithful $\nu\in M_p(G)$ is ergodic.*

Proof. The support $p_\nu=\mathbb{1}_G$ •.

4.3. **Discussion.**

4.3.1. *Classical Version.* A new proof of the classical theorem follows. The necessary conditions are easy. Suppose that a random walk on a classical group given by $\nu \in M_p(G)$ is not ergodic. In the classical case, by Theorem 3.7 v., all quasi-subgroups are subgroups, and $p_\nu \leq \mathbb{1}_H$ for $H < G$ a proper subgroup.

Suppose that the random walk is not concentrated on a subgroup. Then ν is concentrated on a cyclic coset of a proper quasi-subgroup. The proper quasi-subgroup is a proper subgroup $N < G$, and there is a T_ν-cyclic partition of unity $\{p_i\}_{i=0}^{d-1}$, and thus a partition $\biguplus_{i=0}^{d-1} S_i$, with $p_i = \mathbb{1}_{S_i}$. By definition $S_0 = N < G$ a proper subgroup, and $\nu(S_1) = 1$. Using the random variable picture (1), each $\zeta_i \in S_1$, and thus $\xi_k \in S_1^k$. As the walk is not concentrated on a subgroup, it is irreducible, and thus every $s \in G$ is in some $S_1^k = S_k$, where S_k is understood mod d.

Define a map $\theta : G \to C_d$ by $\{S_i\} \to \{i\} \subset C_d$. Elements $s_i \in S_i = S_1^i$ and $s_j \in S_j = S_1^j$ satisfy $s_i s_j \in S_1^i S_1^j = S_1^{i+j} = S_{i+j}$, and thus θ is a homomorphism, and its kernel, $S_0 = N$, is a proper normal subgroup $N \lhd G$, and so $\theta^{-1}(1) = S_1$ is a coset of a proper normal subgroup.

In trying to generalise the above to the quantum case, immediately an issue is that $p_0 = \mathbb{1}_S$ is only a quasi-subgroup $S \subset G$, and not a subgroup. The author is not aware of any theory of cosets of quasi-subgroups, and even if, as could be conjectured, that for a T_ν-cyclic partition of unity $\{p_i\}_{i=0}^{d-1}$:

$$\Delta(p_i) = \sum_{j=0}^{d-1} p_{i-j} \otimes p_j,$$

and some class of quotient of G by the quasi-subgroup S be constructed, such that "$F(G/S)$" $\cong F(C_d)$; or perhaps some class of morphism $p_i \mapsto \delta_i \in F(C_d)$ be constructed, and the notion of a 'normal quasi-subgroup' developed, the contents of Section 4.3.2 suggest that this doesn't go anywhere useful.

4.3.2. *Pure States.* In the proof of the ergodic theorem for random walks on finite classical groups, the proof of necessity does not assume irreducibility when it shows that if $\nu \in M_p(G)$ is concentrated on the coset of a proper normal subgroup, then the random walk is not ergodic. If $\operatorname{supp}\nu \subseteq gN$, for N a proper normal subgroup $N \lhd G$, then the random walk on G given by ν exhibits an obvious periodicity. The standard way to see this is to consider the random variable picture:

$$\xi_k = \zeta_k \cdots \zeta_1,$$

and to note that with the random variables $\zeta_i \in gN$, the random variables $\xi_k \in g^k N$.

It is possible to recast this on the algebra of functions level. For a subgroup $H \leq G$ of a classical group G, by writing down a relation that functions $f \in F(G)$ constant on left cosets satisfy, namely, where $\pi : F(G) \to F(H)$ is the projection onto $F(H)$,

$$(I_{F(G)} \otimes \pi) \circ \Delta(f) = f \otimes \mathbb{1}_H, \tag{17}$$

a copy of $F(G/H)$ is found in $F(G)$. Similarly one can define the functions constant on right cosets, $F(H\backslash G)$.

In the classical case, if $N \lhd G$ is a proper normal subgroup

$$F(G/N) = \bigoplus_{gN \in G/N} \mathbb{1}_{gN} F(G) \mathbb{1}_{gN} \cong \bigoplus_{gN \in G/N} \mathbb{C}\delta_{gN},$$

and the δ_{gN} are minimal projections in $F(G/N)$. To carefully distinguish between elements of $F(G)$ and $F(G/N)$, denote by $\imath : F(G/N) \to F(G)$, $\delta_{gN} \mapsto \mathbb{1}_{gN}$ the inclusion. If $\nu \in M_p(G)$ is concentrated on the coset of a proper normal subgroup, $p_\nu \leq \imath(\delta_{gN})$. Suitably normalised, via $\mathcal{F} : F(G/N) \to F(G/N)'$, the minimal projection $\delta_{gN} \in 2^{G/N}$ defines a pure state $\delta^{gN} \in M_p(G/N)$, and

$$p_\nu \leq \imath(p_{\delta^{gN}}),$$

and of course,

$$(\delta^{gN})^{\star k} = \delta^{g^k N},$$

is also a pure state. In the classical case[1],

$$p_\nu \leq p_\mu \Rightarrow p_{\nu^{\star 2}} \leq p_{\mu^{\star 2}},$$

and so

$$p_{\nu^{\star k}} \leq \imath(p_{\delta^{g^k N}}),$$

which implies that ν is not ergodic.

Consider now the quantum case. Following Wang [40], using the map π that helps define a subgroup of a quantum group in (3.2), the same relations ((17), and its right counterpart) that define functions constant on cosets of subgroups of a classical group, define in the quantum case *-subalgebras $F(G/H)$ and $F(H\backslash G)$, and H is said to be a *normal subgroup* of a quantum group G if these subalgebras coincide.

The question now arises: what is the quantum generalisation of a probability concentrated on a coset of a proper normal subgroup? An obvious generalisation of a δ_{gN} would be a minimal projection $p \in 2^{G/N}$. That $\nu \in M_p(G)$ be concentrated on it would translate to $p_\nu \leq \imath(p)$, where $\imath : F(G/N) \to F(G)$ is the inclusion. Minimal projections $p \in 2^G$ give rise to pure states $\mathcal{F}(p/\int_G p)$.

[1] question: is this true in the quantum case?

The trivial subgroup $\mathbb{C} \cong F(\{e\})$ given by $\pi = \varepsilon$ is a normal subgroup, and, as for all $f \in F(G)$:

$$(I_{F(G)} \otimes \varepsilon) \circ \Delta(f) = f \cong f \otimes \mathbb{1}_{\{e\}},$$

all elements of $F(G)$ are constant on cosets of $\{e\} \lhd G$, and indeed $F(G) \cong F(G/\{e\})$. Take a pure state $\delta \in M_p(G)$ and its associated minimal projection, which is necessarily its support p_δ. Note that $\varepsilon(p_\delta)$ is also a minimal projection in $F(G/\{e\})$, and that $p_\delta \leq \imath(\varepsilon(p_\delta))$ could be to say that δ is supported on a coset of the proper normal subgroup $\{e\} \lhd G$. If the random walk given by a pure state δ were ergodic, then this would be a counterexample to the claim that $\nu \in M_p(G)$ being supported on a coset of a proper normal subgroup is a barrier to ergodicity.

Consider the algebra of functions on $\widehat{S_3}$. Let $\rho : S_3 \to \mathrm{GL}(\mathbb{C}^2)$ be the two-dimensional irreducible representation, and $\xi = (1, \sqrt{2})/\sqrt{3} \in \mathbb{C}^2$ a unit vector. This data defines a state $u \in M_p(\widehat{S_3})$:

$$u(\sigma) = \langle \rho(\sigma)\xi, \xi \rangle.$$

Explicit calculations show that:

$$u = \delta_e + \frac{\sqrt{2}+1}{3}\delta_{(12)} + \frac{\sqrt{2}-1}{3}\delta_{(23)} - \frac{2\sqrt{2}}{3}\delta_{(13)} - \frac{2}{3}\delta_{(123)} - \frac{1}{3}\delta_{(132)}.$$

Note that as $u^{\star k} = u^k$, $u^{\star k} \to \delta_e = \int_{\widehat{S_3}}$, in other words the random walk given by u is ergodic. However, as ρ is irreducible, u is a pure state. Therefore the classical condition for ergodicity, that ν not be concentrated on a coset of a proper normal subgroup, is not in general a barrier for ergodicity for random walks on quantum groups.

This suggests an implication for representation theory. See Section 5.4 for the definition of the Fourier transform of a state $\nu \in M_p(G)$ at a representation $\rho : G \to \mathrm{GL}(H)$, $\widehat{\nu}(\rho)$. The following is an argument of Benjamin Steinberg. Suppose for a classical group G that $\nu \in M_p(G)$ is concentrated on the coset of a proper normal subgroup $N \lhd G$. Let ρ_a be a non-trivial irreducible unitary representation of G/N, viewed as a representation of G. If $m = [G\colon N]$, then for any $k > 0$, $\widehat{\nu}(\rho_a)^{km}(\widehat{\nu}(\rho_a)^*)^{km}$ is the identity matrix and so its trace will not go to zero. See (19) to see that this implies that the random walk is not ergodic. This argument falls down in the quantum case. Where?

5. Partial Results

5.1. Pure States on Kac–Paljutkin and Sekine Quantum Groups. Recall that the algebra of functions on a quantum group has algebra:

$$F(G) \cong \bigoplus_{i=1}^{N} M_{n_j}(\mathbb{C}).$$

At least one of the factors must be one-dimensional to account for the counit, and to gather the one dimensional factors, reorder the index $j \mapsto i$ so that $n_i = 1$ for $i = 1, \dots, m_1$, and $n_i > 1$ for $i > m_1$:

$$F(G) \cong \left(\bigoplus_{i=1}^{m_1} \mathbb{C}e_i \right) \oplus \bigoplus_{i=m_1+1}^{N} M_{n_i}(\mathbb{C}) =: A_1 \oplus B,$$

The pure states of $F(G)$ arise as pure states on single factors. If G is the Kac–Paljutkin quantum group $\mathfrak{G}_0$, $B = M_2(\mathbb{C})$; and if G is a Sekine quantum group Y_n, $B = M_n(\mathbb{C})$. Note further, in these cases, that

$$\Delta(A_1) \subseteq A_1 \otimes A_1 + B \otimes B \text{ and } \Delta(B) \subseteq A_1 \otimes B + B \otimes A_1. \qquad [21, 34] \tag{18}$$

The relations above imply that if $\nu_i := \mathcal{F}(e^i / \int_G e_i)$, $p_A = \sum_{i=1}^{m_1} e_i$, and $p_B \in B$ is the identity in that matrix factor, that for all $k \in \mathbb{N}$,

$$p_{\nu_i^{\star k}} \leq p_A \Rightarrow \nu_i^{\star k}(p_B) = 0,$$

and so the random walk given by ν_i is reducible and so not ergodic. The same relations imply that if $\delta \in M_p(G)$ is a pure state on B, $p_\delta \leq p_B$ that:

$$p_{\delta^{\star 2k}} \leq p_A \text{ and } p_{\delta^{\star(2k+1)}} \leq p_B,$$

and so the random walk given by ρ is not ergodic.

Kac and Paljutkin [21] show that, where n_1 is the number of one-dimensional factors in $F(G)$, whenever B consists of a single factor $M_{n_1}(\mathbb{C})$, the relations (18) hold, and so the random walk given by a pure state on such a quantum group is never ergodic.

5.2. **Zhang Convergence.** The following result is inspired by the classical Markov chain result that a chain with loops is aperiodic (for a random walks on a classical group this implies $e \in \operatorname{supp} \nu$), and the proof of Zhang of this fact for the case of a Sekine quantum group (Prop. 4.1, [41]).

Theorem 5.1. *Let $\nu \in M_p(G)$ be such that $\nu(p_\varepsilon) = \nu(\eta) > 0$. Then the convolution powers $(\nu^{\star k})_{k \geq 1}$ converge.*

Proof. Consider the direct sum decomposition:

$$M_p(G) \subset \mathbb{C}\varepsilon \oplus (\ker \varepsilon)^*,$$

so that

$$\nu = \nu(\eta)\varepsilon + \psi,$$

with $\nu(\eta) > 0$. Note that ε is an idempotent state with density $f_\varepsilon = \eta / \int_G \eta$.

Therefore

$$f_\nu = \frac{\nu(\eta)}{\int_G \eta}\eta + f_\psi \in \mathbb{C}\eta \oplus \ker \varepsilon.$$

An element in a direct sum is positive if and only if both elements are positive. The Haar element is positive and so $f_\psi \geq 0$. Assume that $f_\psi \neq 0$ (if $f_\psi = 0$, then $\psi = 0 \Rightarrow \nu = \varepsilon \Rightarrow \nu^{\star k} = \varepsilon$ for all k and so trivial convergence). As the density of a state,

$$\int_g \left(\frac{\nu(\eta)}{\int_G \eta} \eta + f_\psi \right) = 1 \Rightarrow \int_G f_\psi = 1 - \nu(\eta).$$

Therefore let

$$f_{\tilde{\psi}} := \frac{f_\psi}{\int_G f_\psi} = \frac{f_\psi}{1 - \nu(\eta)},$$

be the density of $\tilde{\psi} \in M_p(G)$. Now explicitly write

$$\nu = \nu(\eta)\varepsilon + (1 - \nu(\eta))\tilde{\psi}.$$

This has stochastic operator

$$T_\nu = \nu(\eta) I_{F(G)} + (1 - \nu(\eta)) T_{\tilde{\psi}}.$$

Let λ be an eigenvalue of T_ν of eigenvector a. This yields

$$\nu(\eta)a + (1 - \nu(\eta)) T_{\tilde{\psi}}(a) = \lambda a,$$

and thus

$$T_{\tilde{\psi}} a = \frac{\lambda - \nu(\eta)}{1 - \nu(\eta)} a.$$

Therefore, as a is also an eigenvector for $T_{\tilde{\psi}}$, and $T_{\tilde{\psi}}$ is a stochastic operator, it follows that

$$\left| \frac{\lambda - \nu(\eta)}{1 - \nu(\eta)} \right| \leq 1$$
$$\Rightarrow |\lambda - \nu(\eta)| \leq 1 - \nu(\eta).$$

This means that the eigenvalues of T_ν lie in the ball $B_{1-\nu(\eta)}(\nu(\eta))$ and thus the only eigenvalue of magnitude one is $\lambda = 1$. By the discussions of Section 2.2, this implies that $(T_\nu^k)_{k \geq 1}$ converges and thus so does $(\nu^{\star k})_{k \geq 1}$ •

5.3. **Freslon's Ergodic Theorem for Random Walks on Duals.** In [19], Amaury Freslon proves the ergodic theorem for random walks on the duals of (possibly infinite) discrete groups. Here is the finite version:

Proposition 5.2. *(Prop. 3.2, [19]) A random walk $u \in M_p(\widehat{G})$ on a finite dual group is ergodic if and only if u does not coincide with a character on a non-trivial subgroup $H < G$* •

The Ergodic Theorem 4.6 allows us to recover Freslon's Ergodic Theorem in the finite case.

Let $u \in M_p(\widehat{G})$, which satisfies $u(\mathbb{1}_{\widehat{G}}) = 1 \Rightarrow u(\delta^e) \cong u(e) = 1$, and also $|u(s)| \leq 1$. Suppose that u is concentrated on a proper quasi-subgroup. That u is concentrated on this quasi-subgroup implies

$$u(\chi_H) = \frac{1}{|H|}\sum_{h\in H} u(\delta^h) = 1,$$

and this implies that $u|_H = 1$, and so u coincides on H with the trivial character $H \to \{1\}$.

Suppose now that u is not concentrated on quasi-subgroup but on a cyclic coset of a quasi-subgroup. Then there exists a quasi-subgroup $p_0 = \chi_H$ and $d \in \mathbb{N}$ such that $T_u^d(\chi_H) = T_{u^d}(\chi_H) = \chi_H$:

$$\begin{aligned} T_{u^d}(\chi_H) &= \left(u^d \otimes I_{F(\widehat{G})}\right)\Delta\left(\frac{1}{|H|}\sum_{h\in H}\delta^h\right) \\ &= \frac{1}{|H|}\sum_{h\in H} u(h)^d \delta^h = \chi_H, \end{aligned}$$

so that each $u(h)$ is a d-th root of unity. Let u be defined by a unitary representation $\rho_u : G \to \mathrm{GL}(H)$ and a unit vector $\chi \in H$. For $h \in H$, following Freslon, apply the Cauchy–Schwarz inequality:

$$|u(h)| = |\langle \rho_u(h)\xi\,\xi\rangle| \leq \|\rho_u(h)\xi\|\|\xi\| = 1,$$

see it is an equality and thus $\rho_u(h)\xi = u(h)\xi$. It follows that

$$u(h_1h_2) = \langle \rho_u(h_1h_2)\xi, \xi\rangle = u(h_1)u(h_2),$$

that is $u|_H$ is a character.

5.4. **Baraquin's Ergodic Theorem.** A tool used in the quantitative analysis of random walks on classical groups is the Upper Bound Lemma of Diaconis and Shahshahani [9]. This tool was extended for use with compact classical groups by Rosenthal [33], finite quantum groups by the author [27], and finally for random walks given by absolutely continuous states on compact quantum groups of Kac type, by Freslon [17]. The upper bound follows an application of the Cauchy–Schwarz inequality to:

$$\|\nu^{\star k} - \pi\|_2^2 = \sum_{\alpha\in\mathrm{Irr}(G)\backslash\{\tau\}} d_\alpha\left[\left(\widehat{\nu}(\alpha)^*\right)^k \widehat{\nu}(\alpha)^k\right]. \tag{19}$$

The map $\|\cdot\|_2 : F(\widehat{G}) \to \mathbb{R}$ here is related to the $\mathcal{L}^2$-norm, for $\varphi \in F(\widehat{G})$ by

$$\|\varphi\|_2^2 := \|f_\varphi\|_{\mathcal{L}_2}^2 = \int_G |f_\varphi|^2.$$

Hence the necessity that the state $\nu \in M_p(G)$ be absolutely continuous (i.e. have a density $f_\nu \in \mathcal{L}^1(G)$, automatic in the finite case).

The set $\text{Irr}(G)$ is an index set for a family of pairwise-inequivalent irreducible unitary representations of the compact quantum group G (the representations are given by corepresentations $\kappa_\alpha : V_\alpha \to V_\alpha \otimes F(G)$). The index τ is for the trivial representation. The dimension $d_\alpha \in \mathbb{N}$ is the dimension of the vector space V_α, while the linear map $\widehat{\nu}(\alpha) \in L(\overline{V})$, the Fourier transform of ν at the representation κ_α, is given by:

$$\widehat{\nu}(\alpha) = (I_{\overline{V_\alpha}} \otimes \nu) \circ \overline{\kappa_\alpha}.$$

Here $\overline{\kappa_\alpha}$ is the representation conjugate to κ_α.

However, for finite quantum groups, of course, all norms are equivalent. Thus (19) can be used qualitatively, to detect if the random walk given by ν is ergodic, and there are a class of states whose ergodicity can be determined quite easily via the upper bound lemma.

Following Freslon [17], consider the *central algebra* of a quantum group, $F(G)_0$, the span of the irreducible characters of G. Where $\{\rho_{ij}^\alpha : i, j = 1, \dots, d_\alpha\}$ are the matrix coefficients of an irreducible representation κ_α, the character of κ_α is given by:

$$\chi_\alpha := \sum_{i=1}^{d_\alpha} \rho_{ii}^\alpha \in F(G),$$

so that $F(G)_0 = \text{span}\{\chi_\alpha : \alpha \in \text{Irr}(G)\}$. Consider a state $\nu \in M_p(G)$ whose density f_ν is in the central algebra:

$$f_\nu = \sum_{\alpha \in \text{Irr}(G)} f_\alpha \chi_\alpha.$$

For such states, it can be shown that the Fourier transform at a representation indexed by α is scalar:

$$\widehat{\nu}(\alpha) = \frac{f_\alpha}{d_\alpha} \cdot I_{d_\alpha} \Rightarrow (\widehat{\nu}(\alpha)^*)^k \, \widehat{\nu}(\alpha)^k = \frac{|f_\alpha|^{2k}}{d_\alpha^{2k}} \cdot I_{d_\alpha},$$

so that, for such a central state:

$$\|\nu^{\star k} - \pi\|_2^2 = \sum_{\alpha \in \text{Irr}(G) \setminus \{\tau\}} d_\alpha^2 \left| \frac{f_\alpha}{d_\alpha} \right|^{2k}.$$

When stating it for the case of a Sekine quantum group, Baraquin (Prop. 3, [2]) all but wrote down the following corollary:

Corollary 5.3. *(Baraquin's Ergodic Theorem) If a random walk on a quantum group G given by $\nu \in M_p(G)$ has density $f_\nu = \sum_{\alpha \in \text{Irr}(G)} f_\alpha \chi_\alpha \in F(G)_0$, then it is ergodic if and only if*

$$|f_\alpha| < d_\alpha,$$

for all non-trivial irreducible representations κ_α •

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