MATH7016: 20% Written Assessment 1 [Ex 62 Marks]

Name:

1. Suppose we have *telemetric* data on the *speed* of a vehicle, collected at 0.1 s intervals. Suppose the first few data points are given by:

t [s]	v [m/s]
0	0
0.1	0.4
0.2	0.9

Assuming we have the rest of the (t, v) data, can we *approximate* the distance travelled by the vehicle?

   (a) If yes, use a method you studied in MATH7016 to approximate the distance travelled after $t = 0.2$ s, given that the distance travelled after $t = 0$ s is zero.

   (b) If no, explain why not.

[5 Marks]

Solution:

2. Consider the displacement, $x(t)$ (in metres), of a body of mass m (in kg), after t seconds, subject to two forces

   - a constant force of $+10$ N, and
   - a damping force, proportional to the velocity.

The initial displacement and initial velocity are both zero.

Formulate this problem as a second order initial value problem.

[5 Marks]

Solution:

3. Calculate the first two non-zero terms of the Maclaurin Series of $y(x) = \cos x$.

[4 Marks]

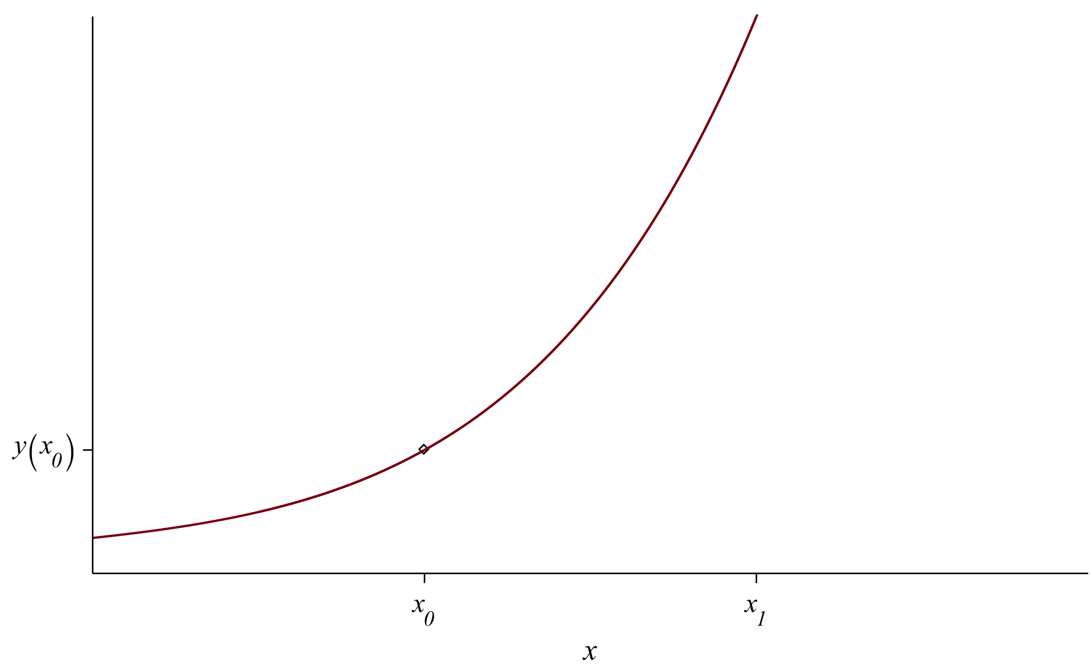
Solution:

4. Consider an initial value problem

$$\frac{dy}{dx} = F(x, y)\,; \;\; y(x_0) = y_0.$$

Euler's Method uses the tangent to the curve at $x = x_0$ to approximate the y-value at $x = x_1 = x_0 + h$.

(a) Draw the tangent at $(x_0, y(x_0))$, the ordinate $y(x_1)$, as well as the ordinate y_1. Also show the error, $\Delta y = |y(x_1) - y_1|$.



Q. 4 Continues overleaf:

(b) Use

$$y - y_1 = m(x - x_1) \tag{1}$$

to find the equation of the tangent at $x = x_0$ in terms of x_0, y_0 and $F(x_0, y_0)$.

(c) Find the value of y at $x = x_0 + h$ in terms of $y(x_0)$, h, and $F(x_0, y_0)$.

(d) Explain, with the aid of a diagram, how a large second derivative causes problems for Euler's Method.

[11 Marks]

Solution:

5. Consider an initial value problem:

$$\frac{dy}{dx} = F(x, y)\,; \;\; y(x_0) = y_0.$$

Using the Three Term Taylor Method with a step-size of h, it can be shown that the local error at step $i+1$, $\varepsilon_{\text{local}}^{i+1}$ satisfies:

$$|\varepsilon_{\text{local}}^{i+1}| \leq \frac{\|y'''\|_{\max} h^3}{6},$$

where $\|y'''\|_{\max}$ is the maximum of the absolute value of the third derivative between $x = x_i$ and $x = x_{i+1}$.

(a) What does it mean to say that the local error is $\mathcal{O}(h^3)$?

(b) Show that if we use the Three Term Taylor Method to approximate $y(x_n) = y(x_0 + n \cdot h)$, that the *global error* is $\mathcal{O}(h^2)$.

(c) What is the effect on the global error if we quarter the step-size?

(d) What is the main disadvantage of the Three Term Taylor Method?

[9 Marks]

Solution:

6. Consider an initial value problem

$$\frac{dV}{dx} = -100\, e^{-x^2}\,; \;\; V(0) = 80$$

This models the shear force, V (in kN), at a distance x (in metres) along a fixed end beam of span 6 m.

Use Heun's Method with a step-size of 0.1 to approximate $V(0.3)$. Use five significant figures for all calculations.

[12 Marks]

Solution:

7. Consider the initial value problem

$$\frac{d^2x}{dt^2} + 2(1 - x(t)^2) \cdot \frac{dx}{dt} + x(t) = 0\,; \;\; x(0) = 0\,, \;\; x'(0) = 1.$$

This is the equation of motion for the *van der Pol oscillator*: a non-conservative oscillator with non-linear damping

Use Euler's Method with a step-size of 0.1 to approximate $x(0.2)$. Use five significant figures for all calculations. [HINT: First write as two first order initial value problems]

[16 Marks]

Solution:

Rough Work:

Useful Formulae

A tables page will also be provided.

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

$$y(x) = y(a) + y'(a)(x-a) + \frac{y''(a)}{2!}(x-a)^2 + \frac{y'''(a)}{3!}(x-a)^3 + \cdots$$

$$y_{i+1} = y_i + h \cdot F(x_i, y_i)$$

$$y_{i+1} = y_i + h \cdot y_i' + \frac{h^2}{2} \cdot y_i''$$

$$\begin{aligned} y_{i+1}^0 &= y_i + h \cdot F(x_i, y_i) \\ y_{i+1} &= y_i + h \cdot \frac{F(x_i, y_i) + F(x_{i+1}, y_{i+1}^0)}{2} \end{aligned}$$

Runge-Kutta Notation

$$y_{i+1} = y_i + k_1 \cdot h$$

where

$$k_1 = F(x_i, y_i)$$

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right) \cdot h$$

where

$$\begin{aligned} k_1 &= F(x_i, y_i) \\ k_2 &= F(x_i + h, y_i + k_1 h) \end{aligned}$$