MATH6040: Test 1 [45 Marks]

Name: Marking Scheme

Answer all questions. Marks may be lost if necessary work is not clearly shown.

PLEASE READ ALL QUESTIONS CAREFULLY.

Roughwork

1. Let $\mathbf{u} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

   (a) Calculate $\mathbf{v} - \mathbf{u}$.

   [1 Mark]

   (b) Find a unit vector in the same direction of $\mathbf{v} - \mathbf{u}$.

   [2 Marks]

   (c) Investigate whether or not

   $$|\mathbf{v} - \mathbf{u}| = |\mathbf{u}| + |\mathbf{v}|.$$

   [3 Marks]

   (d) The following is known as the *Triangle Inequality*

   $$|\mathbf{v} - \mathbf{u}| \leq |\mathbf{u}| + |\mathbf{v}|.$$

   Draw a picture which suggests why the Triangle Inequality is *always* true.

   [1 Mark]

Your Answer:

(a) $\underline{v} - \underline{u} = (-4,2,-4) - (4,-1,1) = (-8,3,-5)$ [1]

(b) $|\underline{v} - \underline{u}| = \sqrt{(-8)^2 + 3^2 + (-5)^2} = 7\sqrt{2} \approx 9.899$ [1]

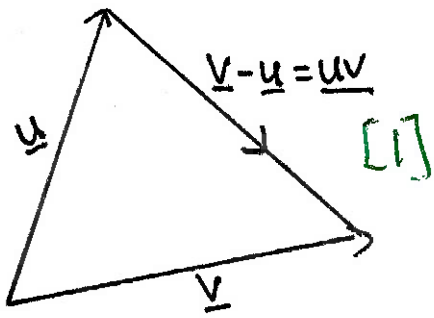
$\Rightarrow \widehat{\underline{v} - \underline{u}} = \dfrac{(-8,3,-5)}{7\sqrt{2}}$ [1]

(c) $|\underline{u}| = \sqrt{(4)^2 + (-1)^2 + (1)^2} = \sqrt{18} = 3\sqrt{2} \approx 4.243$ [1]

$|\underline{v}| = \sqrt{(-4)^2 + 2^2 + (-4)^2} = 6 = 3\sqrt{2}\cdot\sqrt{2} \approx 6.000$

$|\underline{v} - \underline{u}| = 9.899$ [1] [1] $\neq 4.243 + 6.00 = |\underline{u}| + |\underline{v}|$

(d) Note $\underline{v} - \underline{u} = \underline{uv}$:



2. The three points $S(-30, 40, 14)$, $H(-31, 38, 10)$ and $K(-32, 38, 5)$ form a triangle. Determine

(a) the vectors **KS**, **HK** and **HS**.

[3 Marks]

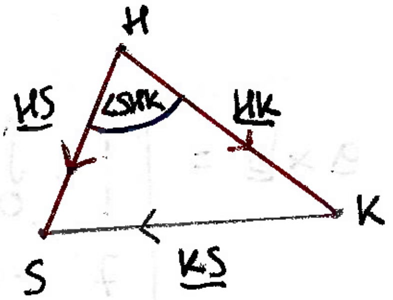
(b) and hence the side lengths of the triangle.

[3 Marks]

(c) the angle SHK.

[5 Marks]

Your Answer:



(a) $\underline{KS} = S - K = (2, 2, 9)$ [1]
$\underline{HK} = K - H = (-1, 0, -5)$ [1]
$\underline{HS} = S - H = (1, 2, 4)$ [1]

(b) $|\underline{KS}| = \sqrt{2^2 + 2^2 + 9^2} = \sqrt{89}$ [1]
$|\underline{HK}| = \sqrt{(-1)^2 + 0^2 + (-5)^2} = \sqrt{26}$ [1]
$|\underline{HS}| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$ [1]

(c) Using $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta$ with $\underline{HS}, \underline{HK}$, noting that [2]
$\underline{HS} \cdot \underline{HK} = -1 + 0 - 20 = -21$ [1]

$\Rightarrow \underline{HS} \cdot \underline{HK} = |\underline{HS}||\underline{HK}|\cos\theta$
$\Rightarrow -21 = \sqrt{21} \cdot \sqrt{26} \cos\theta$ [1]
$\Rightarrow \cos\theta = \dfrac{-21}{\sqrt{21} \cdot \sqrt{26}}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{-21}{\sqrt{21} \cdot \sqrt{26}}\right) \approx 154.0^\circ$
[1]

3. Consider vectors $\mathbf{a} = \mathbf{i} - 3\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - \mathbf{j} - 6\mathbf{k}$.

(a) calculate $\mathbf{a} \times \mathbf{b}$.

[3 Marks]

(b) calculate $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$.

[2 Marks]

(c) what does this imply?

[1 Mark]

(d) explain why $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

[1 Mark]

Your Answer:

(a) $\underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ 7 & -1 & -6 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 1 & 0 \\ 7 & -1 \end{matrix}$ [1]

$= (-21\hat{j} - \hat{k}) - (3\hat{i} - 6\hat{j})$ [1]

$= -3\hat{i} - 15\hat{j} - \hat{k} = (-3, -15, -1)$ [1] → all correct

(b) $\underline{a} \cdot (\underline{a} \times \underline{b}) = (1, 0, -3) \cdot (-3, -15, -1)$

$= -3 + 3 = 0$ [2]

(c) $\underline{a} \perp (\underline{a} \times \underline{b})$ [1]

(d) $\underline{b}$ is also perpendicular to $\underline{a} \times \underline{b}$. [1]

4. Find the value of t such that $\mathbf{a} = (-1, 0, 5)$ is perpendicular to $\mathbf{b} = (-t, 0, -4)$.

[3 Marks]

Your Answer:

$\underline{a} \perp \underline{b} \Rightarrow \underline{a} \cdot \underline{b} = 0$ [1]

$\Rightarrow (-1, 0, 5) \cdot (-t, 0, -4) = 0$

$\Rightarrow t - 20 = 0$ [1]

$\Rightarrow t = 20.$ [1]

5. A weightlifter uses a force $\mathbf{F} = 5\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$ to move a weight from $A(-1, -2, 1.2)$ to $B(-0.7, -1.7, 1.1)$. Find the work done. Suppose that the force is measured in newtons and the distance measured in metres.

[7 Marks]

Your Answer:

[1]

$\underline{d} = \underline{AB} = B - A = (0.3, 0.3, -0.1)$ [1]

$\Rightarrow W = \underline{d} \cdot \underline{F} = (0.3, 0.3, -0.1) \cdot (5, 5, 10)$

[2] $= 1.5 + 1.5 - 1$ [1]

$= 2 \text{ Nm}$

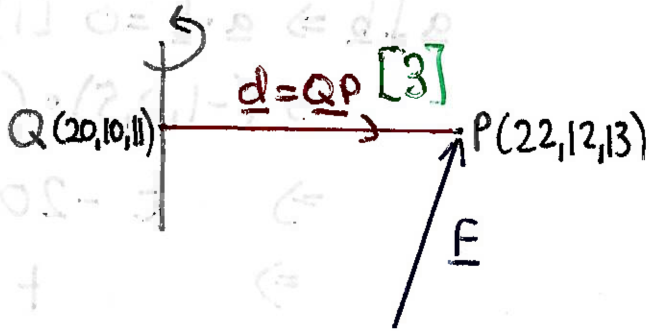
$= 2 \text{ J}.$ [1]

[1] $\Rightarrow$ correct

6. A force $\mathbf{F} = 0\mathbf{i} + 0\mathbf{j} + 98.1\mathbf{k}$ is applied at the point $(22, 12, 13)$. Find the moment of $\mathbf{F}$ about the point $(20, 10, 11)$. You may assume that force is measured in newtons and displacement is measured in metres.

[10 Marks]

Your Answer:



$$\underline{d} = \underline{QP} = P - Q \quad [1]$$
$$= (22,12,13) - (20,10,11)$$
$$= (2,2,2) \quad [1]$$

$$\underline{\tau} = \underline{d} \times \underline{F} \quad [2]$$
$$= \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 98.1 & 0 & 0 \end{array} \right| \quad [1]$$
$$= (196.2\hat{i}) - (196.2\hat{j})$$
$$= 196.2\hat{i} - 196.2\hat{j} \quad [1]$$
$$= (196.2, -196.2, 0) \ \underline{N\ m} \quad [1]$$

MATH6040: Test 1 Scenarios

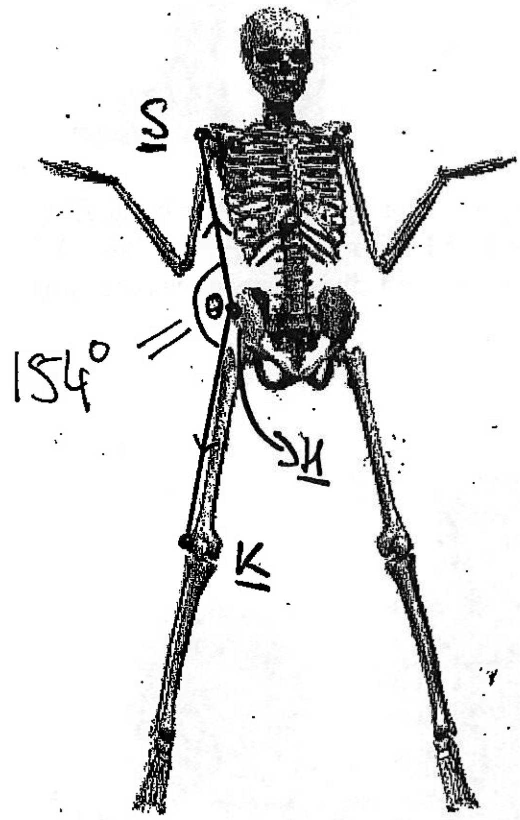
2. Suppose that there are sensors on the body of an individual in the gait analysis lab such that the position of the shoulder is given by $S(-30, 40, 14)$, the position of the hip is given by $H(-31, 38, 10)$ and the position of the knee is given by $K(-32, 38, 5)$. The units of measurement are *decimeters*; 1 dm = .1 m

   Determine

   (a) the vectors **HK** and **HS**.

   (b) the distance from the shoulder to the knee |**SK**|, in decimeters.

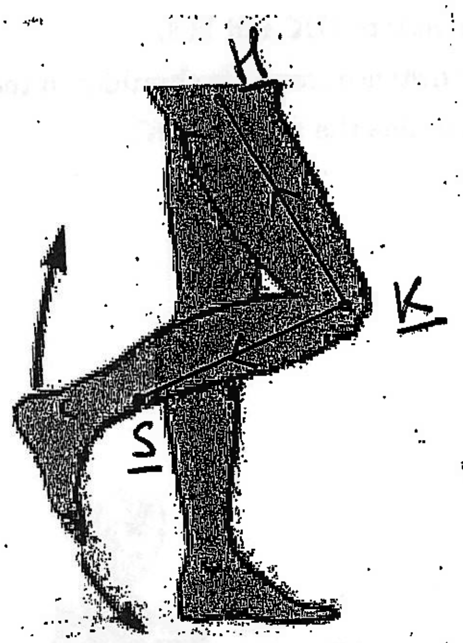
   (c) determine the angle SHK.



4. In an attempt to measure the movement of a patient's lower leg, sensors were placed on the hip, knee and shin. Suppose that hip and knee are fixed at $H(10,-20,10)$ and $K(11,-20,5)$ but the movement of the knee causes the shin sensor to change with time according to $S(11-t,-20,1)$. All measurements are in decimeters.

(i) find the vectors **KH** and **KS**

(ii) find the time t such that the knee is flexed at a right angle.



6. It takes a force of $\mathbf{F} = 0\mathbf{i} + 0\mathbf{j} + 98.1\mathbf{k}$ applied at the hand $H(22,12,13)$ to hold up a 10 kg mass. Find the moment of $\mathbf{F}$ about the elbow $E(20,10,11)$. You may assume that force is measured in Newtons and displacement is measured in decimetres.