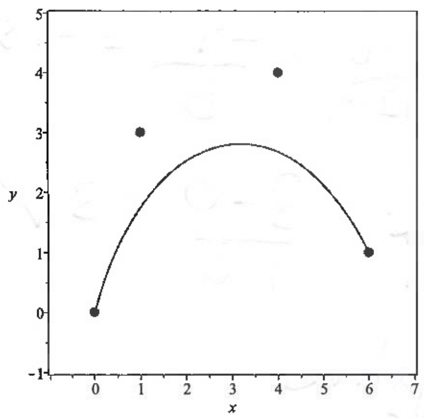
MATH6040: Test 2

Name: Marking Scheme

Answer all questions.

1. No this isn't a sad face, this is a *Bézier curve*, a type of parametric curve, popularised in the 1960s by Renault engineer Pierre Bézier, who used them for designing curves for automobile bodywork. This particular curve is given by

$$x = 3t + 6t^2 - 3t^3 \qquad ; \qquad y = 9t - 6t^2 - 2t^3.$$



The four points are $P_1(0,0)$, $P_2(1,3)$, $P_3(4,4)$, $P_4(6,1)$. The parameter t runs between zero and one such that $(x,y) = P_1(0,0)$ at $t = 0$ and $(x,y) = P_4(6,1)$ at $t = 1$.

(a) Find $\dfrac{dy}{dx}$.

[3 Marks]

(b) Hence find the slope of the tangent to the curve at

i. $P_1(0,0)$ i.e. $t = 0$.

ii. $P_4(6,1)$ i.e. $t = 1$.

[3 Marks]

(c) Show that the slope of the line segment $[P_1P_2]$ is equal to the slope of the tangent to the curve at P_1 as calculated in (b) i.

[1 Mark]

(d) Is the second derivative positive or negative t between zero and one? Explain your answer by appealing to the graph.

[2 Marks]

Solution:

a) $\dfrac{dy}{dx} = \dfrac{9 - 12t - 6t^2}{3 + 12t - 9t^2}$ 1 1 1

ii) 1

i $\dfrac{dy}{dx} = \dfrac{9}{3} = 3$ 1

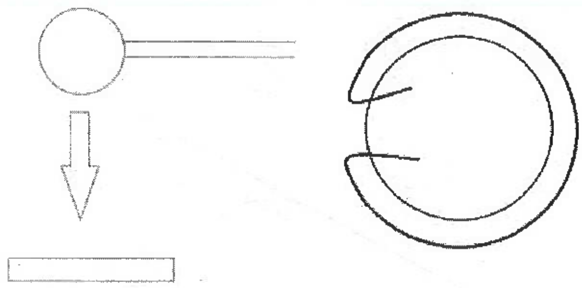
ii $m_2 = \dfrac{9 - 12 - 6}{3 + 12 - 9} = \dfrac{-9}{6} = -1.5$ 1

c) $m_{P_1P_2} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - 0}{1 - 0} = 3$ ✓ 1

d) $\cap \Rightarrow$ negative

1 1

2. A prototype of a hydropower device was designed by a team of engineers. The device consists of a water inflow to a spherical balloon. When the spherical balloon reaches a certain mass, it is dropped and the kinetic energy converted to electrical energy.



Schematic 'Cup'

The balloon is held by a 'cup' whose radius increases to accommodate the balloon. Initially the radius grows quickly but as the balloon increases in volume this speed reduces.

If the water is flowing into the spherical balloon at a rate of 0.015 $m^3\ s^{-1}$, what is the rate of change of the radius of the balloon when the radius is equal to 0.5 m?

$$\left[\text{HINT: } V(r) = \frac{4}{3}\pi r^3\right]$$

[7 Marks]

Solution:

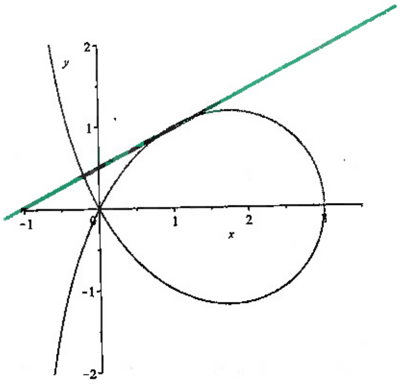
$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \quad (2) \quad \Rightarrow \frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dr}} \quad (1) \quad = \frac{0.015}{\frac{4}{3}\pi 3r^2} \quad (1)\,(1)$$

$$= \frac{0.015}{4\pi(0.5)^2}$$

$$\approx 0.004775 \text{ m/s} \quad (1) \quad (1)$$

3. The *Trisectrix of Maclaurin* is given by

$$y^2 \cdot (1+x) + x^3 - 3x^2 = 0.$$



(a) Use implicit differentiation to find $\frac{dy}{dx}$. [6 Marks]

(b) Hence, find the equation of the tangent to the curve at $(1, 1)$. [3 Marks]

Solution:

(a) $\underbrace{y^2(1) + (1+x)\cdot 2y\cdot\frac{dy}{dx}}_{1} + 3x^2 - 6x = 0$ 1 1 1

$\Rightarrow \frac{dy}{dx}(2y)(1+x) = 6x - 3x^2 - y^2$ 1

$\Rightarrow \frac{dy}{dx} = \frac{6x-3x^2-y^2}{2y(1+x)}$ 1

(b) $m = \frac{6-3-1}{2(2)} = \frac{2}{4} = 1/2$ 1 1

$y - 1 = \frac{1}{2}(x-1) \Rightarrow y = \frac{1}{2}x - 1/2 + 1$ 1

$\Rightarrow y = \frac{1}{2}x + 1/2$

4. Given $z = f(x, y) = x^2y^2 + \cos(y)$, find:

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \, \partial y}.$$

[6 Marks]

Solution:

$$\frac{\partial z}{\partial x} = y^2 \cdot 2x = 2xy^2 \quad 2$$

$$\frac{\partial z}{\partial y} = x^2 \cdot 2y - \sin y = 2x^2y - \sin y \quad 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2x^2y - \sin y\right) = 2y \cdot 2x - 0$$

$$= 4xy \quad 2$$

5. For components of uniform density, the mass is given by density times volume. Therefore if the density of a component is known, the volume is required to calculate the mass.

Suppose a very small cylindrical component is to be used in a medical device. **Use differentials** to approximate the error in the calculation of the volume if the radius is measured to be 6 mm = 0.006 m with an error of 5×10^{-6} m and the height measured to be 21 mm =0.021 m with an error of 2×10^{-6} m. Note

$$V = \pi r^2 h$$

Present your answer in the form

$$V = V_0 \pm \Delta V.$$

[8 Marks]

Solution:

$$V_0 = \pi (0.006)^2 (\overset{0.021}{\cancel{5\times10^{-6}}}) \approx \cancel{5.65487\times10^{-10}}\ m^3$$
$$= 2.3750\times10^{-6}\ m^3 \ 1$$

$$\Delta V \approx |dV|_{max} = \left|\frac{\partial V}{\partial r}\right| \Delta r + \left|\frac{\partial V}{\partial h}\right| \Delta h \quad 2$$
1 1
$$= \pi h . 2r\,\Delta r + \pi r^2 1 . \Delta h$$
$$= \pi (0.021)(2\times0.006)\,5\times10^{-6} + \pi (0.006)^2\,2\times10^{-6}$$
$$\approx 4.18\times10^{-9} \approx 4\times10^{-9}$$
1 1

$$V = [2.375 \pm 0.004]\times10^{-6}\ m^3$$
1