Marking Scheme: Summer 2015

Find

$$\int 20\,t\,e^{-10t}\,dt.$$

[7 Marks]

Solution: Let $u = 20t$ [1] so that $du = 20\,dt$ and $dv = e^{-10t}\,dt$ [1] so that $v = -\frac{1}{10}e^{-10t}$ [1]. Using integration by parts

$$\begin{aligned}\int u\,dv &= 20t\left(-\frac{1}{10}e^{-10t}\right) - \int\left(-\frac{1}{10}e^{-10t}\right)20\,dt \qquad [2]\\ &= -2te^{-10t} + 2\int e^{-10t}\,dt\\ &= -2te^{-10t} + 2\left(-\frac{1}{10}e^{-10t}\right) \qquad [1]\\ &= -2te^{-10t} - \frac{1}{5}e^{-10t} + C \qquad [1]\end{aligned}$$

Exercises

1. Find $\int x\cos x\,dx$ and check your solution. **Ans:** $x\sin x + \cos x + C$
2. $\int \ln x\,dx$ **Ans:** $x(\ln x - 1) + C$
3. Find $\int x\ln x\,dx$. **Ans:** $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$
4. $\int \frac{\ln x}{x^2}\,dx$ **Ans:** $-\frac{1+\ln x}{x} + C$
5. $\int_0^2 xe^{2x}\,dx$ **Ans:** $\frac{1}{4}(3e^4+1)$

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6. * Find $\int \sin^{-1}x\,dx$. **Ans:** $x\sin^{-1}(x) + \sqrt{1-x^2} + C$
7. * Find $\int \theta\sec^2\theta\,d\theta$. **Ans:** $\theta\tan(\theta) - \ln|\sec\theta| + C$
8. * $\int x\arctan x\,dx$ **Ans:** $\frac{1}{2}(x^2+1)\arctan x - \frac{x}{2} + C$

[illegible] — Antiderivatives involving Inverse Trigonometric Functions

This section is hard but v. important for third year. L7.

The u-substitution method and Parts replaces complicated antiderivatives by simpler ones, but one must then be able to evaluate those simpler integrals. This is often done by using a table of standard integrals such as the list in the mathematical tables. It is not necessary to memorize all of these, but one should recognize each one of them if it arises when attempting to antidifferentiate some function.

In a table of standard integrals, quadratic expressions always appear in one of the forms $x^2 + a^2$, $x^2 - a^2$ or $a^2 - x^2$, where $a \in \mathbb{R}$ is some constant. These are related to the inverse trigonometric functions via Pythagoras Theorem.

We use implicit differentiation to find the derivative of $\sin^{-1}(x)$ and $\tan^{-1}(x)$.

4.2.1 Proposition

We have that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

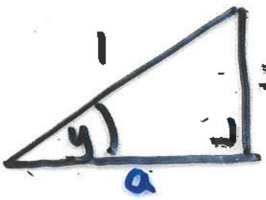
Proof. We start with $y(x) = \sin^{-1}(x)$: $\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}(x)$

Therefore we have that $x = \sin(y(x))$. Now differentiate implicitly with respect to x:

$\Rightarrow 1 = \cos(y)\cdot\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

We want our derivative to be in terms of x. If we put y in a right-angle triangle with hypotenuse one, Pythagoras gives us $\cos y$ in terms of x:



$1^2 = x^2 + a^2$

$\Rightarrow a^2 = 1 - x^2$

$\Rightarrow a = \sqrt{1-x^2}$

$\sin^2 y + \cos^2 y = 1$

$\Rightarrow \cos^2 y = 1 - \sin^2 y$

$\Rightarrow \cos y = \sqrt{1-\sin^2 y}$

$= \sqrt{1-x^2}$

Therefore $\cos y = \sqrt{1-x^2}$ and we are done.

These are the formulae that appear in your mathematical tables:

4.2.2 Proposition

We have that

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Proof. We can show the second one by differentiating the right-hand side using a Chain Rule. Note first that

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2},$$

so that

$$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{a}\frac{1}{1+\left(\frac{x}{a}\right)}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)$$
$$= \frac{1}{a^2+x^2}.$$

The first antiderivative is left as an exercise. You need

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}.$$

$\to a^2+x^2$

$\to a^2-x^2$

If you have a quadratic (with an x^2) that is not a sum or difference of two squares (i.e., if an x term appears also) then $\boxed{\text{complete the square}}$ — that is write

$$\pm x^2 + px + q = a^2 \pm (x+b)^2$$

for some a, $b \in \mathbb{R}$, where the $\pm$ depend on whether we will be integrating an integrand with a^2+x^2 or a^2-x^2.

$+x^2 \longrightarrow a^2+(x+b)^2$

$-x^2 \longrightarrow a^2-(x+b)^2$

Examples

1. Complete the square:

$$x^2+x+\frac{5}{2}$$

Solution: Do what you want to do:

$$x^2+1x+\frac{5}{2} \overset{!}{=} a^2+(x+b)^2 = a^2+b^2+\underbrace{2b}_{=1}x+x^2$$

Now equate coefficients:

- $2b=1 \Rightarrow b=\frac{1}{2}$

- $a^2+b^2=\frac{5}{2} \Rightarrow a^2+\frac{1}{4}=\frac{5}{2} \Rightarrow a^2=\frac{9}{4} \Rightarrow a=\pm\sqrt{\frac{9}{4}}=+\frac{3}{2}$

$$x^2+x+\frac{5}{2}=\left(\frac{3}{2}\right)^2+\left(x+\frac{1}{2}\right)^2$$

2. Find $I = \int \frac{1}{x^2-12x+41}\,dx$.

Solution: Looking at this note that

$$I \sim \int \frac{1}{a^2+x^2}\,dx \text{ and perhaps } I \sim \int \frac{1}{a^2+(x+b)^2}\,dx.$$

First we must complete the square

$$x^2-12x+41 \overset{!}{=} a^2+(x+b)^2 = a^2+b^2+2bx+x^2$$

- $2b=-12 \Rightarrow b=-6$

- $a^2+b^2=41 \Rightarrow a^2+36=41 \overset{-36}{\Rightarrow} a^2=5 \Rightarrow a=\pm\sqrt{5}$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

Therefore the antiderivative is

$$I = \int \frac{1}{(\sqrt{5})^2 + (x-6)^2}\,dx.$$

$\frac{dw}{dx} = 1$

Use the substitution $w = x - 6 \Rightarrow dx = dw$:

$$I = \int \frac{1}{(\sqrt{5})^2 + w^2}\,dw = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{w}{\sqrt{5}}\right)$$

$a = \sqrt{5}$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x-6}{\sqrt{5}}\right) + C$$

3. Summer 2014 Evaluate

$-x^2 \to a^2 - (x+b)^2$

$$I = \int \frac{1}{\sqrt{15 + 2x - x^2}}\,dx.$$

Solution: There is no direct integration but there is a possible manipulation. There is a quadratic under a square-root in the denominator. This should lead us to think of

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C \text{ but also } \int \frac{1}{\sqrt{a^2 - (x+b)^2}}\,dx.$$

Complete the square:

$$\boxed{15} + 2x - x^2 \doteq a^2 - (x+b)^2 = a^2 - (x^2 + 2bx + b^2)$$

$$= \boxed{a^2 - b^2} - 2bx - x^2$$

- $-2b = 2$
$\Rightarrow b = -1$

- $a^2 - b^2 = 15$
$\Rightarrow a^2 - 1 = 15$
$\Rightarrow a^2 = 16$
$\Rightarrow a = \pm\sqrt{16} = +4$

Now rewrite the integral and note that it looks like the integral of $\sin^{-1}(x)$ so we do a substitution

$$I = \int \frac{1}{\sqrt{4^2 - (x-1)^2}}\,dx$$

Let $w = x - 1$
$\Rightarrow \frac{dw}{dx} = 1$
$\Rightarrow dw = dx$

$$= \int \frac{1}{\sqrt{4^2 - w^2}}\,dw$$

$$= \sin^{-1}\left(\frac{w}{4}\right)$$

$$= \sin^{-1}\left(\frac{x-1}{4}\right) + C$$

Remark
As far as MATH6040 is concerned, these 'complete-the-square', inverse-trig-integrals are easily summarised using:

$$\int \frac{1}{\sqrt{\cdots - x^2}} \Rightarrow \sin^{-1} \Rightarrow a^2 - (x+b)^2,$$
$$\int \frac{1}{x^2 + \cdots} \Rightarrow \tan^{-1} \Rightarrow a^2 + (x+b)^2.$$

4. **Winter 2012** Determine the following integral

$$\int_{-2}^{1} \frac{1}{x^2+6x+16}\,dx$$

Solution: Can this integral be done directly? Well it is $1/(x^2+\cdots)$ so, in the context of MATH6040, an inverse tan and so need the manipulation known as completing the square:

$$x^2 + 6x + \boxed{16} \stackrel{!}{=} a^2 + (x+b)^2 = \boxed{a^2+b^2} + 2bx + x^2$$

$\cdot\ 2b = 6$
$\Rightarrow b = 3$
$\div 2$

$\cdot\ a^2 + b^2 = 16 \Rightarrow a^2 + 9 = 16 \underset{-9}{\Rightarrow} a^2 = 7$
$\underset{\sqrt{}}{\Rightarrow} a = \pm\sqrt{7}$

$$\int \frac{1}{x^2+6x+16}\,dx = \int \frac{1}{(\sqrt{7})^2 + (x+3)^2}\,dx$$

Now this yields to the substitution $w = x+3$. We have $\frac{dw}{dx} = 1$ and so $dw = dx$:

$$= \int \frac{1}{(\sqrt{7})^2 + w^2}\,dw = \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{w}{\sqrt{7}}\right)$$

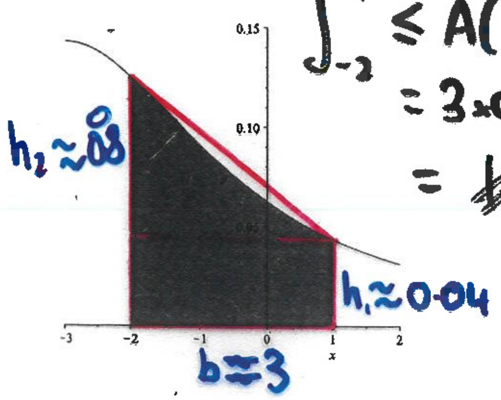
This can be antidifferentiated directly... it is an inverse tan, $\tan^{-1}$ with $a = \sqrt{7}$:

$$\Rightarrow \int_{-2}^{1} \frac{1}{x^2+6x+16}\,dx = \left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x+3}{\sqrt{7}}\right)\right]_{-2}^{1}$$

$$= \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{1+3}{\sqrt{7}}\right) - \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{-2+3}{\sqrt{7}}\right)$$

$$\approx 0.2362$$

RAD



$$\int_{-2}^{1} \leq A(\square) + A(\triangleright)$$
$$= 3 \times 0.04 + \tfrac{1}{2}3(0.08)$$
$$= \cancel{1.32}\ 0.243$$

Figure 4.3: This area is *concave* and so we can use a trapezoid to find an *upper bound*.

Marking Scheme: Autumn 2015

Find

$$\int \frac{7}{x^2-12x+37}\,dx.$$

[6 Marks]

Solution: Complete the square

$$\begin{aligned} x^2-12x+37 &\stackrel{!}{=} (x+a)^2+b^2 \\ &= x^2+2ax+a^2+b^2, \end{aligned}$$

which gives $2a \stackrel{!}{=} -12 \Rightarrow a = -6$ [1] and

$$\begin{aligned} a^2+b^2 &\stackrel{!}{=} 37 \\ \Rightarrow b^2 &= 37-a^2 = 37-36 = 1 \\ \Rightarrow b &= \pm 1 \qquad [1] \\ \Rightarrow x^2-12x+37 &= (x-6)^2+1, \end{aligned}$$

and so we have

$$\begin{aligned} \int \frac{7}{x^2-12x+37}\,dx &= 7\int \frac{1}{(x-6)^2+1^2}\,dx \qquad [1] \\ &= 7\cdot\frac{1}{2}\cdot\tan^{-1}\left(\frac{w}{1}\right) \qquad [1] \\ &= 7\tan^{-1}(x-6)+C. \qquad [1] \end{aligned}$$

Exercises

1. Prove that $\dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1+x^2}$. [Similar to the proof of the first part of Proposition 4.2.1: this is actually Section 3.3 material].

2. Find/evaluate the integral

   (a) $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{6}{\sqrt{1-t^2}}\,dt$ Ans: π.

   (b) $\displaystyle\int_0^1 \frac{4}{t^2+1}\,dt$ Ans: π.

Q. 1-2: Complete the Square. Q. 4-6 Find the antiderivative.

1. x^2+x+1 **Ans:** $\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$.

2. $-x^2+5x-2$ **Ans:** $\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{5}{2}\right)^2$.

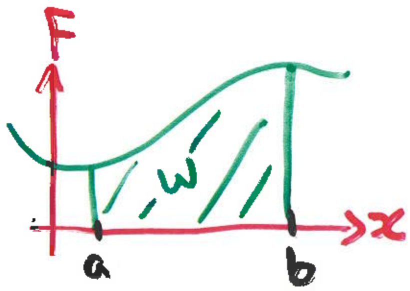
3. **Summer 2012** Evaluate $\int_{-1}^{1}\frac{1}{x^2+8x+25}\,dx$ **Ans:** $\frac{1}{3}\tan^{-1}\left(\frac{5}{3}\right)-\frac{\pi}{12}$.

4. **Winter 2017** Evaluate $\int_{-2}^{1}\frac{1}{\sqrt{21-4x-x^2}}\,dx$ **Ans:** $\sin^{-1}(3/5)\approx 0.6435$.

5. $\int\frac{dx}{\sqrt{15+2x-x^2}}$ **Ans:** $\arcsin\left(\frac{x-1}{4}\right)+C$

6. $\int\frac{dx}{x^2-x+2}$ **Ans:** $\frac{2}{\sqrt{7}}\arctan\left(\frac{2x-1}{\sqrt{7}}\right)+C$

7. $\int\frac{1}{2x^2+8x+40}\,dx$ **Ans:** $\frac{1}{8}\tan^{-1}\left(\frac{x+2}{4}\right)+C$



4.3 Work

4.3.1 Work done by a Force

If we have a force $F(x)$ that is given as a function of position, then the work done by the force in moving the object from $x=a$ to $x=b$ is given by

$$\boxed{W=\int_a^b F(x)\,dx}\sim\sum F(x)\,\Delta x=\sum(\Delta W). \tag{4.1}$$

This is also examinable in MATH6040. It will appear on the exam paper as $W=\int F\,dx$.

1. **Summer 2014** Suppose that the force on an object at a distance x metres from a point O, measured in kilo-newtons, is given by

$$F(x)=10x\sin x.$$

Find the work done in moving the object from O to a distance of 15π metres from O.

Solution The work is given by the integral of force:

$$W=\int_0^{15\pi}\underbrace{10x}_{u}\cdot\underbrace{\sin x\cdot dx}_{dv}$$

RADIANS

$$\left(=10\int_0^{15\pi}x\cdot\sin x\cdot dx\right)$$

PARTS: $\int u\cdot dv=u\cdot v-\int v\cdot du$

This requires an integration by parts:

$u = 10x \Rightarrow \frac{du}{dx} = 10(1) = 10$

Let $u = 10x \Rightarrow du = 10\,dx$ and $\int dv = \int \sin x\,dx \Rightarrow v = -\cos x$ so that the anti-derivative is:

$$\int 10x \cdot \sin x \cdot dx = 10x \cdot (-\cos x) - \int (-\cos x) \cdot 10\,dx$$
$$= -10x \cdot \cos x + 10 \sin x$$

and so the work done is:

$$W = [-10x \cdot \cos x + 10 \cdot \sin x]_0^{15\pi}$$
$$= [-10(15\pi) \cdot \cos(15\pi) + 10\sin(15\pi) - (-10(0) \cdot \cos 0 + 10 \sin 0)]$$
$$= 150\pi \approx 471.2 \text{ KN m}$$

J, kJ and N m are also OK if the appropriate unit conversion was done.

2. **Summer 2015** Suppose that the force on an object at a distance x metres from a point O, measured in newtons, is given by

$$F(x) = \frac{1}{\sqrt{5 + 4x - x^2}} = \frac{1}{\sqrt{3^2 - (x-2)^2}}.$$

i. *Complete the square* of $5 + 4x - x^2$.

Solution: We want

$$5 + \boxed{4}x - x^2 \stackrel{!}{=} a^2 - (x+b)^2$$
$$= a^2 - b^2 \boxed{-2b}x - x^2$$

so we want $-2b \stackrel{!}{=} 4 \Rightarrow b = -2$ and

$$a^2 - b^2 = 5$$
$$\Rightarrow a^2 - 4 = 5$$
$$\Rightarrow a^2 = 9 \Rightarrow a = \pm\sqrt{9} = +3$$

$$5 + 4x - x^2 = 3^2 - (x-2)^2$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

ii. Hence, find the work done in moving the object from $x = 2$ m (from O) to $x = 5$ m (from O).

Solution: We have

$$W = \int_2^5 \frac{1}{\sqrt{3^2-(x-2)^2}}\,dx = \left[\sin^{-1}\left(\frac{x-2}{3}\right)\right]_2^5 \quad \text{RAD}$$

Let $w = x - 2$ so that $dx = dw$:

$$= \int \frac{1}{\sqrt{3^2-w^2}}\,dw = \sin^{-1}\left(\frac{w}{3}\right) = \left[\sin^{-1}\left(\frac{x-2}{3}\right)\right]_2^5$$

$$= \sin^{-1}\left(\frac{5-2}{3}\right) - \sin^{-1}\left(\frac{2-2}{3}\right)$$

$$= \frac{\pi}{2}\ \text{N m}$$

Marking Scheme: Autumn 2014

Suppose that the force on an object at a distance x metres from a point O, measured in kN, is given by

$$F(x) = 0.5x \ln x.$$

Find the work done in moving the object from a point $x = 1$ metres from O to a point a distance of $x = 2$ metres from O.

[7 Marks]

Solution: The work is given by the integral of force:

$$W = \overbrace{\int_1^2 0.5x \ln x\,dx}^{[1]} = \frac{1}{2}\int_1^2 x \ln x\,dx.$$

This requires an integration by parts [1]. Let $u = \ln x \Rightarrow du = \frac{dx}{x}$ [1] and $dv = x\,dx \Rightarrow v = \frac{x^2}{2}$ [1] so that the antiderivative is:

$$\frac{1}{2}\left(\ln x\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\left(\frac{dx}{x}\right)\right) = \frac{1}{2}\left(\frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx\right)$$

$$= \frac{1}{4}x^2\ln x - \frac{1}{4}\cdot\frac{x^2}{2} = \frac{1}{4}x^2\ln x - \frac{1}{8}x^2.\ [1]$$

$$\Rightarrow W = \left[\frac{1}{4}x^2\ln x - \frac{1}{8}x^2\right]_1^2$$

$$= \left[\left(\frac{1}{4}(2^2\ln 2) - \frac{1}{8}(2^2)\right) - \left(\frac{1}{4}((1)^2\ln 1) - \frac{1}{8}(1)^2\right)\right]\ [1] = ((\ln 2 - \frac{1}{2}) - (-\frac{1}{8}))$$

$$= \ln 2 - \frac{3}{8}\ \text{kN m} \approx 0.318\ \text{kN m}$$

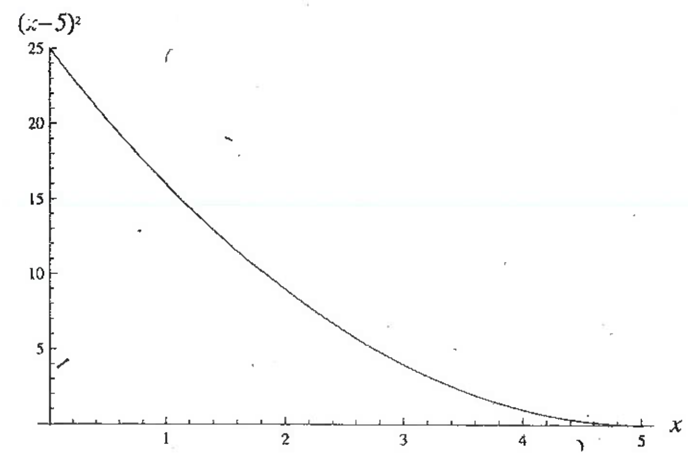


Figure 4.8: A plot of $y=(x-5)^2$. Ans: $\frac{125}{3}$

4.4.2 Volume & Centre of Gravity of a Surface of Revolution → like from a lathe

If we rotate a region under a function $y=f(x)$ about the x-axis, then a solid is generated:

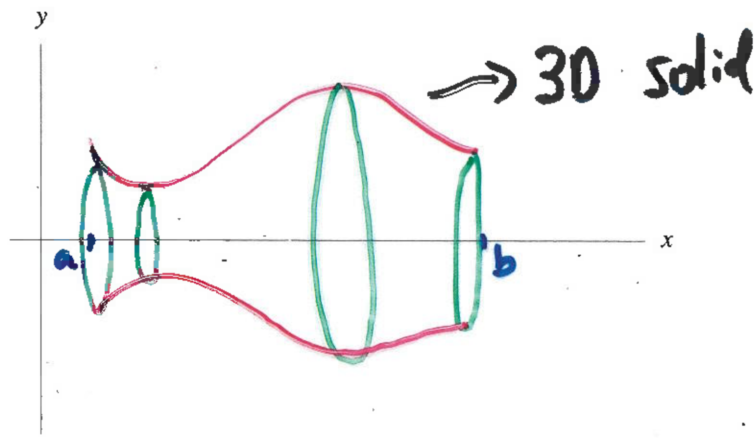
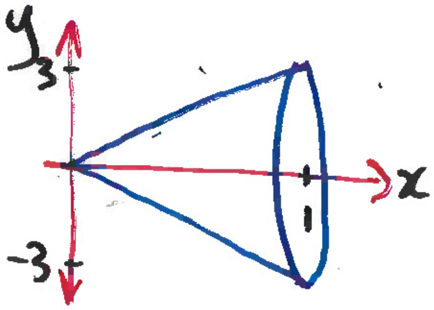


Figure 4.9: The volume of the solid generated by rotating the curve $y=f(x)$ — between $x=a$ and $x=b$ — is given by $V=\int_a^b \pi y^2\,dx$ (which is on the exam paper).

Examples

1. Summer 2015 Describe the solid generated by rotating the curve $y=3x$ about the x-axis between $x=0$ and $x=1$. Find the volume of this solid.

   Solution: Note that the curve is a line as it is of the form $y=mx+c$ so that the solid is a cone (of radius three and height one).



The volume is given by

$$V=\int \pi y^2 dx = \int_0^1 \pi (3x)^2 dx = \int_0^1 \pi 9x^2 dx = \left[\pi 9 \frac{x^3}{3}\right]_0^1$$

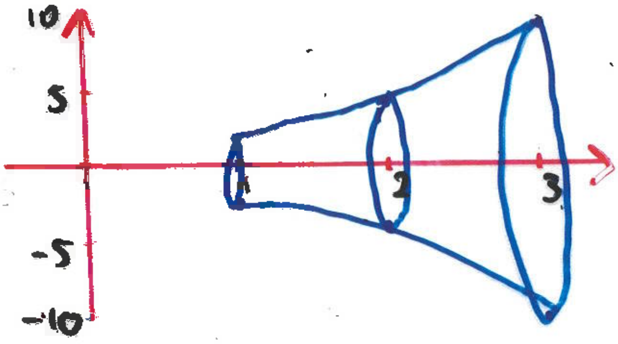
$$= \pi 9 \frac{(1)^3}{3} - \pi 9 \frac{(0)^3}{3} = 3\pi$$

2. **Winter 2012** Sketch and determine the volume of the solid formed when the portion of the curve $y = x^2 + \frac{1}{x}$ lying between $x = 1$ and $x = 3$ is rotated about the x-axis.

Solution: We can do a rough sketch by plotting points:

x	1	2	3
y	2	4.5	9.33

Now we plot:



We must calculate the integral

$$V=\int \pi y^2 dx = \int_1^3 \pi \left(x^2+\frac{1}{x}\right)^2 dx$$

Direct from tables? Manipulation? We can rewrite the integrand as

$$\left(x^2+\frac{1}{x}\right)^2 = \left(x^2+\frac{1}{x}\right)\left(x^2+\frac{1}{x}\right) = x^4 + x + x + \frac{1}{x^2}$$

$$= x^4 + 2x + x^{-2}$$

Now we compute

$$V = \left[\pi\left(\frac{x^5}{5} + 2\frac{x^2}{2} + \frac{x^{-1}}{-1}\right)\right]_1^3$$

$$= \left(\pi\left(\frac{(3)^5}{5} + \frac{2(3)^2}{2} + \frac{(3)^{-1}}{-1}\right) - \pi\left(\frac{(1)^5}{5} + \frac{2(1)^2}{2} + \frac{(1)^{-1}}{-1}\right)\right)$$

$$\approx 179.3$$

Our very last task it to calculate the centre of gravity of a surface of revolution:

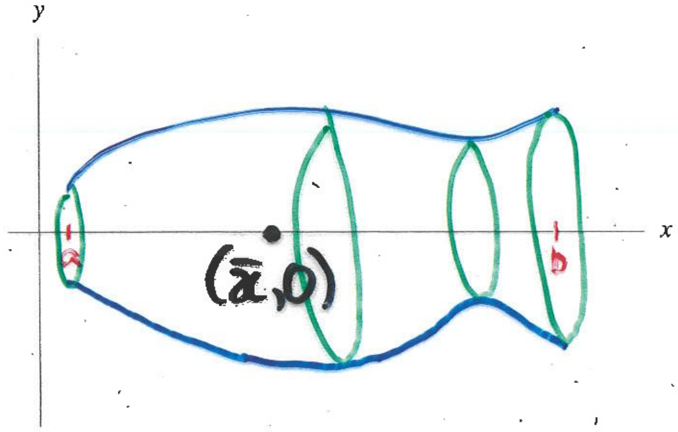


Figure 4.10: How can we find the centre of gravity of the volume generated by rotating the curve $y = f(x)$ about the x-axis between $x = a$ and $x = b$? The centre of gravity will be at a point on the x-axis (why?) — we denote that point by $(\bar{x}, 0)$.

It turns out that the answer to this question is given by the coordinates

$$\bar{x} = \frac{\int xy^2\,dx}{\int y^2\,dx}\ ,\ \bar{y} = 0 \tag{4.3}$$

where $y = f(x)$. *These formulae will be given to you in the exam.* I would recommend strongly that you evaluate each of the two integrals separately. Note that $\int y^2\,dx = \frac{V}{\pi}$ and indeed

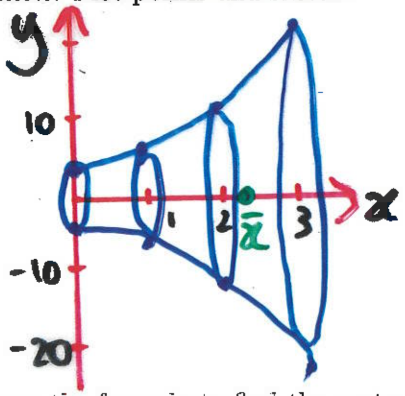
$$\bar{x} = \frac{\int xy^2\,dx}{\int y^2\,dx} = \frac{\int \pi xy^2\,dx}{\int \pi y^2\,dx} = \frac{\int \pi xy^2\,dx}{V}.$$

Autumn 2012

Suppose that the bounded area between the curve $y = 2x^2 + 3$, the x-axis, and the ordinates at $x = 0$ and $x = 3$ is rotated around the x-axis. Sketch and determine the coordinates of the centre of gravity of the solid generated.

x	0	1	2	3
y	3	5	11	21

Solution: Plot points and sketch:



We use the formula to find the centre of gravity. First we look at $\int xy^2\,dx$.

$$\int xy^2\,dx = \int_0^3 x(2x^2+3)^2\,dx$$

This integral yields quite nicely to the substitution $w = 2x^2 + 3$ but you should look for a manipulation before a substitution. Hence we multiply out the integrand carefully:

$$(2x^2+3)^2 = (2x^2+3)(2x^2+3) = 4x^4 + 6x^2 + 6x^2 + 9$$

$$\Rightarrow x \cdot (2x^2+3)^2 = 4x^5 + 12x^3 + 9x$$

Now we can integrate from $x = 0$ to $x = 3$:

$$\int xy^2\,dx = \left[4\frac{x^6}{6} + 12\frac{x^4}{4} + 9\frac{x^2}{2}\right]_0^3$$

$$= \left(\frac{4(3)^6}{6} + \frac{12(3)^4}{4} + \frac{9(3)^2}{2}\right) - \left(\frac{4(0)^6}{6} + \frac{12(0)^4}{4} + \frac{9(0)^2}{2}\right)$$

$$\approx 769.5$$

Now we look at $\int y^2\,dx$. Luckily we have $(2x^2+3)^2$ already so we can work away:

$$\int y^2\,dx = \left[4\frac{x^5}{5} + 12\frac{x^3}{3} + 9x\right]_0^3$$

$$= \left(\frac{4(3)^5}{5} + \frac{12(3)^3}{3} + 9(3)\right) - \left(\frac{4(0)^5}{5} + \frac{12(0)^3}{3} + 9(0)\right)$$

$$= 329.4$$

Putting it all together we have that the centre of gravity is found at

$$\bar{x} = \frac{\int xy^2\,dx}{\int y^2\,dx} = \frac{769.5}{329.4}$$

$$\approx 2.336$$