Some Unresolved and Unexplored Aspects of Random Walks on Quantum Groups

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The slides are dense, and unsuitable for note taking: slides available now at jpmccarthymaths.com

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Given a finite group, G, and elements $s_i \in G$ chosen according to a fixed probability distribution ($\nu \in M_p(G)$), the random variables ξ_i are given by:

$$\xi_i := s_i \cdots s_1 \cdot s_0.$$

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Denote by $\pi \in M_p(G)$ the uniform or *random* distribution. The distance to random is measured by the *total variation distance*:

$$\|\nu^{\star k} - \pi\|_{\mathrm{TV}} = \sup_{S \subset G} |\nu^{\star k}(S) - \pi(S)|$$

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the algebra of functions on G. Apply the functor composition to the morphisms m, e, and $^{-1}$ to give maps Δ (comultiplication), ε (counit), and S (antipode). The image of the commutative diagrams expressing associativity, identity, and inverses, are commutative diagrams called coassociativity, the counital property, and the antipodal property:

$$(\Delta \otimes I_{F(G)}) \circ \Delta = (I_{F(G)} \otimes \Delta) \circ \Delta$$

$$(\varepsilon \otimes I_{F(G)}) \circ \Delta = I_{F(G)} = (I_{F(G)} \otimes \varepsilon) \circ \Delta$$

$$M \circ (S \otimes I_{F(G)}) \circ \Delta = \eta_{F(G)} \circ \varepsilon = M \circ (I_{F(G)} \otimes S) \circ \Delta$$

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We will also refer to *compact* quantum groups via their algebra of continuous functions, $C(G)$. Their definition may be motivated by the theorem that

$$\text{semigroup \& cancellation} \;\Rightarrow\; \text{group} .$$

Random Walks on Finite Quantum Groups [FG05]

A probability ν on a classical group gives rise to a *state* on $F(G)$:
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Note that a (compact) quantum group also has a "uniform distribution". Given by the *Haar state*, $\int_G$, it is also invariant in the sense that $\int_G \star\nu = \int_G = \nu\star\int_G$ for all $\nu \in M_p(G)$.

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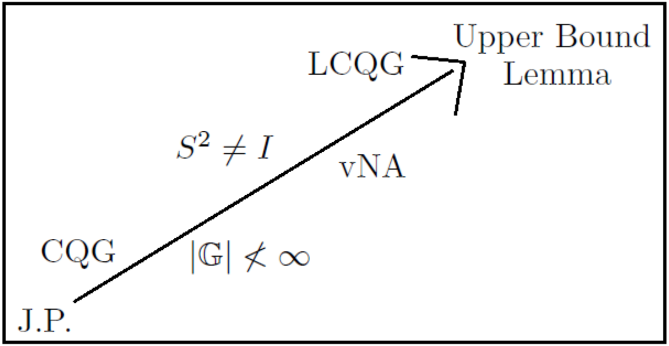

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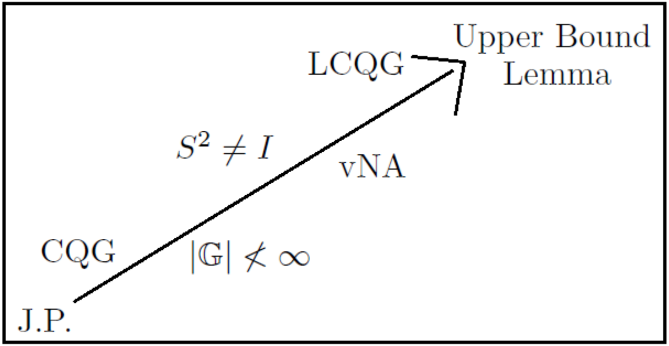

- Still teaching at CIT, a "technological university"; still trying to pursue research.

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Using the quantum identification subsets $\longleftrightarrow$ projections, the distance between probabilities ν, $\mu \in M_p(G)$ is given by the *total variation distance*:

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Of primary interest are random walks which ergodic; that *converge to random* in the sense that $\nu^{\star k} \to \pi := \int_G$.

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Let $\mathrm{Irr}(G)$ be an index set for a family of pairwise-inequivalent irreducible unitary representations of a (finite) quantum group G. Representations of G are *corepresentations* of $F(G)$, $\kappa_\alpha : V_\alpha \to V_\alpha \otimes F(G)$, and $d_\alpha := \dim V_\alpha$.

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$$F(G) \to \bigoplus_{\alpha \in \text{Irr}(G)} L(\overline{V}_\alpha)\,,\ a_\nu \mapsto \widehat{\nu},$$

defined for each $\alpha \in \text{Irr}(G)$:

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Theorem ([McC17])

$$\|\nu^{\star k} - \pi\|^2 \leq \frac{1}{4} \sum_{\alpha \in Irr(G) \setminus \{\tau\}} d_\alpha \operatorname{Tr}\left[\left(\widehat{\nu}\,(\alpha)^*\right)^k \left(\widehat{\nu}\,(\alpha)\right)^k\right].$$

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Amaury Freslon [AF18A] extended the upper bound lemma to compact quantum groups of Kac type ($S^2 = I_{C(G)}$). For the total variation distance, the projections are in $\mathcal{L}^\infty(G)$, and the driving probability $\nu \in M_p(G)$ must therefore extend to $\mathcal{L}^\infty(G)$, and also have an $\mathcal{L}^1$-density $a_\nu \in \mathcal{L}^1(G)$.

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Lower Bounds for the distance to random can be found by choosing a suitable test projection, or a suitable test function via the presentation

$$\left\|\nu^{\star k} - \pi\right\| = \frac{1}{2} \sup_{\|\phi\|_\infty \leq 1} \left|\nu^{\star k}(\phi) - \pi(\phi)\right|.$$

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- free wreath products $\widehat{\Gamma} \wr_* S_N^+$, including quantum reflection groups H_N^{s+} [AF18B]
- duals of discrete groups, $\widehat{\Gamma}$, including for $\Gamma = \mathbb{F}_N$ the free group on N generators, [AF18C]

Aspects as yet Unresolved and Unexplored in the Quantum Setting

Some things to look at:

- Spectral Analysis of the Stochastic Operator for *Reversible* Walks
- Ergodicity
- Irreducibility
- A Sufficient Condition for Convergence?
- Periodicity
- Shannon Entropy
- Non-Kac Case?
- Homogeneous Spaces

Some “low-hanging fruit”; some hard problems.

Spectral Analysis for Reversible Walks

Let $\nu \in M_p(G)$ define a random walk on a (compact) quantum group G. Define the *stochastic operator* $P_\nu : C(G) \to C(G)$ by

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For $\mu \in M_p(G)$, $P_\nu^T \mu = \nu \star \mu$. A standard treatment of (finite) classical Markov chains is to do spectral analysis on stochastic operators $P \in L(F(X))$. Let $\lambda_\star$ be the second largest (in magnitude) eigenvalue of P_ν.Two classical results that might be straightforward propositions for random walks on a *finite* quantum group are

$$\|\nu^{\star k} - \pi\| \leq C\,\lambda_\star^k,$$

and (reversible/symmetric walks $\nu = \nu \circ S$)

$$\|\nu^{\star k} - \pi\|^2 \leq \frac{|G| - 1}{4} \lambda_\star^{2k}.$$

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$$d(k+1) = \frac{1}{2}\|P_\nu^T(\nu^{\star k} - \pi)\|_{\ell_1} \leq \|P_\nu^T\|_1 \cdot d(k),$$

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On the bus this morning, I realised for $b \in F(G)$, and ν with a density:

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$s(k)$ is monotonic decreasing. Where $\mathcal{F} : a_\nu \mapsto \nu$, and $\star_A$ the convolution in $F(G)$:

$$s(k+1) = \|P_{\mathcal{F}(S(a_\nu))}(a^{\star_A k} - \mathbb{1}_G)\|_\infty \leq \|P_{\mathcal{F}(S(a_\nu))}\|_\infty \cdot s(k),$$

and as $F(G)$ together with $\|\cdot\|_\infty$ is a C^*-algebra, and $P_{\mathcal{F}(S(a_\nu))}$ unital and positive, the norm is given by $\|P_{\mathcal{F}(S(a_\nu))}(\mathbb{1}_G)\|_\infty = 1$.

Total Variation Distance Monotonic Decreasing?

On the bus this morning, I realised for $b \in F(G)$, and ν with a density:

$$\|P_\nu b\|_1 = \|S(a_\nu) \star_A b\|_1 \underset{SW}{\leq} \|S(a_\nu)\|_1 \|b\|_1 = \|b\|_1 \qquad \bullet$$

In the (finite) quantum case, where

$$s(k) := \|\nu^{\star k} - \pi\|_\infty := \|a_{\nu^{\star k}} - \mathbb{1}_G\|_\infty ,$$

$s(k)$ is monotonic decreasing. Where $\mathcal{F} : a_\nu \mapsto \nu$, and $\star_A$ the convolution in $F(G)$:

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The Upper Bound Lemma might possibly help show that $t(k) := \|\nu^{\star k} - \pi\|_2$ is monotonic decreasing

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Things are not so straightforward for random walks on quantum groups.

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Already for the Kac-Paljutkin Quantum Group $\mathfrak{G}_0$ of order eight, there are two idempotent states that are not the Haar state on any quantum subgroup [AP96]. It is an important problem to find necessary and sufficient conditions on ν for the random walk driven by ν to be ergodic as was already noted in [FS08].

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Baraquin [IB18] uses the Upper Bound Lemma to find necessary and sufficient conditions for ergodicity for random walks on the Kac-Paljutkin quantum group $\mathfrak{G}_0$ and the Sekine quantum groups Y_n, arising from the Fourier Transform ($\mathcal{F}(a_\nu) = \nu$) of linear combinations of irreducible characters.

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These conditions are given by inequalities between the absolute values of the coefficients of $\mathcal{F}^{-1}(\nu)$ in the character-linear-expansion, and the dimension of the representation of the associated character.

Irreducibility

Define the support of a $\nu \in M_p(G)$ as the smallest projection p_ν such that $\nu(p_\nu) = 1$. *Perhaps* the appropriate definition for irreducibility (finite) is

$$\forall \text{ projections } p \in F(G), \quad \exists\, k \in \mathbb{N} \quad : \quad \nu^{\star k}(p) \neq 0.$$

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$$\imath : F(H) \hookrightarrow \ker \pi_H \oplus F(H) \subset F(G);\ a \mapsto 0 \oplus a.$$

If $p_\nu \leq \mathbb{1}_H := \imath\pi_H(\mathbb{1}_G)$ for some quantum subgroup H of G, then $\nu(\mathbb{1}_H) = \nu(\imath\pi_H(\mathbb{1}_G)) = 1$, so that ν is supported on H.

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$$\begin{aligned}
(\mu \star \nu)\imath\pi_H &= ((\mu\imath\pi_H) \otimes (\nu\imath\pi_H)) \circ \Delta \circ \imath\pi_H \\
&= (\mu\imath \otimes \nu\imath)(\pi_H \circ \pi_H)\Delta \circ \imath\pi_H \\
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\end{aligned}$$

Franz & Skalski [FS09] show that idempotent states on finite quantum groups correspond to *pre-subgroups*. Perhaps irreducibility might be equivalent to not concentrated on a pre-subgroup?

A Sufficient Condition for Convergence?

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$$F(Y_n) = \left(\bigoplus_{i,j \in \mathbb{Z}_n} \mathbb{C}e_{(i,j)} \right) \oplus M_n(\mathbb{C}).$$

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$$\{e_{(i,j)} \,:\, i,j \in \mathbb{Z}_n\} \cup \{E_{ij} \,:\, i,j = 1,2,\dots,n\}$$

denoted by

$$\{e^{(i,j)} \,:\, i,j \in \mathbb{Z}_n\} \cup \{E^{ij} \,:\, i,j = 1,2,\dots,n\}.$$

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Zhang [HZ18] proves that for a random walk on Y_n driven by

$$\nu = \sum_{i,j\in\mathbb{Z}_n} \alpha_{(i,j)} e^{(i,j)} + \sum_{r,s=1}^{n} \kappa_{rs} E^{rs},$$

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Could this classically true result be true generally in the quantum case? For finite quantum groups, at least, there must be a one-dimensional factor such that

$$F(G) = \mathbb{C}e_1 \oplus \ker\varepsilon, \qquad \varepsilon(a_1 e_1 \oplus b) = a_1,$$

so in the finite case, considering the dual basis, the conjecture is that if $\nu = \alpha_1 e^1 + \cdots$, that $\alpha_1 \neq 0$ implies convergence of the convolution powers.

Periodicity

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Consider the Kac-Paljutkin quantum group of order eight [KP66], with algebra of functions $F(\mathfrak{G}_0) = \mathbb{C}^4 \oplus M_2(\mathbb{C})$ with basis $\{e_1, e_2, e_3, e_4, E_{11}, E_{12}, E_{21}, E_{22}\}$ and elements of the dual basis given by e^i ($e^1 = \varepsilon$) and E^{ij}.

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$$(e^2)^{\star k} = \begin{cases} e^2, & \text{if } k \text{ odd} \\ e^1, & \text{if } k \text{ even.} \end{cases}$$

Is it the case that e^2 is concentrated on the coset of a proper normal subgroup?

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In the classical case, the random walk is periodic if and only if ν is concentrated on such a coset. Is it the case in the quantum case that concentrated on such a coset implies periodicity?

Diameter of Finite Quantum Group?

In the classical case volume and diameter bounds can be used to analyse convergence rates.

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A naïve approach to the (finite) case would look as follows. Let p_k be the support of $\nu^{\star k}$, and define $V(k) := \dim \operatorname{ran} p_k$ be the volume, and D the diameter to be the minimum k such that $V(k) = |G| := \dim F(G)$.

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The classical study of random walks uses these ideas to produce bounds on the distance to random. Typically such random walks do *not* exhibit the cut-off phenomenon.

Convolution Factorisations

Consider the group S_n and symmetric probabilities $\nu_i \in M_p(S_n)$ uniform on $\{(i,i),\dots,(i,n)\}$. The uniform/random/Haar distribution on S_n has a convolution factorisation

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It may be deduced — even in the quantum case — that if π has a convolution factorisation, then, of course,

$$\sum_{\alpha \in \mathrm{Irr}(G)\setminus\{\tau\}} d_\alpha \operatorname{Tr}\left[\left(\prod_{i=m}^{1} \widehat{\nu_i}(\alpha)^*\right)\left(\prod_{i=m}^{1} \widehat{\nu_i}(\alpha)\right)\right] = 0,$$

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and, on the other hand that $\prod_{i=m}^{1} \widehat{\nu_i}(\alpha) \neq 0$ implies that there is not a convolution factorisation.

Shannon Entropy

Classically, the Relative Shannon Entropy ‘distance’

$$S(\nu, \pi) = \int_G a_\nu \ln(a_\nu)$$

has the property that $S(\delta^g, \pi) = \ln |G|$, $S(\pi, \pi) = 0$, and $S(\nu^{\star k}, \pi)$ is monotonic decreasing in k. There is also a relation between $S(\nu^{\star k}, \pi)$ and $k\, S(\nu, \pi)$ (that I cannot quite write down).

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Considering S as a distance to random is somewhat moving away from the Diaconis–Shahshahani approach, but it is used in the study of random walks on groups, and Crann and Kalanter [CK14] amongst others have worked on entropy in the context of quantum groups.

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It might also be interesting to see what Fourier Theory can say in light of calculations such as:

$$P_\nu(\nu_k) = \frac{k+1}{k}\cdot\nu_{k+1} - \frac{1}{k}\cdot\nu.$$

Non-Kac Case (Freslon)?

Crucial to the work thus far is the result that

$$\|\nu - \pi\| = \frac{1}{2}\|a_\nu - \mathbb{1}_G\|_1.$$

There is a proof for finite G [McC19], and [AF18A] has the more general result for compact Kac G.

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If the previous question is solved, then one can consider random walks on compact quantum groups which are not of Kac type. The Upper Bound Lemma will then involve matrices Q_α measuring the modular theory of the Haar state and some (but not all) dimensions in the formulas must be replaced by quantum dimensions.

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For estimates, the main problem here is to define explicit central states since there is no Haar-state preserving conditional expectation onto the central algebra (where the analysis is tractable). However, there are tools from monoidal equivalence to do this.

Homogeneous Spaces

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A random walk on G given by $\xi_1, \xi_2, \ldots$ induces a Markov chain on X given by $\xi_1 x_0, \xi_2 x_0, \ldots$.

The action $G \overset{\alpha}{\curvearrowright} X$ induces a representation ρ of G on $F(X)$ by $\rho(g)f(x) = f(\alpha(g^{-1}, x))$. This representation splits into a direct sum of irreducible representations ρ^β indexed by $\mathrm{Irr}(\rho)$.

Homogeneous Spaces

Let $G \overset{\alpha}{\curvearrowright} X$ be a transitive action of a finite group on a finite set X. Fix an $x_0 \in X$ and consider the isotropy subgroup $N \subset G$.

The standard isomorphism $X \ni gx_0 \overset{\varphi}{\longmapsto} gN \in G/N$ respects the action. A probability $\nu \in M_p(G)$ induces a probability $\tilde{\nu} \in M_p(X)$ by

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Diaconis [D88] exploits representation theory to prove for a right N-invariant $\nu \in M_p(G)$, π the uniform distribution on X, an upper bound lemma for this Markov chain on X, and shows that when (G, N) is a *Gelfand pair*, that the Fourier analysis becomes 'simple'.

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Fin

J'ai un epsilon de francais... Merci Beaucoup!