One has to choose u and dv correctly for the method to work. In the last example, if we had taken $u = 1$ and $dv = \ln(x)\,dx$, we would have been stuck because to continue we need to know v, and finding this is the same problem as evaluating the original anti-derivative.

Similarly, a poor choice of u and dv can make things worse instead of better. Consider $J = \int xe^x\,dx$. If we set $u = e^x$, $dv = x\,dx$, then $du = e^x\,dx$, $v = x^2/2$, so

$$\int u\,dv = J = uv - \int v\,du = \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2e^x\,dx.$$

While this equation is true, it is of no help to us since we have replaced the original integral by a more difficult one. If instead we had started by choosing $u = x$ and $dv = e^x\,dx$, then integration by parts works.

$$\int u\,dv = u\cdot v - \int v\,du$$

4. **Autumn 2013** Evaluate $\int_1^e \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv}$.

Solution: Here it is not clear that an integration by parts is necessary... surely this is in the tables? It's is not actually so let us try an integration by parts. Let $I = \int \ln x\,dx$. First up; choose u according to LIATE. That is $u = \ln x$... and dv? Now we have

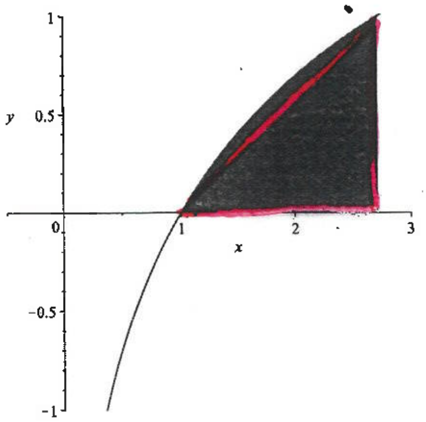
$u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow du = \dfrac{1}{x}\cdot dx$ $\qquad dv = dx$, $v = \int dv = \int dx = x$

Now using the formula

$$\int u\cdot dv = \ln x\cdot x - \int x\cdot\frac{1}{x}\,dx = x\cdot\ln x - \int 1\,dx = x\cdot\ln x - x$$

However this is an integral with limits so we do top limit minus bottom limit:

$$\int_1^e \ln x\cdot dx = [x\cdot\ln x - x]_1^e = (e\cdot\ln e - e) - (1\cdot\ln 1 - 1) = 1.$$



$$\int_1^e \geq A(\Delta) = \frac{1}{2}(e-1)(1) \approx 0.86$$

Figure 4.2: The shape is *convex* and so we can get an *upper bound* using a triangle; note that e is a number, $e \approx 2.718$.

5. Evaluate $\int_0^{\pi/2} x \sin 2x \, dx$.

Solution: Firstly we will find an anti-derivative by finding

$v = \int dv$

$$I = \int \overset{u}{x} \underbrace{\sin(2x)\, dx}_{dv},$$

and worry about the limits later. A w-substitution $w = 2x$ leads to $I = \frac{1}{2}\int w \sin w \, dw \ldots$ we choose $u = x$ by LIATE. Hence we have $dv = \sin 2x \, dx$. We want to use $\int u \, dv = uv - \int v \, du$ so will need v and du. Differentiating u and integrating dv does this for us:

$$V = \int \sin(2x)dx = \int \sin(w) \cdot \frac{dw}{2} = \frac{1}{2}\int \sin(w)dw = +\frac{1}{2}(-\cos w) = -\frac{1}{2}\cos(2x)$$

$w = 2x \Rightarrow \frac{dw}{dx} = 2 \Rightarrow dw = 2dx \Rightarrow \frac{dw}{2} = dx$

$u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

Now we use the formula:

$$\int u \cdot dv = x\left(-\frac{1}{2}\cos(2x)\right) - \int\left(-\frac{1}{2}\cos 2x\right) dx$$

$$= -x \cdot \frac{\cos(2x)}{2} + \frac{1}{2}\int \cos(2x)\, dx$$

We have that $\int \cos 2x \, dx = \frac{1}{2}\sin 2x$ so we have

$$\frac{1}{2}\int \cos(2x)\, dx = \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right)$$

and so

$$I = -x\frac{\cos 2x}{2} + \frac{1}{4}\sin 2x.$$

Now we must plug in the limits to evaluate the integral:

$$\int_0^{\pi/2} x \cdot \sin(2x)dx = \left[-x \cdot \frac{\cos 2x}{2} + \frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= \left(-\frac{\pi}{2} \cdot \frac{\cos \pi}{2} + \frac{1}{4}\sin \pi\right) - \left(-0 + \frac{1}{4}\sin 0\right)$$

$$= \frac{\pi}{4}.$$

where we used the fact that $\cos \pi = -1$, $\sin \pi = 0$ and $\sin 0 = 0$.

4.4 Mensuration

4.4.1 Area & Centroid of a Bounded Region

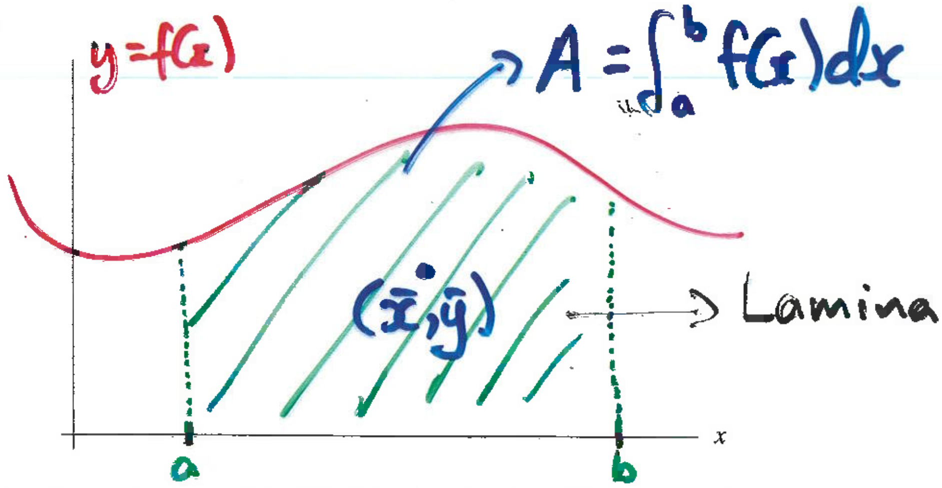


Figure 4.4: Imagine the region as a thin 3D object: a *lamina*. The area under a *positive* function between points $x = a$ and $x = b$ is given by the integral $\int_a^b f(x)\,dx$. This allows us to find the area of bounded curves by the additivity of area.

Suppose that we have an area bounded by the graph of a function as shown:

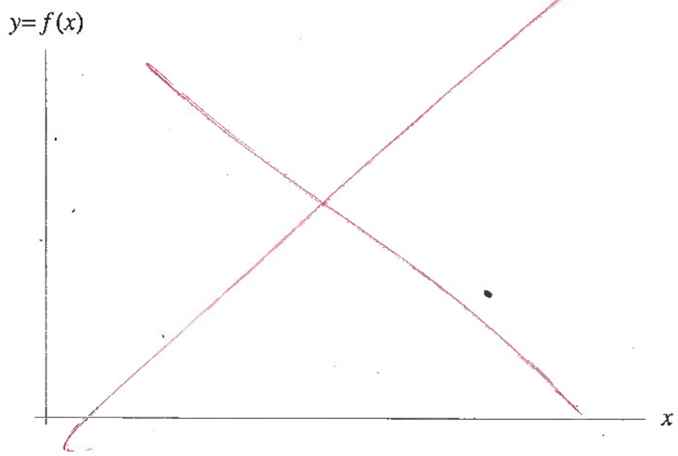


Figure 4.5: How do we find the centroid of the bounded region (lamina)?

Suppose that we wanted to balance the lamina on a point. Where can this point be found? It turns out that the answer to this question is given by the coordinates

$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx} = \frac{\int xy\,dx}{A}\;,\;\bar{y} = \frac{\frac{1}{2}\int y^2\,dx}{\int y\,dx} = \frac{\frac{1}{2}\int y^2,dx}{A} \tag{4.2}$$

where $y = f(x)$. Note that $A = \displaystyle\int y\,dx$. These formulae will be given to you in the exam and are derived in these notes in an appendix if you are so inclined. Typically you will have found the denominators, the area of the bounded region, in part (i).

Winter 2012

i. Determine the area between the curve $y = 2x^2 + 3$, the x-axis, and the ordinates at $x = 0$ and $x = 3$.

Solution: The first thing to do is to sketch a graph of the function. There are various ways of doing this but what we should note here is that $y = 2x^2 + 3$ is a 'happy', $\bigcup$ quadratic. The next thing to note is that everything is positive. Therefore, roughly, the function looks like:

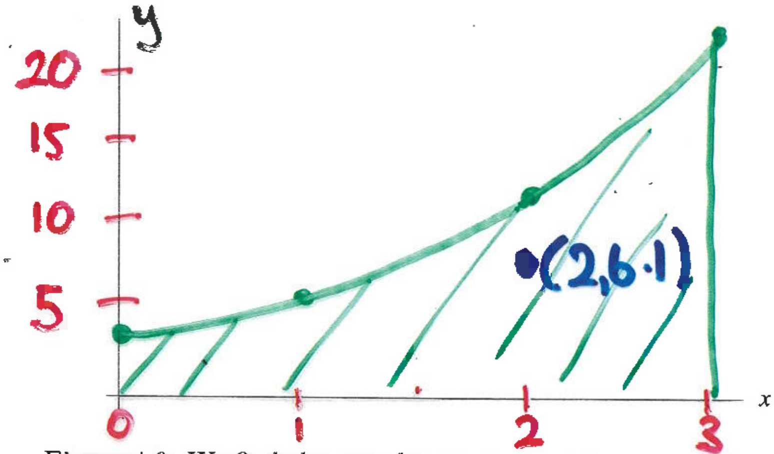


Figure 4.6: We find the area by integrating from $x = 0$ to $x = 3$.

If necessary, you can plot points to get an idea of what the curve looks like, e.g.

x	0	1	2	3
y	3	5	11	21

Therefore the area is given by

$$A = \int y\,dx = \int_0^3 (2x^2+3)\,dx = \left[2\frac{x^3}{3} + 3x\right]_0^3$$

$$= \left(2\frac{(3)^3}{3} + 3(3)\right) - (0+0) = 27$$

ii. Determine the centroid of the bounded region between the curve $y = 2x^2 + 3$, the x-axis, and the ordinates at $x = 0$ and $x = 3$.

Solution: We use the formulae $\bar{x} = \dfrac{\int xy\,dx}{\int y\,dx}$ and $\bar{y} = \dfrac{\frac{1}{2}\int y^2\,dx}{\int y\,dx}$. Recall that $A = \int y\,dx = 27$.

First we look at $\int xy\,dx$.

$$= \int_0^3 x\cdot(2x^2+3)\,dx = \int_0^3 (2x^3+3x)\,dx$$

$$= \left[2\frac{x^4}{4} + 3\frac{x^2}{2}\right]_0^3$$

$$= \left(2\frac{(3)^4}{4} + 3\frac{(3)^2}{2}\right) - (0+0)$$

$$= 54$$

$$(2x^2+3)\cdot(2x^2+3) = 4x^4+6x^2+6x^2+4$$

Now we look at $\int y^2\,dx$

$$= \int_0^3 (2x^2+3)^2\,dx = \int_0^3 (4x^4+12x^2+9)\,dx$$

$$= \left[4\frac{x^5}{5} + 12\cdot\frac{x^3}{3} + 9x\right]_0^3$$

$$= \left(4\frac{(3)^5}{5} + 12\frac{(3)^3}{3} + 9(3)\right) - (0+0+0)$$

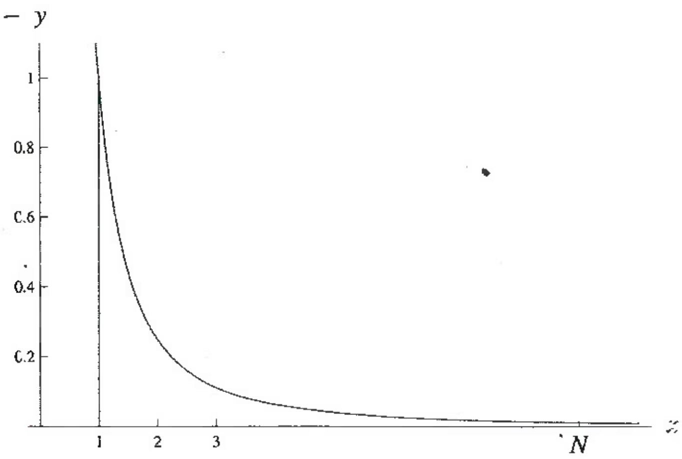
$$= 329.4$$

Now putting it all together we have

$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx} = \frac{54}{27} = 2 \quad ; \quad \bar{y} = \frac{\frac{1}{2}\int y^2\,dx}{\int y\,dx} = \frac{\frac{1}{2}(329.4)}{27} = 6.1$$

Marking Scheme: Autumn 2014

Let N be a constant and consider the region $\mathcal{R}$ bounded by the curve $y = \frac{1}{x^2}$ between $x = 1$ and $x = N$ as shown.



i. Show that the area of $\mathcal{R}$ is $1 - \frac{1}{N}$.

[2 Mark]