3.5 Applications to Error Analysis

Differentials

For a differentiable function $y = f(x)$ of a single variable x, we define the differential $dx = \Delta x$ to be an independent variable; that is, $dx = \Delta x$ can be given the value of any real number. Differentiable functions are *locally approximately linear*: the tangent at x approximates the function well near x. The differential of y is then defined by:

$$dy = f'(x)\Delta x$$

$$f'(x_0) = \frac{dy}{\Delta x}$$

$$\Delta f \approx dy = f'(x_0)\Delta x$$

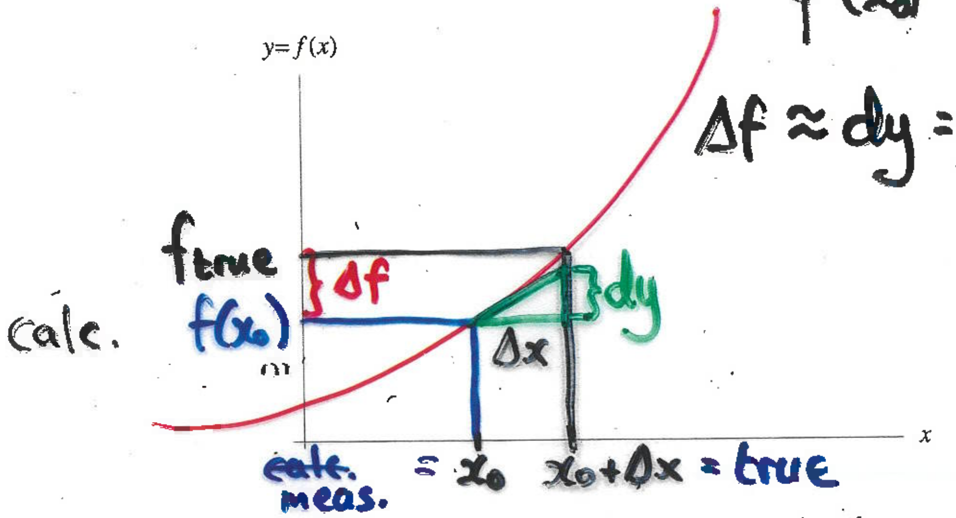


Figure 3.24: The differential estimates the actual change in y, Δy, due to a change in x. $x_0 \to x_0 + \Delta x$. For small changes in x, the differential is approximately equal to the actual change in y: $dy \approx \Delta y$.

For a differentiable function of two variables $z = f(x, y)$, we define the differentials Δx and Δy to be independent variables and the differential dz estimates the change in z when x changes to $x + \Delta x$ and y changes to $y + \Delta y$:

$$\Delta z \approx dz = \Delta z_x + \Delta z_y = \underbrace{\frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y}_{\text{differential}}.$$

Δz_x *is the error in the calculation of z due to the error in the measurement of x, Δx* and similarly for Δz_y. This extends to three or more independent variables as expected.

Propagation of Errors

Suppose we have a physical property P related to two other properties A and B by:

$$P = P(A, B)$$

Now suppose we measure A and B and record values A_0 and B_0 with associated errors ΔA and ΔB. We can now keep track of the errors in the *calculation* of P due to errors in the *measurement* of A and B by knowing *"how much P will change due to small changes in A (and/ or B) between $A - \Delta A$ and $A + \Delta A$ (and $B - \Delta B$ and $B + \Delta B$)"*. The *max absolute value differential*, $|dP|_{\max}$, gives an estimate of this error, ΔP:

$$\Delta P \approx |dP|_{max} = |P_A|\Delta A + |P_B|\Delta B$$

error in calculation; $P_A = \frac{\partial P}{\partial A}$; ΔA, ΔB: errors in meas.

The reason that we use absolute values is because we don't want errors to cancel each other out. The partial derivatives are evaluated at the measurements and this formula is *not* in the tables.

Typically we will have a quantity $P = P(A, B)$ with given measurements A_0 & B_0 with errors ΔA & ΔB and our job will be to present the calculation of P as

$$P = P(A_0, B_0) \pm \Delta P.$$

It is good practise to

1. round the error ΔP to *one significant figure* $\Delta P \approx |dP|_{max}$ roughly.
2. match the number of decimals/precision of P with ΔP

Examples

1. The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm. Use differentials to estimate the maximum error in the calculated volume of the cone.

   Solution: $V = \frac{1}{3}\pi r^2 h \Rightarrow V_0 = \frac{1}{3}\pi(10)^2(25) \approx 2617.99 \approx 2620$

$$\Delta V \approx |dV|_{max} = |V_r|\Delta r + |V_h|\Delta h$$

$$= \frac{1}{3}\pi h \cdot 2r \cdot \Delta r + \frac{1}{3}\pi r^2 \cdot 1 \cdot \Delta h$$

$$= \frac{1}{3}\pi(25)(2(10))0.1 + \frac{1}{3}\pi(10)^2 \cdot 1 \cdot 0.1$$

$$\approx 63 \approx 60$$

$$V = (2,620 \pm 60)\ cm^3$$

$$\frac{\Delta V}{V_0} \approx 2\%, \quad \frac{\Delta r}{r_0} \approx 1\%, \quad \frac{\Delta h}{h_0} \approx 0.4\%$$

2. In an experiment to measure the acceleration due to gravity, g, a physicist dropped an object in a vacuum tube of length ℓ, and measured the time taken for the object to reach the bottom, to calculate g, the physicist used the formula:

$$g = \frac{2\ell}{t^2}. = 2\ell \cdot t^{-2}$$

   If the length of the vacuum tube was measured as 10 m with an error of $\Delta\ell = 10$ cm$=0.1$ m, and the time was measured to be 1.43 s with an error of $\Delta t = 0.01$ s; use differentials to estimate a range of values of g.

Solution: Our best guess of $g = 2\ell t^{-2}$ is just

$$g_0 = 2(10)(1.43)^{-2} \approx 9.78 \approx 9.8$$

Now we use

$$\Delta g \approx |dg|_{\max} = \left|\frac{\partial g}{\partial \ell}\right| \Delta\ell + \left|\frac{\partial g}{\partial t}\right| \Delta t. = |g_L|\Delta L + |g_t|\Delta t$$

We calculate, being careful to note $\Delta\ell = 0.1$ m.

$$\Delta g \approx |dg|_{\max} = |2t^{-2}(1)|\Delta L + |2\ell\cdot(-2t^{-3})|\Delta t$$

$$= 2t^{-2}\cdot\Delta L + 4\ell\cdot t^{-3}\cdot\Delta t$$

$$= 2(1.43)^{-2} 0.1 + 4(10)(1.43)^{-3}\cdot 0.01$$

so we have

$$\approx 0.2346 \approx 0.2$$

$$g = (9.8 \pm 0.2)\ \text{m/s}^2$$

3. Suppose that the maximum deflection, in millimetres, of a simply supported beam of span L, measured in metres, under a constant load w, measured in kilonewtons per metre, is given by

$$\delta = \frac{wL^4}{7680} = \frac{1}{7680} w\cdot L^4.$$

Use differentials to approximate the error in the calculation of δ, $\Delta\delta$, when the span, L, is measured to be 6 m with an error of 1 cm = 0.01 m and the load, w, is measured to be 12 kN m^{-1} with an error of 0.12 kN m^{-1}.

Present your answer in the form

$$\delta = \delta_0 \pm \Delta\delta.$$

Solution: Our best approximation of δ is

$$\delta_0 = \frac{12(6)^4}{7680} \approx 2.025 \approx 2.03$$

We have

$$\Delta\delta \approx |d\delta|_{\max} = |\delta_w|\Delta w + |\delta_L|\Delta L$$

We have

$$\Delta\delta \approx \frac{1}{7680} L^4 \cdot 1 \cdot \Delta W + \frac{1}{7680} W \cdot 4L^3 \Delta L$$

$$= \frac{6^4}{7680} 0.12 + \frac{12 \times 4 \times 6^3}{7680} \times 0.01$$

$$\approx 0.03375 \approx 0.03$$

$$\delta = (2.03 \pm 0.03) \text{ mm}$$

This procedure generalises in the obvious way to more than two variables.

Worked Examples

1. If square beams of length L are manufactured from an alloy, then the maximum deflection under a uniform loading of W kN m^{-1} in millimetres is given by

$$\delta = \frac{WL^4}{25920}.$$

Use differentials to estimate a range of values for δ where the loading was measured to be 40 kN m^{-1} and the length was measured to be 6 m with maximum errors of 0.1 kN m^{-1} and 0.001 m.

Solution: First of all our best guess of δ is given by

$$\delta = \left.\frac{WL^4}{25920}\right|_{(W,L)=(40,6)} = \frac{(40)(6)^4}{25920} = 2 \text{ mm}.$$

We have that

$$\begin{aligned}
\Delta\delta \approx |d\delta|\text{max} &= \left|\frac{\partial\delta}{dW}\right|\Delta W + \left|\frac{\partial\delta}{\partial L}\Delta L\right| \\
&= \left.\frac{L^4}{25920}\right|_{(W,L)=(40,6)}(0.1) + \left.\frac{4WL^3}{25920}\right|_{(W,L)=(40,6)}(0.001) \\
&= \frac{(6)^4}{25920}(0.1) + \frac{4(40)(6^3)}{25920}(0.001) \\
&= 0.00633 \\
\Rightarrow \delta &= (2.000 \pm 0.006) \text{ mm}.
\end{aligned}$$

2. **Winter 2017** For a spring made from hardened steel, with 10 active coils, the spring constant, k, measured in N m^{-1}, depends on the radius of the wire in metres, r, and the radius of the coils in metres, R. This relationship is given by

$$k = (5 \times 10^9) \cdot \frac{r^4}{R^3}.$$

Revision of Antidifferentiation

Some Conceptional Basics

Roughly, if a real-valued function f is positive on $[a, b] := a \leq x \leq b$ then the integral of f on $[a, b]$ is the area under the curve $y = f(x)$ between $x = a$ and $x = b$. This can be made as rigourous as required:

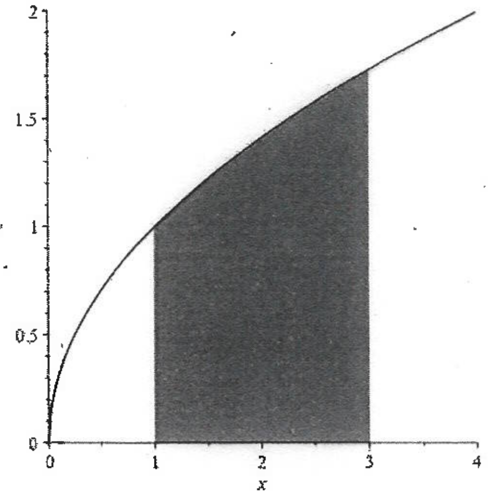


Figure 4.1: If this is the graph of $y = f(x)$ then $\int_1^3 f(x)\,dx$ is the area shaded.

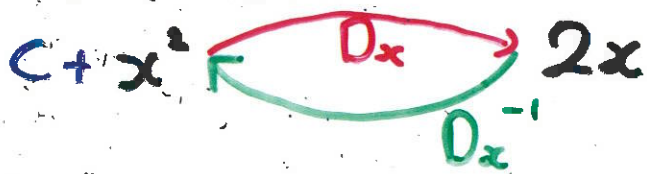
Using the Second Fundamental Theorem of Calculus, to calculate integrals one needs to antidifferentiate. Recall that to differentiate a function f we find the function f', the derivative of f, whose value at $x = a$, $f'(a)$, is the slope of the tangent to $y = f(x)$ at $x = a$. For example, the derivative of x^2 is $2x$ and we write

$$\frac{d}{dx}(x^2) = 2x$$

This $\dfrac{d}{dx}$ is an *operator* that takes as an input a function (of x), and outputs another function of x (the derivative of the input). Now antidifferentiating is doing this in reverse. So an antiderivative of $2x$ is x^2 and, for the moment, we may write

$$\left(\frac{d}{dx}\right)^{-1}(2x) = x^2$$

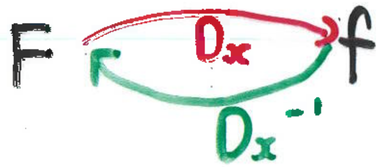
Antidifferentiating is like running differentiation backwards:



The only problem is that this isn't quite the whole story... because e.g. $2x$ has more than one antiderivative, for example $x^2 + 3$, $x^2 - 4$ to name but two. Indeed every function given by $f(x) = x^2 + C$ is an antiderivative of $2x$. To make all this precise we need the language of equivalence classes but actually for the purposes of integration it doesn't matter if you use $x^2 + 7$, $x^2 - \pi$, or $x^2 + 10^{67890}$ — for the purposes of integration you get the same answer.

This C is usually called the *constant of integration*. Were it up to me I would call it the constant of antidifferentiation.

Therefore we may as well think of the antiderivative of $2x$ — for the purposes of integration — as x^2+0, i.e. just x^2 with no C (the why will be revealed).



Second Fundamental Theorem of Calculus

If F is an antiderivative of f (so that $F'=f$), then

Equivalently,
$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b)-F(a)$$

In words, to calculate an integral, find an antiderivative, then do '(substitute) top limit minus bottom limit'. To calculate the integral of f we need to antidifferentiate it: we need to find an F such that $F'=f$

We said earlier it doesn't matter which antiderivative we use. It can be shown that all antiderivatives of a function f differ by a constant (their graphs are the same except shifted up or down — same derivatives of means same slopes: they are 'parallel') and so all are of the form $F+C$ for a constant C. So if F is an antiderivative of f and we use $F+C$ instead of F we find the integral equal to

i.e. the same thing as we would had without using the $+C$.

Therefore, all in all, we need to be able to antidifferentiate if we want to calculate integrals. Perhaps, amongst other reasons, because antidifferentiation is not a 'perfect' inverse of differentiation (we get a 'family' of antiderivatives rather than just one) the notation $\left(\frac{d}{dx}\right)^{-1}$ is not used. Instead the notation $\int dx$ is used (note the lack of limits). This can be read as 'antidifferentiate' ($\int$), with respect to x) (dx). This 'antiderivative operator' isn't exactly an operator in the sense that $\frac{d}{dx}$ is but it makes a little sense to write something like:

$$\int f(x)\,dx := \left(\frac{d}{dx}\right)^{-1} f(x).$$