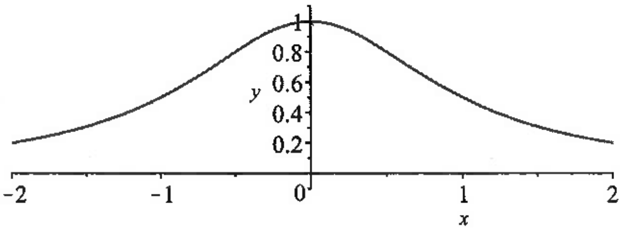
MATH6040: Test 2 Ex38

Name: Marking Scheme.

Answer all questions.

1. The *Witch of Agnesi* is defined parametrically by

$$x = t \ , \text{ and } \ y = \frac{1}{1+t^2} = (1+t^2)^{-1}.$$



(a) Find $\dfrac{dy}{dx}$.

[4 Marks]

(b) Hence find the slope of the tangent line to the curve at $\left(1, \dfrac{1}{2}\right)$.

[2 Marks]

(c) Is the second derivative positive or negative at $t = 0$? Explain your answer either by appealing to the graph or by calculating $\dfrac{d^2y}{dx^2}$.

[2 Marks]

Solution:

a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ (1) $= \dfrac{-(1+t^2)^{-2} \cdot 2t}{1}$ (1) (1) (1) $= -\dfrac{2t}{(1+t^2)^2}$

b) At $(1, 1/2)$, $x = t = 1$ (1)

$\Rightarrow m = -\dfrac{2(1)}{(1+1^2)^2}$ (1) $= -\dfrac{2}{4} = -1/2$

c) $t = 0 \Rightarrow (x,y) = (0,1)$, graph has $\cap$ (1) shape and so $\dfrac{d^2y}{dx^2} < 0$. (1)

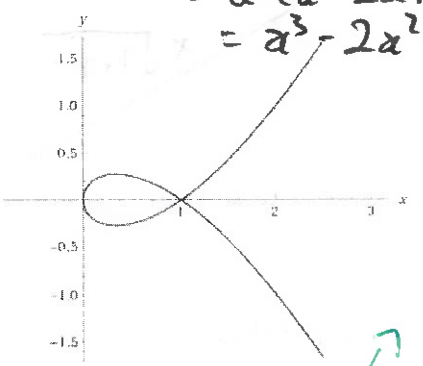
Extra Space:

3. Tschirnhaus' Cubic is given by

$$3y^2 = x(x-1)^2.$$

$$= x\cdot(x^2-2x+1)$$

$$= x^3-2x^2+x \quad 1$$



$$6y\cdot\frac{dy}{dx} = x\cdot 2(x-1)+(x-1)^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x(x-1)+(x-1)^2}{6y}$$

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

[6 Marks]

(b) Hence, find the equation of the tangent to the curve at $\left(2, \sqrt{\frac{2}{3}}\right)$.

[3 Marks]

Solution:

a) $3\cdot 2(y)\frac{dy}{dx} = 3x^2-4x+1$ 1   1 +1   1   or product rule

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2-4x+1}{6y} \quad 1$$

b) $m = \left.\frac{dy}{dx}\right|_{(2,\sqrt{2/3})} = \frac{3(2)^2-4(2)+1}{6\sqrt{2/3}} = \frac{5}{6}\sqrt{\frac{3}{2}} = \sqrt{\frac{75}{\cancel{36}\,72}} = \sqrt{\frac{25}{\cancel{12}\,24}}$ 1

$$= \frac{5}{2\sqrt{6}} \quad 1$$

$\approx \cancel{1.443}$

≈ 1.021

$$\Rightarrow y-y_1 = m(x-x_1)$$

$$\Rightarrow y-\sqrt{\frac{2}{3}} = \frac{5}{2\sqrt{6}}(x-2) \quad 1$$

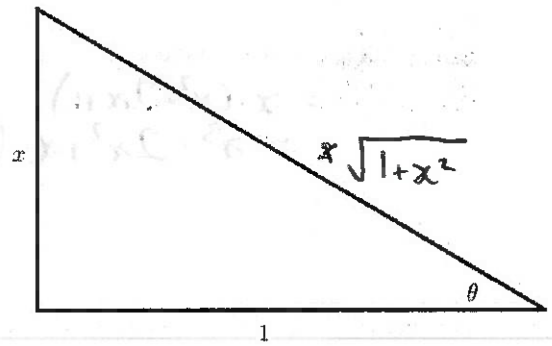
$$\sqrt{\frac{2}{3}}-\frac{5}{\sqrt{6}}$$

$$\Rightarrow y = \frac{5x}{2\sqrt{6}} - \frac{5}{\sqrt{6}} + \sqrt{\frac{2}{3}} \Rightarrow y = \frac{5}{2\sqrt{3}}x + \cancel{\frac{1}{\sqrt{3}}(\sqrt{2}-5)}$$

$$\Rightarrow y \approx \cancel{1.443}x - 2.070$$

$$1.020x - 1.225$$

2. A component in a device consists of a right-angled triangle of hypotenuse ~~2 cm~~ $\sqrt{1+x^2}$, base 1 cm, and a variable height x.



The engineer needs to control how fast the angle changes. Note $h = x$

(a) Show that $x = \tan\theta$.

(b) Show that if $x = 1$ cm, then $\theta = \pi/4$.

(c) Suppose that the height x is decreasing at a rate of 0.1 cm per second. How fast is the angle, θ, changing if $x = 1$ cm?

$$\left[\text{HINT: } (\tan\theta)' = \sec^2(\theta),\ \sec\theta = \frac{1}{\cos\theta}.\right]$$

[7 Marks]

Solution:

a) $\tan\theta = \frac{x}{1} \Rightarrow x = \tan\theta$ 1

b) $1 = \tan\theta \underset{\tan^{-1}}{\Rightarrow} \tan^{-1}(1) = \theta \Rightarrow \theta = \pi/4$ 1

c) $\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} \Rightarrow -0.1 = \sec^2\theta \cdot \frac{d\theta}{dt}$ 1 1 1

$\Rightarrow \frac{d\theta}{dt} = \frac{-.1}{\cancel{co}\sec^2\theta} = -.1\cos^2\theta \Big|_{\theta=\pi/4} = -0.05$ rad/s 1 1 bonus all correct

4. Verify that $u(x,y) = 2x + xy + x^2$ is a solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\,\partial y} - \frac{1}{x}\frac{\partial u}{\partial y} - 2 = 0.$$

[6 Marks]

Solution:

$$\frac{\partial u}{\partial x} = 2\cdot(1) + y\cdot(1) + 2x = 2 + y + 2x \quad 1$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2 + y + 2x) = 0 + 0 + 2 = 2 \quad 1$$

$$\frac{\partial u}{\partial y} = 0 + x(1) + 0 = x \quad 1$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(x) = 1 \quad 1$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - \frac{1}{x}\frac{\partial u}{\partial y} - 2$$

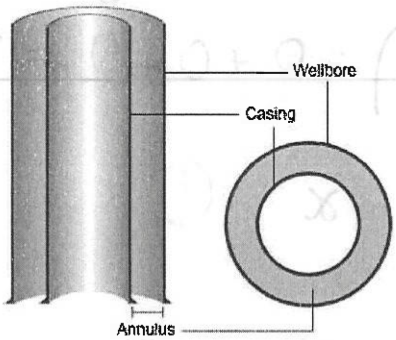
$$= 2 + 1 - \frac{1}{x}\cdot x - 2$$

$$= 3 - 1 - 2 = 0 \checkmark$$

1 1

5. For an *annulus*, the second moment of area, through the central axis (rotating around the x-axis), measured in cm^4, is related to the outer radius R and inner radius r, both measured in centimetres. This relationship is given by

$$I = \frac{\pi}{4}(R^4 - r^4).$$



Use differentials to approximate the error in the calculation of I, ΔI, when the outer radius R, is measured to be 40 cm with an error of 4 cm and the inner radius, r, is measured to be 35 cm with an error of 3 cm.

Present your answer in the form

$$I = I_0 \pm \Delta I.$$

$I_0 = \frac{\pi}{4}(40^4 - 35^4) \approx 832000$ 1

$\approx 10^6 = 1000000$

[8 Marks]

Solution: $\Delta I \approx |dI|_{max} = \left|\frac{\partial I}{\partial R}\right| \Delta R + \left|\frac{\partial I}{\partial r}\right| \Delta r$ 1

No absolute value -2

$= \left|\frac{\pi}{4} \cdot 4R^3\right| \Delta R + \left|-\frac{\pi}{4}(4r^3)\right| \Delta r$ 1 1

$= \pi(40^3) \times 4 + \pi(35^3) \times 3 \approx 1.2 \times 10^6 \approx 10^6 = 1000000$ 1 1

$I = (1000000 \pm 1000000)\ \text{cm}^4.$ 1 1

Bonus 2 for noticing $\Delta I \geq I_0$

not above 8