MATH7021: Assignment 2 — Remarks for Poorly Answered Questions

April 15, 2019

1 General Remarks

- regardless of whether we are solving an ordinary differential equation (ode) where the independent variable is t (time) or x (distance), Laplace methods only apply to positive values of the independent variable; i.e. $t \geq 0$ or $x \geq 0$. Therefore if you are ever plotting a solution to an ode solved using Laplace, $t < 0$ or $x < 0$ should not be included (even if the formal 'solution' (e.g. $x(t) = \sin t$) makes sense for $t < 0$).

- given that ye had the time, it would have been a good idea to check your answers. This was a lesson that was not learnt from the first assignment. This would have saved a lot of us lost marks. You can check your answers by seeing if

  A. they satisfy the differential equation[1],

  B. they satisfy the initial conditions.

  You will probably not have the time to check answers in the final exam.

- We need to be careful when (linearly) taking the Laplace transform of a constant times a derivative. For example, suppose you are taking the Laplace transform of $-6f''(t)$, where $f(0) = -1$ and $f'(0) = 2$, here is good practise

$$\begin{aligned} \mathcal{L}\left\{-6f''(t)\right\} &= -6\mathcal{L}\{f''(t)\} \\ &= -6(s^2F(s) - sf(0) - f'(0)) \\ &= -6(s^2F(s) - s(-1) - 2) \\ &= -6(s^2F(s) + s - 2) \\ &= -6s^2F(s) - 6s + 12. \end{aligned}$$

  Overkill? Well how about:

$$\begin{aligned} \mathcal{L}\left\{-6f''(t)\right\} &= -6(s^2F(s) - sf(0) - f'(0)) \\ &= -6(s^2F(s) - s(-1) - 2) \\ &= -6s^2F(s) - 6s + 12. \end{aligned}$$

  However this is what some of us are doing:

$$\begin{aligned} \mathcal{L}\left\{-6f''(t)\right\} &= -6 \cdot s^2F(s) - sf(0) - f'(0) \\ &= -6 \cdot s^2F(s) - s(-1) - 2 \\ &= -6s^2F(s) - 6s + 12. \end{aligned}$$

[1] Cf. Q.1(c) (ii)

This is very dangerous (without the bracket) and you are very likely (and some of us have) done the following:

$$\begin{aligned}\mathcal{L}\left\{-6f''(t)\right\} &= -6\cdot s^2F(s) - sf(0) - f'(0)\\ &= -6\cdot s^2F(s) - s(-1) - 2\\ &= -6s^2F(s) + s - 2,\end{aligned}$$

which is wrong... you need the bracket: technically without it, you are wrong:

$$a\cdot b + c = ab + c \neq ab + ac = a(b+c),$$

although if you end up with the correct answer I leave you off... but many of ye are not ending up with the correct answer so I would like to see the bracket to avoid this.

Question-Specific Remarks

1. (a) ii. <u>Show that we have $|x(t)| \leq 1$ for all t.</u>

Remark: A number of us made the same serious logical error here that ye made in Assignment 1... We need to prove the result for *ALL* t. Showing the result for a few cases, say $t = 0, 1, 2$, isn't sufficient to prove the result for all possible times, t.

For example, suppose you were asked to prove the following 'theorem':

All Brazilians are good at soccer.

Pointing out a few Brazilians that are good is not sufficient to prove that *all* Brazilians are good.

The correct answer here is that

$$x(t) = \cos(\omega t),$$

for some ω; and for <u>all</u> $x\in\mathbb{R}$

$$-1 \leq \cos x \leq 1;$$

in other words cosine only takes values between -1 and 1. Therefore the absolute value of cosine is always between 0 and 1 and so $|x(t)| \leq 1$ for *all* t.

iii. <u>Explain this result $|x(t)| \leq 1$.</u>

What I needed was something like:

The mass begins from rest at a distance of one from equilibrium. $|x(t)| \leq 1$ means that the mass is never more than a distance of one from equilibrium and in fact oscillates between $x = -1$ and $x = 1$.

Note with no damping there is no energy loss and the oscillations continue forever.

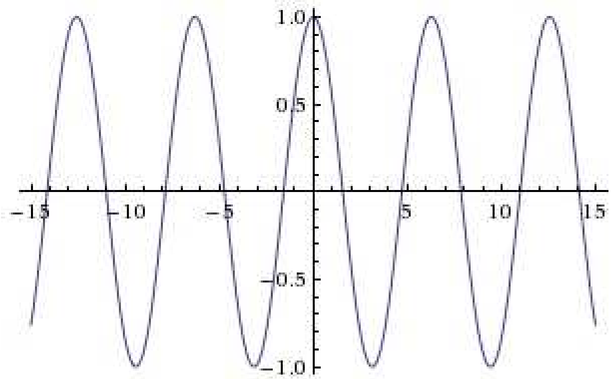

Figure 1: Cosine only takes on values between −1 and 1

2. (a) Watch the video and <u>write down:</u>
   i. <u>for what range of distances do we have underdamping</u>
   ii. <u>for what distance do we have critical damping</u>
   iii. <u>for what range of distances do we have overdamping?</u>

   Those of us who answered:
   i. 20 to 2.5 mm
   ii. 2.5 to 1.5 mm
   iii. 1.5 to 0 mm

   or even
   i. 20 to 2.5 mm
   ii. 2 mm
   iii. 1.5 to 0 mm,

   did not get the full mark. You need to understand that CRITICAL damping only occurs at one distance and everything else is either overdamping or underdamping (no matter how insignificant); i.e. at 2.25 mm, although not shown in the video, is underdamped.

   (c) i. The biggest issue here was student's getting to, say,

$$Y(s) = \frac{\frac{4}{s} + 2s + 2}{(s+4)^2}.$$

   People from here went on to try and find the Rule II partial fraction expansion:

$$Y(s) \overset{!}{=} \frac{A}{s+4} + \frac{B}{(s+4)^2}.$$

   You see on p.104 that to find a partial fraction expansion, that $Y(s)$ must be a *rational* function. This means that

$$Y(s) = \frac{p(s)}{q(s)},$$

   with p and q *polynomials*, sums of positive powers of s. The top of $Y(s)$, $\frac{4}{s} + 2s + 2$ is *not* a polynomial because it is equal to

$$4s^{-1} + 2s + 2,$$

and the s^{-1} is not a positive power of s. There are various ways to avoid this, the easiest being to take the $Y(s)$ above and to multiply above and below by s:

$$Y(s) = \frac{\frac{4}{s} + 2s + 2}{(s+4)^2} \cdot \frac{s}{s} = \frac{4 + 2s^2 + 2}{s \cdot (s+4)^2},$$

showing that Y actually requires a Rule I and a Rule II.

iii. What type of damping will the bridge undergo if the wind suddenly stops? If the wind stops then you are left with:

$$\frac{d^2y}{dt^2} + b \cdot \frac{dy}{dt} + c \cdot y(t) = 0,$$

i.e. a damped harmonic oscillator. If we do the $b^2 - 4ac$ analysis:

- $b^2 - 4ac < 0 \Rightarrow$ underdamped
- $b^2 - 4ac = 0 \Rightarrow$ critically damped
- $b^2 - 4ac > 0 \Rightarrow$ overdamped,

we can see the damping is critical.

You can also see this by looking at the solution:

$$y(t) = Be^{-at} + Cte^{-at} + A.$$

There are no oscillations — with no sine nor cosine there are no oscillations — and we don't have a two speed convergence (two exponentials). Therefore it is critically damped.