1. (a) i. Simplify the following expression as much as possible

$$\frac{x^2 - 16}{x - 4}.$$

ii. Solve for y:

$$\frac{3 + 5y}{y} = 2y.$$

iii. Write

$$\left(\frac{a^7 a^3}{a^2}\right)^3$$

in the form a^p, where p is a rational number.

[11 Marks]

(b) Solve for x:

i. $4^{1-2x} = 3^{4x+1}$.

ii. $\log_5(x + 7) + \log_5(x - 3) = 2\ \log_5(x)$

[10 Marks]

(c) Suppose you have an alphabet of size 26 and you want at least 2,000,000 distinct passwords. What should the minimum length restriction on your passwords?

[4 Marks]

2. (a) Simplify $A \cup \overline{(A \cap \overline{B})}$ using only *laws of sets*. Identify the laws used in each step of your solution.

[8 Marks]

(b) Let $A = \{1, 2, 3, 4, 5\}$. Define a relation R on A by the following:

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4), (5,5)\}.$$

You may assume that R is a *transitive* relation.

i. Graphically represent the relation R using a *digraph*.

ii. Hence, or otherwise, determine if R is:

A. reflexive. Justify your answer.

B. symmetric. Justify your answer.

iii. Is R an equivalence relation? If you answer yes, write down the equivalence classes of R. If you answer no, justify your answer.

[8 Marks]

(c) Consider $\{0, 1\}^6$ the bit strings of length six and let $U = \{a, b, c, d, e, f\}$.

i. Let $X \subset U$ be given by $X = \{a, b, c, f\}$. What is the bit string representation of X?

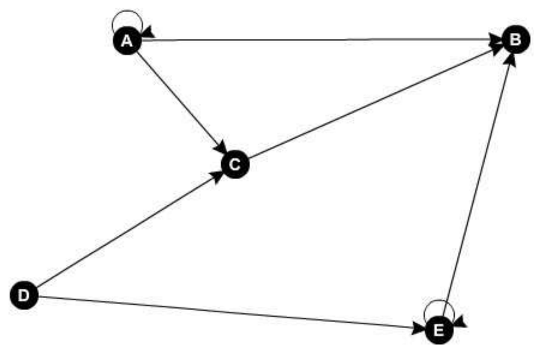
ii. How many bit strings of length six are there?

iii. Hence, or otherwise, write down $|\mathcal{P}(U)|$.

iv. How many subsets of size two does U have?

[9 Marks]

3. (a) Consider the following digraph:



i. Write down the set of vertices V and the set of edges E.

ii. The set E is a relation on V. Is this relation reflexive? Justify your answer.

[5 Marks]

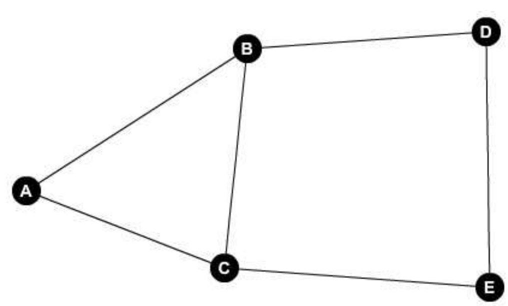
(b) Consider an undirected graph G.

i. What is a Hamiltonian Cycle on G?

ii. State Dirac's Theorem.

iii. Is it possible for a graph to have a Hamiltonian Cycle without satisfying the conditions of Dirac's Theorem?

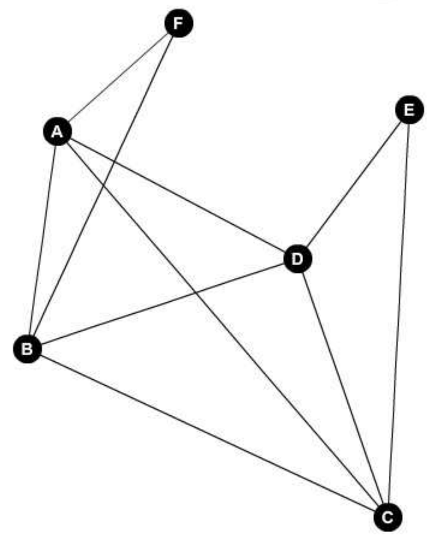
iv. Find a Hamiltonian Cycle in the graph below:



v. Does this graph satisfy the conditions of Dirac's Theorem? Justify your answer.

[10 Marks]

(c) Consider a computer network given by the following (undirected) graph:



i. Give the degree of each vertex.

ii. Is the graph connected? Give a reason for your answer.

iii. Is the graph a tree? Give a reason for your answer.

iv. Does the graph have an Euler Cycle? If yes, either justify your answer or find an Euler Cycle. If no, explain your answer.

v. Does the graph have a Hamiltonian Cycle? If yes, find one. If not, explain your answer.

[10 Marks]

4. (a) Let $X = \{0, 1, 2, 3, 4\}$ and $Y = \{8, 9, 10, \ldots, 16\}$. Define $f : X \to Y$ as $f(x) = 2x + 8$. List the ordered pairs of the relation that define this function.

[4 Marks]

(b) Show that the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 2$ in not onto.

[4 Marks]

(c) Let f be the function that maps strings of characters and blank spaces onto strings of characters by removing all blank spaces and vowels. For example, f("dog cat") = "dgct". Let g be the function that maps strings of characters onto integers such that the value of a string is simply the number of characters (including blanks) in the string.

i. What is f("Michael D Higgins")?

ii. What is g("Michael D Higgins")?

iii. What is $(g \circ f)$("Michael D Higgins")?

[6 Marks]

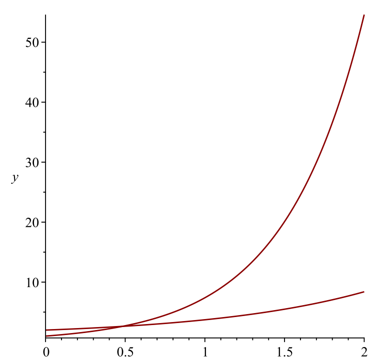
(d) Let $B = \{0, 1\}$. There are four functions $B \to B$. For each of the four functions:

i. List the ordered pairs of the relation that define the function,

ii. Represent the function using either an arrow diagram or a graph.

[8 Marks]

(e) Below see a plot of the graphs of the $\mathbb{R} \to \mathbb{R}$ functions $f(x) = 1 + e^x$ and $g(x) = e^{2x}$. Copy briefly into your answer booklet and label each curve appropriately:



[3 Marks]

Tables

Indices and Logarithms

$a^p a^q = a^{p+q}$

$\frac{a^p}{a^q} = a^{p-q}$

$(a^p)^q = a^{pq}$

$a^0 = 1$

$a^{-p} = \frac{1}{a^p}$

$a^{\frac{1}{q}} = \sqrt[q]{a}$

$a^{\frac{p}{q}} = \sqrt[q]{(a)^p} = (\sqrt[q]{a})^p$

$(ab)^p = a^p b^p$

$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

$\log_a(xy) = \log_a x + \log_a y$

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a(x^q) = q\log_a x$

$\log_a 1 = 0$

$\log_a\left(\frac{1}{x}\right) = -\log_a x$

$a^x = y \Leftrightarrow \log_a y = x$

$\log_a(a^x) = x$

$a^{\log_a x} = x$

$\log_b x = \frac{\log_a x}{\log_a b}$

Sets

Name	Equality	
Double Complement Law	$\overline{(\overline{A})} = A$	
Identity Laws	$A \cap U = A$	$A \cup \varnothing = A$
Annihilation Laws	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Inverse/Complement Laws	$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$
Idempotent Laws	$A \cup A = A$	$A \cap A = A$
Commutative Laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
DeMorgans Laws	$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$	$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
Absorption Laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Associative Laws	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributive Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$