Leaving Cert Applied Maths: Some Notes

Golden Rule of Applied Maths: Draw Pictures!

Numbered equations are in the tables.

Proportionality:

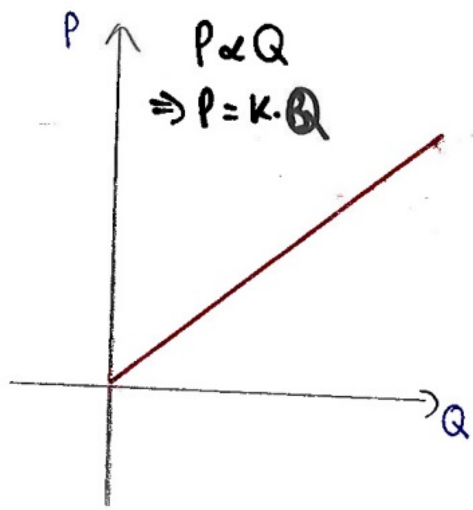

Figure 1: Note that $P = kQ \sim y = mx + c$ except $c = 0$ — the line goes through the origin — and the slope is denoted by k.

1 Constant Acceleration Motion

The usual set up is that we choose $s = 0$ when $t = 0$ so that s represents the displacement from a starting point O. If the acceleration, a, is constant between times $t = 0$ and t then, where $u = v(0)$ is the initial velocity, and $v = v(t)$ is the final velocity, then we have

$$v = u + at \tag{1}$$

$$s = ut + \frac{1}{2}at^2 \tag{2}$$

$$v^2 = u^2 + 2as \tag{3}$$

$$s = \left(\frac{u + v}{2}\right)t \tag{4}$$

We can consider displacement and velocity as functions of time, t; $s = s(t)$ and $v = v(t)$ respectively. Therefore, for example, $s(n) - s(n - 1)$ is the distance travelled in the nth second.

The area under the velocity-time curve is the distance travelled. This is a consequence of the Differential Equations fact that:

$$\frac{ds}{dt} = v(t) \Rightarrow s(t:\, t_1 \to t_2) = \int_{t_1}^{t_2} v(t)\,dt,$$

and the integral is equal to the area under the curve.

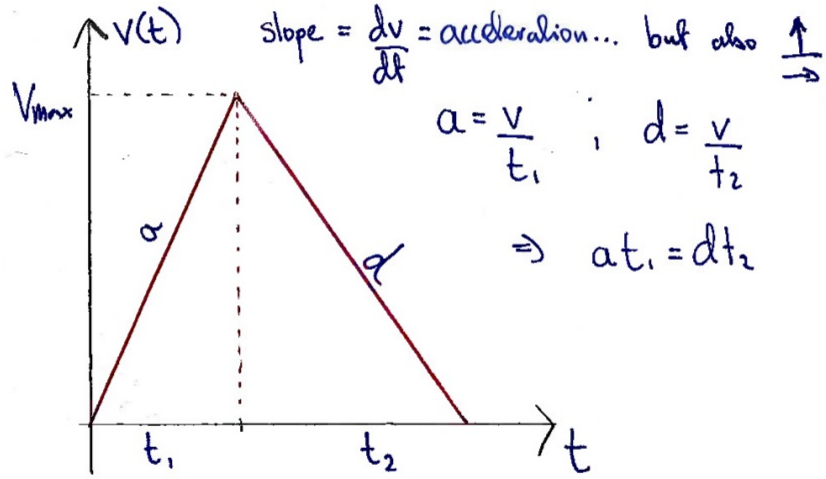

Figure 2: When there is acceleration followed immediately by deceleration, calculate v_{max} in two different ways.

What can come into this question is *power* and *work*. Work is force by distance:

$$W = F \cdot s \tag{5}$$

Power is the rate at which work is done. Rate of change is derivative:

$$P = \frac{dW}{dt}.$$

If the force F is constant we have

$$\begin{aligned} P &= \frac{d}{dt}(F \cdot s) \\ &= F \cdot \frac{ds}{dt}. \end{aligned}$$

As we will see again later, $\frac{ds}{dt}$ is the rate of change of distance, s, in other words velocity, so we have:

$$F = F \cdot v. \tag{6}$$

Another small remark is that retardation is negative acceleration. So, for example, if the acceleration is $a = -2$ m s^{-2}, the retardation is 2 m s^{-2}.

2 Relative Velocity

A big formula here is:

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B \tag{7}$$

The $\mathbf{V}_A$ is the velocity of A while $\mathbf{V}_{AB}$ is the velocity of A *relative to* B (as if B were fixed at the origin).

All of these are vector quantities so have both x and y components. The vector of magnitude one in the x-direction is $\hat{\imath}$ and the vector of magnitude one in the y-direction is $\hat{\jmath}$ so these vectors are going to be in the form

$$\mathbf{V}_i = v_x\hat{\imath} + v_y\hat{\jmath}.$$

The direction of vectors can be found via trigonometry (once a picture is drawn). The magnitude of vectors can be found using Pythagoras:

$$|a\hat{\imath} + b\hat{\jmath}| = \sqrt{a^2 + b^2}. \tag{8}$$

The other big formula is

$$\mathbf{R}_{AB} = \mathbf{R}_A - \mathbf{R}_B \tag{9}$$

where $\mathbf{R}_A$ is the *displacement* of A and $\mathbf{R}_{AB}$ is the displacement of A *relative* to B (as if B were fixed at the origin).

If you are asked to find the shortest distance between A and B in subsequent motion there are three options. You must have calculated $\mathbf{V}_{AB}$ and know $\mathbf{R}_{AB}$ (the displacement of A relative to B). Place B at the origin.

1. **Trigonometry:** To use trigonometry you will have to have $\mathbf{R}_{AB} = \pm k\hat{\imath}$ or $\pm k\hat{\jmath}$.

2. **Coordinate Geometry:** Extend the vector $\mathbf{V}_{AB}$ to a line. Find the slope using $m = \frac{\text{rise}}{\text{run}}$ and write in the form

   $$ax + by + c = 0.$$

   Recall that B is at the origin, $(0, 0)$. The shortest distance between a point (x_1, y_1) and a line $ax + by + c = 0$ is the perpendicular distance:

   $$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \underset{(x_1,y_1)=(0,0)}{=} \frac{|c|}{\sqrt{a^2 + b^2}}. \tag{10}$$

3. **Vector/Calculus:** The position of A relative to B at time t is given by the vector:

   $$\mathbf{R}_{AB}(t) = \mathbf{R}_{AB} + \mathbf{V}_{AB} \cdot t,$$

   where $\mathbf{R}_{AB}$ is the *initial* displacement of A relative to B. The distance from A to B is the magnitude of this vector:

   $$d(t) = |\mathbf{R}_{AB} + \mathbf{V}_{AB} \cdot t|.$$

This can be found in terms of t (using Pythagoras) and is of the form:

$$d(t) = \sqrt{f(t)},$$

for some function/expression in t, $f(t)$. Recall that the minimum of a function $d(t)$ is found where the derivative is equal to zero. It is possible to differentiate $\sqrt{f(t)}$ but it is easier instead to minimise the *square* of the distance:

$$d(t)^2 = f(t).$$

Find the t that does this, then substitute into $d(t)$.

If you are asked to find for how long ship A is within a distance r of B there are three options. You must have calculated $\mathbf{V}_{AB}$ and know $\mathbf{R}_{AB}$ (the displacement of A relative to B.) Place B at the origin.

1. **Trigonometry:** To use trigonometry you will have to have $\mathbf{R}_{AB} = \pm k\hat{\imath}$ or $\pm k\hat{\jmath}$. Draw a circle of radius r around B and get cracking.

2. **Coordinate Geometry:** Extend the vector $\mathbf{V}_{AB}$ to a line. Find the slope using $m = \frac{\text{rise}}{\text{run}}$ and write in the form

$$y = mx + c \tag{11}$$

Recall that B is at the origin, $(0, 0)$. Draw a circle of radius r around B. Such a circle has equation

$$x^2 + y^2 = r^2 \tag{12}$$

Solve the simultaneous equations (your picture should have the line intersecting the circle at two points)

$$\begin{aligned} y &= mx + c \\ x^2 + y^2 &= r^2 \end{aligned}$$

to find the two pairs of coordinates where A is a distance of r from B. Usually the question is concerned with for how long is A within a distance r of B. Using the distance formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{13}$$

to find this distance and use

$$v = \frac{d}{t}, \tag{14}$$

to find t.

3. **Vector/Calculus:** The position of A relative to B at time t is given by the vector:

$$\mathbf{R}_{AB}(t) = \mathbf{R}_{AB} + \mathbf{V}_{AB} \cdot t,$$

where $\mathbf{R}_{AB}$ is the *initial* displacement of A relative to B. The distance from A to B is the magnitude of this vector:

$$d(t) = |\mathbf{R}_{AB} + \mathbf{V}_{AB} \cdot t|.$$

This can be found in terms of t (using Pythagoras) and is of the form:

$$d(t) = \sqrt{f(t)}.$$

We are interested in this less than r:

$$d(t) = \sqrt{f(t)} \leq r.$$

Square both sides and you have a quadratic inequality. Draw a rough sketch of the quadratic and proceed from there to find the two times. Therefore the difference between these two times will be for how long A will have been within a distance r of B.

For two objects to be on *collision* course the direction of $\mathbf{V}_{AB} = v_x\hat{\imath} + v_y\hat{\jmath}$ (where A is going, relative to B) must be equal and opposite to $\mathbf{R}_{AB} = r_x\hat{\imath} + r_y\hat{\jmath}$ (where A *is*, relative to B). This is achieved either by:

$$\text{direction of } \mathbf{V}_{AB} \sim -\frac{v_y}{v_x} = \frac{r_y}{r_x} \sim \text{ direction of } \mathbf{R}_{AB},$$

equivalently

$$\mathbf{V}_{AB} = -k \cdot \mathbf{R}_{AB},$$

for some positive constant k.

For wind and currents, the most important equation is

$$\mathbf{V}_P = \mathbf{V}_{PW} + \mathbf{V}_W. \tag{15}$$

Here $\mathbf{V}_P$ is the *actual* velocity of the person, $\mathbf{V}_{PW}$ is the velocity of the person *relative* to the wind/water, and $\mathbf{V}_W$ is the velocity of the wind/water.

Occasionally it is necessary to solve

$$a \cdot \cos(\theta) \pm b \cdot \sin(\theta) = c.$$

There are two methods:

1. Write the equation in the form

   $$\sin(\theta \pm \phi) = k.$$

   `https://jpmccarthymaths.com/2018/01/29/writing-a-cosine-plus-sine-as-a-sine/`

2. There is an alternative. Rewrite as

   $$a \cdot \cos(\theta) = c \mp b \cdot \sin(\theta).$$

   Now square both sides:

   $$a^2 \cdot \cos^2(\theta) = c^2 \mp 2bc\sin(\theta) + b^2 \cdot \sin^2(\theta).$$

   Now this looks worse but using

   $$\sin^2(\theta) + \cos^2(\theta) = 1 \Rightarrow \cos^2(\theta) = 1 - \sin^2(\theta), \tag{16}$$

   we can write this as an equation in $\sin(\theta)$ alone:

   $$a^2 \cdot (1 - \sin^2(\theta) = c^2 \mp 2bc\sin(\theta) + b^2 \cdot \sin^2(\theta).$$

   Now let $s = \sin(\theta)$:

   $$a^2 \cdot (1 - s^2) = c^2 \mp 2bc \cdot s + b^2 \cdot s^2.$$

   This is just a quadratic in s.

3 Projectile Motion

On the flat, with the following sign conventions:

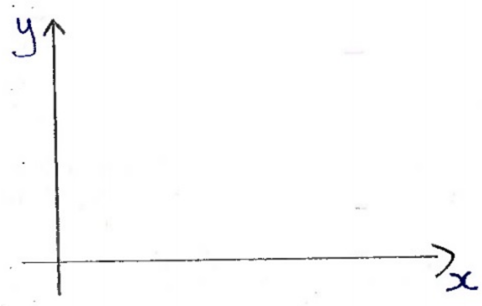

Figure 3: 'Up' is positive is the y-sense, and, in this set up, 'Right' is positive in the x-sense.

Therefore the acceleration in the x-direction is $a_x = 0$, the acceleration in the y-direction is $-g$, so using equations (1) and (2):

$$\begin{aligned} v_x &= u_x + a_x t \Rightarrow v_x = u_x \\ v_y &= u_y + a_y t \Rightarrow v_y = u_y - gt \\ s_x &= u_x t + \frac{1}{2}a_x t^2 \Rightarrow s_x = u_x t \\ s_y &= u_y t + \frac{1}{2}a_y t^2 \Rightarrow s_y = u_y t - \frac{1}{2}gt^2 \end{aligned}$$

Range is s_x when $s_y = 0$.

Maximum height is s_y when $v_y = 0$.

You may have to use

$$\frac{1}{\cos(\theta)} = \sec(\theta), \tag{17}$$

together with

$$\sin^2(\theta) + \cos^2(\theta) = 1, \tag{18}$$

and

$$\sec^2(\theta) = 1 + \tan^2(\theta). \tag{19}$$

Indeed we have to be aware that there are various trigonometric identities in the tables that we may have to use.

Note that we can end up with quadratics in $\tan(\theta)$:

$$a \cdot \tan^2(\theta) + b \cdot \tan(\theta) + c = 0.$$

Just let $t = \tan(\theta)$ and solve the quadratic:

$$a \cdot t^2 + b \cdot t + c = 0,$$

find the two roots t_1 and t_2 and find the corresponding angles using

$$\tan(\theta_i) = t_i \Rightarrow \theta_i = \tan^{-1}(t_i).$$

Note that s_x and s_y can be viewed as functions $s_x = s_x(t)$ and $s_y = s_y(t)$.

For projectiles up/down an inclined plane let the inclined plane be the s_x- and s_y-axes. Using trigonometry we can find both a_x and a_y.

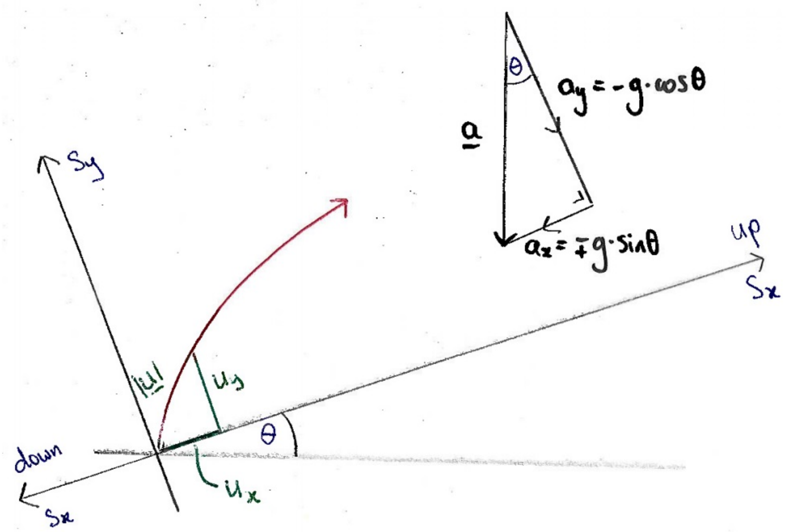

Figure 4: The velocity vector $\mathbf{u}$ makes an angle with the inclined plane.

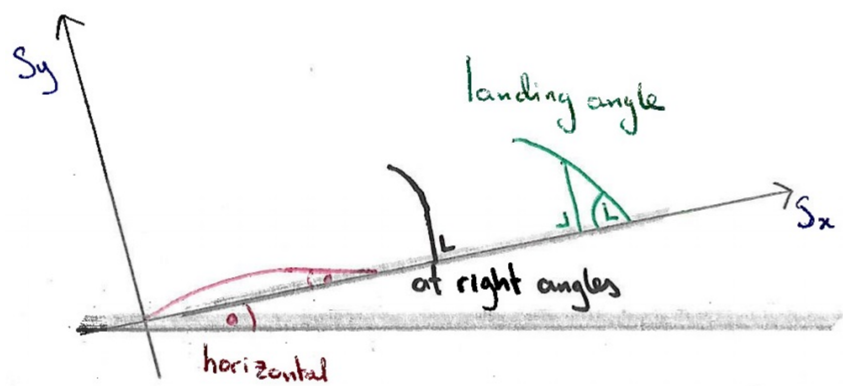

Figure 5: The landing angle L is given by $\tan(L) = -\frac{v_y}{v_x}$. If the projectile lands at right angles then $v_y = 0$. If the projectile hits the inclined plane while travelling horizontally, via a 'Z', the landing angle and angle of incline are equal.

If you want to find the maximum range up/down the inclined plane write

$$R = k \cdot \sin(\phi)$$

and note that $\sin(\phi)$ has a maximum value of one and this occurs when $\phi = 90°$.

4 Newton's Laws

The force of gravity is called *weight* and is given by:

$$W = mg. \tag{20}$$

The force of friction is proportional to the *normal force*[1] R:

$$\text{Friction} = \mu R.$$

Newton's Second Law states that

$$\sum_i F = ma_i, \tag{21}$$

where i is a given direction. If there is no movement in a particular direction i:

$$v_i = 0 \Rightarrow \Delta(v_i) = 0 \Rightarrow a_i \Longrightarrow \sum_i F = 0.$$

A *lift* just adds additional acceleration.

When working with a *wedge*:

1. Draw all the forces on the question paper:
   (a) weights
   (b) reactions $\rightarrow$ don't forget the normal force on a particle also acts on the wedge
   (c) frictions
2. Draw all accelerations on the question paper.
3. For each particle (including the wedge)
   (a) Look at all relevant directions
   (b) Find acceleration in that direction
   (c) Write $\sum_i F = ma_i$ for that direction.

[1]the force that occurs almost whenever an object does not move 'down through' a medium

5 Impacts & Collisions

For *conservative systems*, the *mechanical energy*,

$$T = \text{K.E.} + \text{P.E.} = \frac{1}{2}mv^2 + mgh,$$

is conserved (kept at a constant). So the initial mechanical energy:

$$T_0 = \text{K.E.}_0 + \text{P.E.}_0 = \frac{1}{2}mv_0^2 + mgh_0$$

is equal to the final mechanical energy:

$$T_1 = \text{K.E.}_1 + \text{P.E.}_1 = \frac{1}{2}mv_1^2 + mgh_1.$$

Impulse is

$$m\mathbf{v} - m\mathbf{u} = \Delta p = \text{ change in momentum.}$$

For one-dimensional, *direct*, collisions, choose a positive direction and apply:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \tag{22}$$

$$v_1 - v_2 = -e(u_1 - u_2) \tag{23}$$

For two-dimensional, *oblique*, collisions the situation looks like:

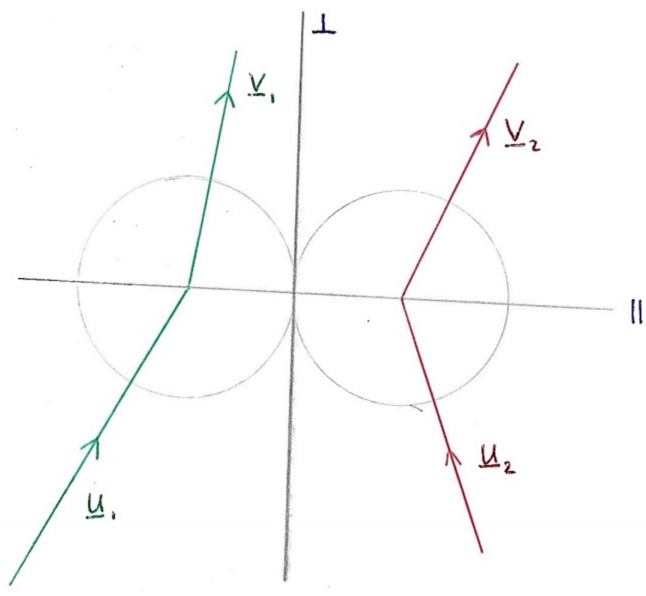

Figure 6: Choose the parallel axis, $\|$, to be the x-axis, and so associated with $\hat{\imath}$. The perpendicular axis, $\perp$, is the y-axis, associated with $\hat{\jmath}$.

Note first that there are no forces in the $\perp$-direction and so:

$$\sum_{\perp} F = 0 \Rightarrow a_\perp = 0 \Rightarrow \Delta(v_\perp) = 0,$$

so that in both cases, $v_y = u_y$ — the velocity in the y-direction does not change. Use this in conjunction with equations (22) and (23).

6 Differential Equations

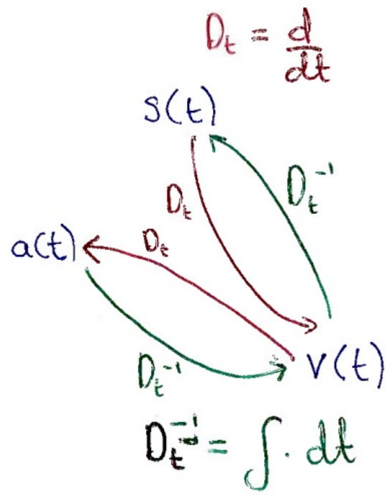

Figure 7: The... *differential*... relationship between displacement/distance, $s(t)$; velocity/speed, $v(t)$; and acceleration $a(t)$.

Therefore we have

$$\begin{aligned} v(t) &= \frac{ds}{dt} \\ \Rightarrow a(t) &= \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}. \end{aligned}$$

If we write $v = v(s(t))$, and use the Chain Rule

$$\begin{aligned} a(t) &= \frac{d}{dt}\left(v(s(t))\right) \\ &= \frac{dv}{ds} \cdot \frac{ds}{dt} \\ \Rightarrow a(t) &= v \cdot \frac{dv}{ds}. \end{aligned}$$