Gaussian Elimination Exercises

1. Write a system of linear equations corresponding to each of the following augmented matrices.

$$(i)\left[\begin{array}{ccc|c} 1 & -1 & 6 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & -1 & 0 & 1 \end{array}\right] \quad (ii)\left[\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \\ 0 & -1 & 1 & 3 \end{array}\right].$$

2. **Autumn 2013** A corporation wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15 and 20 passengers, respectively.

   (a) Identify three variables x, y and z.

   (b) Write down the corresponding *linear system* and find the system's *solution set* $\mathcal{S}$ including the real parameter $t \in \mathbb{R}$. **Ans:** $(x, y, z) = (-8 + t, 20 - 2t, t)$.

   (c) Find how many of each type of plane could be leased by finding *all positive, whole number* solutions. **Ans:** $(0, 4, 8)$, $(1, 2, 9)$ or $(2, 0, 10)$.

3. **Summer 2012** Use *Gaussian Elimination* methods to determine the *solution set* $\mathcal{S}$ of the following system of linear equations.

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 8 \\ -x & - & y & + & 2z & = & -6 \\ x & + & 3y & + & 4z & = & 10 \end{array}$$

   Ans: $(x, y, z) = (4 + 5t, 2 - 3t, t)$.

4. **Autumn 2012** Use *Gaussian Elimination* methods to solve the following system of linear equations.

$$\begin{array}{rcrcrcr} 3x & + & 4y & - & z & = & -17 \\ 2x & + & y & + & z & = & 12 \\ x & + & y & - & 2z & = & -21 \end{array}.$$

   Verify your solution by substitution.

5. Find all the solutions (if any) of each of the following systems of linear equations using augmented matrices and Gaussian elimination:

$$(i)\ \begin{array}{c} x + 2y = 1 \\ 3x + 4y = -1 \end{array} \quad (ii)\ \begin{array}{c} 3x + 4y = 1 \\ 4x + 5y = -3 \end{array} \quad (iii)\ \begin{array}{c} 3x - 2y = 5 \\ -12x + 8y = 16 \end{array}$$

$$(iv)\ \begin{array}{c} 2x + y + z = -1 \\ x + 2y + z = 0 \\ 3x - 2z = 5 \end{array} \quad (v)\ \begin{array}{c} -2x + 3y + 3z = -9 \\ 3x - 4y + z = 5 \\ -5x + 7y + 2z = -14 \end{array} \quad (vi)\ \begin{array}{c} 3x - 2y + z = -2 \\ x - y + 3z = 5 \\ -x + y + z = -1 \end{array}$$

6. **Autumn 2013** Apply **only** the *Gauss-Jordan Method* to solve the system of linear equations

$$\begin{aligned} x + y - z &= 1 \\ x + 2y - 2z &= 0 \\ -2x + y + z &= 1. \end{aligned}$$

   Ans: $(x, y, z) = (2, 2, 3)$.

7. * Consider the following statements about a system of linear equations with augmented matrix A. In each case decide if the statement is true, or give an example for which it is false:

   (a) If the constants are all zero then the only solution is the zero solution (all variables equal to zero).

   (b) If the system has a non-zero solution, then the constants are not all zero.

   (c) If the constants are all zero and there exists a solution, then there are infinitely many solutions.

   (d) If the constants are all zero and if the row-echelon form of A has a row of zeros, then there exists a non-zero solution.

8. * Suppose that $A\mathbf{x} = \mathbf{b}$ is a linear system written in augmented matrix form $[A|\mathbf{b}]$. Explain why the solution set of $[A|\mathbf{b}]$ is the same as the solution set of $E_i[A|\mathbf{b}]$, for $i = 1, 2, 3$ where $E_1 = r_i \leftrightarrow r_j$, $E_2 := r_i \to kr_i$ and $E_3 := r_i \to r_i + r_j$ are the elementary row operations.

Selected Answers & Solutions:

1 (i) Letting the first column be x, the second y and the third z:

$$\begin{aligned} x - y + 6z &= 0 \\ y &= 3 \\ 2x - y &= 1. \end{aligned}$$

(ii)

$$\begin{aligned} 2x - y &= -1 \\ -3x + 2y + z &= 0 \\ -y + z &= 3. \end{aligned}$$

5 (i)

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & -1 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -4 \end{array}\right] \xrightarrow{r_2 \times -1/2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right].$$

Hence $y = 2$ and $x = 1 - 2y = -3$.

(ii) $x = -17, y = 13$.

(iii) No solutions.

(iv) $x = 1/9$, $y = 10/9$, $z = -7/3$.

(v) $x = -21 - 15t$, $y = -17 - 11t$, $z = t$ for $t \in \mathbb{R}$.

(vi) $x = -7$, $y = -9$, $z = 1$.

7 (a) No

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has non-zero solutions. For example $x = 1$, $y = -1$.

(b) No. The above example is a counter-example.

(c) No.

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

has a unique solution $x = y = 0$.

(d) No.

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has the unique solution $x = y = 0$.