Random Walks on Finite Quantum Groups

Diaconis-Shahshahani Upper Bound Lemma for Finite Quantum Groups

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The Classical Problem

Given a finite group, G, and elements $\zeta_i \in G$ chosen according to a fixed probability distribution ($\nu \in M_p(G)$), the sequence of random variables $\{\xi_i\}_{i=1}^{k}$ given by:

$$\xi_i := \zeta_i \cdot \dots \cdot \zeta_2 \cdot \zeta_1,$$

is called a *(right-invariant) random walk on G driven by ν*.

The distribution of ξ_k is given by

$$\underbrace{\nu \star \cdots \star \nu \star \nu}_{k \text{ copies}} =: \nu^{\star k},$$

where

$$\mu \star \nu(s) = \sum_{t \in G} \mu(st^{-1})\nu(t).$$

Denote by $\pi \in M_p(G)$ the uniform or *random* distribution. The distance to random is measured by the *total variation distance*:

$$\|\nu^{\star k} - \pi\|_{\mathrm{TV}} = \sup_{S \subset G} |\nu^{\star k}(S) - \pi(S)| = \frac{1}{2}\|\nu^{\star k} - \pi\|_{\ell^1}.$$

Diaconis-Shahshahani Theory

Every group representation $\rho : G \to \mathrm{GL}(V)$ splits into a direct sum of irreducible representations where $\rho^\alpha : G \to \mathrm{GL}(V_\alpha)$ with $\dim(V_\alpha) =: d_\alpha$.

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Definition: (used by Diaconis) The *Fourier Transform of* $\nu \in M_p(G)$ is a linear map:

$$\widehat{\nu} \in \bigoplus_{\nu \in \mathrm{Irr}(G)} L(V_\alpha);$$

where the *Fourier Transform of* ν at *the representation* ρ^α is:

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Upper Bound Lemma: Where τ is the *trivial representation*,

$$\|\nu^{\star k} - \pi\|_{\mathrm{TV}}^2 \leq \frac{1}{4} \sum_{\alpha \in \mathrm{Irr}(G) \setminus \{\tau\}} d_\alpha \mathrm{Tr}\left[\left(\widehat{\nu}(\alpha)^*\right)^k \widehat{\nu}(\alpha)^k \right].$$

(Diaconis & Shahshahani (1981))

Examples

- Simple Random Walk on Circle, $\mathbb{Z}_n$ — step left/right with equal probability; close to random in $\mathcal{O}(n^2)$ steps.
- Random Walk on the Hypercube, $\mathbb{Z}_2^n$ — stick or move to one of the nearest neighbours with equal probability, $\frac{1}{n+1}$; close to random in $\mathcal{O}\left(\frac{1}{4}n\ln n\right)$ steps.
- Random Transposition Shuffle of S_n — swap two cards chosen at random; close to random in $\mathcal{O}\left(\frac{1}{2}n\ln n\right)$ steps.

Diaconis (1988)

Quantum Diaconis-Shahshahani Theory?

Consider again the Upper Bound Lemma:

$$\|\nu^{\star k}-\pi\|^2 \leq \frac{1}{4}\sum_{\alpha\in \mathrm{Irr}(G)\backslash\{\tau\}} d_\alpha \mathrm{Tr}\left[\left(\widehat{\nu}\left(\alpha\right)^*\right)^k \widehat{\nu}\left(\alpha\right)^k\right].$$

Note there is no reference to points in the space G. While it appears that $\widehat{\nu}(\alpha)$ is defined with respect to points, it is actually a sum over points, a rôle played by the Haar state h:

$$\underbrace{\frac{1}{|G|}\sum_{t\in G} f(t)}_{\text{classical: references points } t\in G} = \underbrace{h(f)}_{\text{quantum: no reference to points}}.$$

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...and quantum groups have (co)representations. (This work uses the $\kappa : V \to V\otimes F(\mathbb{G})$, $e_j \mapsto \sum_i e_i \otimes \rho_{ij}$ formulation).

Random Walks on Finite Quantum Groups

Definition: The algebra of functions on a (finite) quantum group $\mathbb{G}$, is a finite-dimensional C^*-Hopf algebra $A =: F(\mathbb{G})$.

Early work on quantum stochastic processes by various authors led to random walks on duals of compact groups (particularly Biane) and other examples, but Franz & Gohm (2005) define with clarity a *random walk on a finite quantum group*.

Definition: The sequence of random variables $\{j_i\}_{i=0}^{k}$

$$j_k = \Delta^{(k)} : F(\mathbb{G}) \to \bigotimes_{k+1 \text{ copies}} F(\mathbb{G}),$$

with distributions given by

$$\Psi_k \circ j_k = (\nu^{\otimes k} \otimes \varepsilon) \circ \Delta^{(k)} \cong \nu^{\star k}, \qquad (\nu \in M_p(\mathbb{G})),$$

is called a *right-invariant random walk on $\mathbb{G}$ driven by ν.*

Distance to Random

Definition: The random distribution on a finite quantum group, $\mathbb{G}$, is given by the Haar state:

$$\pi := \int_{\mathbb{G}} := h : F(\mathbb{G}) \to \mathbb{C}.$$

Therefore the distance to random might be defined with respect to some norm, $\|\cdot\|$, on the quantum group ring $\mathbb{C}\mathbb{G}$:

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The very important question:

> *What are necessary and sufficient conditions on $\nu \in M_p(\mathbb{G})$ to ensure that $\nu^{\star k} \to \pi$?*

is open. In the classical case ν must not be concentrated on a subgroup (irreducibility) nor a coset of a normal subgroup (aperiodicity).

Distance to Random

For $\|\nu^{\star k} - \pi\|$ to be called a quantum total variation distance, such a norm must have three properties:

- Agreement in the classical case:

$$\|\nu^{\star k} - \pi\| = \|\nu^{\star k} - \pi\|_{\mathsf{TV}},$$

- A Cauchy–Schwarz-type inequality (UBL is for a $\|\cdot\|_2$):

$$\|\nu^{\star k} - \pi\| \leq c\|\nu^{\star k} - \pi\|_2,$$

- For lower bounds, a presentation as a supremum:

$$\|\nu^{\star k} - \pi\| = \sup_{s\in S} F(s, \nu^{\star k} - \pi).$$

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The bijective map $\mathcal{F} : F(\mathbb{G}) \to \mathbb{C}\mathbb{G}$, defined by

$$\mathcal{F}(a)b = \int_{\mathbb{G}} ba,$$

allows the norm of elements of $\mathbb{C}\mathbb{G}$ to be calculated back in $F(\mathbb{G})$, and indeed the norm

$$\|\nu^{\star k} - \pi\| := \frac{1}{2}\|\mathcal{F}^{-1}(\nu^{\star k} - \pi)\|_1^{F(\mathbb{G})},$$

has the three properties required in order for $\|\nu^{\star k} - \pi\|$ to be defined as *a* quantum total variation distance.

Distance to Random

Agreement in the classical case follows from $\mathcal{F}^{-1}(\delta^s) = |G|\delta_s$ and $(\nu^{\star k} - \pi)(\delta_s) \in \mathbb{R}$ for $\nu \in M_p(G)$.

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Using the map $\mathcal{F} : F(\mathbb{G}) \to \mathbb{C}\mathbb{G}$ an (unnormalised) Haar integral $\widehat{h} : \mathbb{C}\mathbb{G} \to \mathbb{C}$ on $\mathbb{C}\mathbb{G}$ may be defined by:

$$\widehat{h}(\mathcal{F}(a)) = \varepsilon(a)\,; \qquad \text{e.g. } \widehat{h} = \varepsilon \circ \mathcal{F}^{-1},$$

and used to define a two-norm on $\mathbb{C}\mathbb{G}$:

$$\|\mu\|_2^{\mathbb{C}\mathbb{G}} = \sqrt{\widehat{h}(|\mu|^2)}.$$

Distance to Random

Consider the two-norm on $F(\mathbb{G})$:

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and using a standard non-commutative $\mathcal{L}^p$-space Cauchy–Schwarz Inequality:

$$\begin{aligned} \|\nu^{\star k} - \pi\| &\leq \frac{1}{2}\|\mathcal{F}^{-1}(\nu^{\star k} - \pi)\|_2^{F(\mathbb{G})} \\ &= \frac{1}{2}\|\nu^{\star k} - \pi\|_2^{\mathbb{C}\mathbb{G}}. \end{aligned}$$

The Upper Bound Lemma expresses the square of $\|\nu^{\star k} - \pi\|_2^{\mathbb{C}\mathbb{G}}$ as a sum over non-trivial representations.

Distance to Random

There is also a supremum-presentation for lower bounds:

$$\|\nu^{\star k} - \pi\| = \frac{1}{2} \sup_{\phi \in F(\mathbb{G}) : \|\phi\|_\infty^{F(\mathbb{G})} \leq 1} |\nu^{\star k}(\phi) - \pi(\phi)|.$$

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The guise of the classical definition favoured by probabilists

$$\|\nu^{\star k}-\pi\|_{\mathrm{TV}}=\sup_{S\subset G}|\nu^{\star k}(S)-\pi(S)|,$$

may be adapted to the quantum case. Define a subset $\mathbb{S}\subset\mathbb{G}$ by a projection $\mathbb{1}_{\mathbb{S}}$; and $\mathbb{1}_{\mathbb{G}}=1_{F(\mathbb{G})}$. Then $\phi=2\mathbb{1}_{\mathbb{S}}-\mathbb{1}_{\mathbb{G}}$ is a suitable 'test function' and

$$\|\nu^{\star k}-\pi\|\geq|\nu^{\star k}(\mathbb{1}_{\mathbb{S}})-\pi(\mathbb{1}_{\mathbb{S}})|.$$

Diaconis-Van Daele Theory

Definition: (Simeng Wang (2014)) The *Fourier Transform of* $\nu \in M_p(\mathbb{G})$ is a linear map:

$$\widehat{\nu} \in \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} L(V_\alpha),$$

where the *Fourier Transform of* ν at *the representation* κ_α is given by:

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The map $\mathcal{F} : F(\mathbb{G}) \to \mathbb{C}\mathbb{G}$ also allows a *Fourier Transform of* $a \in F(\mathbb{G})$ *at the representation* κ_α to be defined:

$$\widehat{a}(\alpha) := \widehat{\mathcal{F}(a)}(\alpha).$$

Diaconis-Van Daele Theory

Using the results of Van Daele (concerning the map $\mathcal{F}$), it can be shown that the properties of the classical Fourier Transform $\widehat{\nu}(\rho^\alpha)$, that are used to prove the classical Upper Bound Lemma, are *also* shared by the quantum $\widehat{\nu}(\kappa_\alpha)$.

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For example, the sum over irreducible representations comes from the classical

$$\nu(\delta_e) = \frac{1}{|G|} \sum_{\alpha \in \mathrm{Irr}(G)} d_{\alpha} \operatorname{Tr}\left[\widehat{\nu}(\alpha)\right];$$

which has a generalisation to finite quantum groups:

$$\widehat{h}(\nu) = \sum_{\alpha \in \mathrm{Irr}(G)} d_{\alpha} \operatorname{Tr}\left[\widehat{\nu}(\alpha)\right].$$

Upper Bound Lemma

Leaning heavily on the Kac condition and the traciality of $\int_{\mathbb{G}}$, the Upper Bound Lemma for Finite Quantum Groups follows in a similar manner to that of the classical result of Diaconis and Shahshahani.

In the notation that is used, the classical Upper Bound Lemma:

$$\|\nu^{\star k}-\pi\|^2 \leq \frac{1}{4}\sum_{\alpha\in \mathrm{Irr}(G)\backslash\{\tau\}} d_\alpha \operatorname{Tr}\left[\left(\widehat{\nu}\left(\alpha\right)^*\right)^k \widehat{\nu}\left(\alpha\right)^k\right],$$

and the quantum Upper Bound Lemma:

$$\|\nu^{\star k}-\pi\|^2 \leq \frac{1}{4}\sum_{\alpha\in \mathrm{Irr}(\mathbb{G})\backslash\{\tau\}} d_\alpha \operatorname{Tr}\left[\left(\widehat{\nu}\left(\alpha\right)^*\right)^k \widehat{\nu}\left(\alpha\right)^k\right].$$

are essentially the same thing.

A Walk on Sekine Quantum Group

The Sekine Quantum Group (1996) $\mathbb{KP}_n$ of order $2n^2$ may be realised on

$$F(\mathbb{KP}_n) = \left(\bigoplus_{i,j \in \mathbb{Z}_n} \mathbb{C} e_{(i,j)} \right) \oplus \mathrm{M}_n(\mathbb{C}),$$

with a suitably defined comultiplication. The quantum group $\mathbb{KP}_2$ is commonly mistaken for the Kac-Paljutkin quantum group of order 8. In fact $\mathbb{KP}_2 \cong \widehat{D_4}$.

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Restrict to n odd and consider the random walk driven by

$$\nu = \frac{1}{4}(e^{(0,1)} + e^{(1,0)} + E^{11} + E^{12} + E^{21} + E^{22}) \in M_p(\mathbb{KP}_n).$$

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Franz & Skalski (2009) contains formulae which can be used to understand $\nu^{\star k}$.

A Walk on the Sekine Quantum Groups

These formulae show that

$$\nu^{\star k}(E_{ij}) = 0 \qquad \text{if } |i-j| \neq 0, 1;$$

i.e. off the subdiagonal, the superdiagonal and the diagonal.

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Regarding the open question mentioned earlier, how can one tell if $\nu^{\star k} \to \pi$? In theory, the Upper Bound Lemma can be used in specific cases to answer this very question and indeed in this case

$$\|\nu^{\star k} - \pi\|^2 \leq \sum_{\alpha \in \text{Irr}(\mathbb{KP}_n)\backslash\tau} d_\alpha \operatorname{Tr}\left[(\widehat{\nu}(\alpha)^*)^k \, \widehat{\nu}(\alpha)^k\right] \underset{k\to\infty}{\to} 0.$$

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One must control terms at $2n-1$ one dimensional representations and $\binom{n}{2}$ two dimensional representations. This convergence shows that, with respect to the dual basis:

$$\nu^{\star k} \to \pi = \frac{1}{2n^2} \sum_{i,j \in \mathbb{Z}_n} e^{(i,j)} + \frac{1}{2n} I_n.$$

A Walk on the Sekine Quantum Groups

Upper Bound: For $k = \frac{n^2}{80} + \alpha n^2$ with $\alpha \geq 1$ and $n \geq 7$

$$\|\nu^{\star k} - \pi\| \leq \frac{111}{100} e^{-\alpha \pi^2}.$$

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It should be possible to reduce α.

The upper bound for the walk on is dominated by $(n-1)/2$ representations in particular. The upper bound for these terms:

$$2\sum_{v=1}^{\frac{n-1}{2}}\cos^{4k-2}\left(\frac{\pi v}{n}\right)\leq 4e^{-\pi^2(2k-1)/n^2};$$

is quite sharp. The lower bound

$$\left|\frac{1}{2n}-\nu^{\star k}\left(\mathbb{1}_{\mathbb{C}E_{\frac{n+1}{2},\frac{n+1}{2}}}\right)\right|,$$

in particular might deserve further analysis.

Some Questions

- Irreducibility is harder than the classical case (where 'not concentrated' on a subgroup is enough). Can anything be said about aperiodicity in the quantum case? (U. Franz).
- Can a lower bound for the studied random walk on $\mathbb{KP}_n$, comparable with the upper bound for $\mathcal{O}(n^2)$ steps, be found? Using a matrix element as a test element, independent of n, crudely, $\|\nu^{\star k} - \pi\| \geq \frac{1}{2}\left(\frac{3}{4}\right)^k$.
- Find a random walk on the family of Sekine quantum groups with a driving probability exhibiting the *cut-off phenomenon*. Perhaps with $\nu_n \in M_p(\mathbb{KP}_n)$ exhibiting n-dependence.
- Extend the Upper Bound Lemma to compact quantum groups (with the Kac Property) driven by states $\nu \in M_p(\mathbb{G})$ of the form $\mathcal{F}(a)$. In the classical case, conjugate-invariant driving probabilities lead to a somewhat-tractable upper bound.
- Find convergence rates for quantum analogues of classical random walks; e.g. the random transposition shuffle.

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