Here $F(s-a)$ is the Laplace Transform of $f(t)$ evaluated at the shifted $s-a$ rather than the usual s. Looking in the opposite direction (s-domain back to t-domain):

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\} = e^{at}f(t). \tag{3.15}$$

In this example we have $\dfrac{2}{(s+2)^5}$ looking the like the Laplace transform

except shifted to

$$2F(s-(-2)) = 2\frac{1}{(s-(-2))^5}. \tag{3.16}$$

Now what is the inverse Laplace transform of $\dfrac{1}{s^5}$. Well looking at the table of Laplace transforms we see

$$\mathcal{L}\{t^N\} = \frac{N!}{s^{N+1}}. \tag{3.17}$$

It follows that to go backwards (s-domain $\to$ t-domain; i.e. the inverse Laplace transform) you have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^N}\right\} = \frac{t^{N-1}}{(N-1)!}. \tag{3.18}$$

My recommendation is that you write this on your set of Laplace transform tables as soon as you come into the exam.

Anyway, the inverse Laplace transform of $\dfrac{1}{s^5}$ is therefore

which implies that

4. $\dfrac{107+45s+s^2-7s^3}{\underset{\mathrm{I}}{(s-2)}\underset{\mathrm{I}}{(s+3)}\underset{\mathrm{III}}{(s^2+25)}}$.

Solution: There is nothing like this in the tables so we have to find the partial fraction expansion. Note that $s^2+25 \geq 25 > 0 \Rightarrow$ no real roots $\Rightarrow$ no real factors. We may *not* use the cover-up method as the denominator does not factor as a product of distinct linear terms:

$$\frac{\boxed{107+45s+s^2-7s^3}}{(s-2)(s+3)(s^2+25)} \overset{!}{=} \frac{A}{s-2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+25}$$

We want to find the A, B, C and D that does this (guaranteed to be unique) so we write the partial fraction expansion as a fraction again:

$$= \frac{\boxed{A(s+3)(s^2+25)+B(s-2)(s^2+25)+(Cs+D)(s-2)(s+3)}}{(s-2)(s+3)(s^2+25)}$$

So we want

$$107 + 45s + s^2 - 7s^3 = A(s+3)(s^2+25) + B(s-2)(s^2+25) + (Cs+D)(s-2)(s+3).$$

Now these are equal for all values of s so we can look at (why these numbers?)

- Killer $s = 2$: $\Rightarrow 107 + 45(2) + (2)^2 - 7(2)^3 = A(2+3)(2^2+25) + 0 + 0$

  $\Rightarrow 145A = 145$

  $\Rightarrow \underline{A = 1}$

- Killer $s = -3$:

$\Rightarrow 107 + 45(-3) + (-3)^2 - 7(-3)^3 = 0 + B(-3-2)((-3)^2+25) + 0$

$\Rightarrow -170B = 170$

$\Rightarrow \underline{B = -1}$

- Kills C $s = 0$:

$\Rightarrow 107 + 0 + 0 + 0 = 1(0+3)(0+25) + (-1)(0-2)(0+25) + (0+D)(-2)(3)$

$\Rightarrow 107 = 75 + 50 - 6D \Rightarrow 6D = 18 \Rightarrow \underline{D = 3}$

- Simplest left $s = 1$:

$107 + 45 + 1 - 7 = 1(4)(26) + (-1)(-1)(26) + (C+3)(-1)(4)$

$\Rightarrow 146 = 104 + 26 - 4C - 12$

$\Rightarrow 4C = -28 \Rightarrow \underline{C = -7}$

So we want to find the inverse Laplace transform of

$$F(s) = \frac{1}{s-2} - \frac{1}{s+3} + \frac{3-7s}{s^2+25} \tag{3.19}$$

The first two terms are fine; they are in the tables:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Now what about $\frac{3-7s}{s^2+25}$. It kind of looks like

$$\frac{K}{s^2+K^2}$$

It also looks like

$$\frac{s}{s^2+K^2}$$

Any ideas?

$$\frac{3-7s}{s^2+5^2} = \frac{3}{s^2+5^2} - \frac{7s}{s^2+5^2} = \frac{3}{5}\,\frac{5}{s^2+5^2} - 7\cdot\frac{s}{s^2+5^2}$$

Now we can apply the inverse Laplace transform:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+5^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\}. \\ &= e^{2t} - e^{-3t} + \frac{3}{5}\sin(5t) - 7\cos(5t) \end{aligned}$$

5. $F(s) = \dfrac{(s+1)^2}{s^4}$.

Solution: This looks very difficult but we can actually do a little manipulation to make this work out easier:

Now we just apply the inverse Laplace transform recalling that $\mathcal{L}^{-1}\{1/s^N\} = t^{N-1}/(N-1)!$:

6. $F(s) = \dfrac{2s+1}{s^2+6s+13}$.

Solution: We want to do a partial fraction expansion. Factorise $s^2+6s+13$:

This is bad. However can we see that $s^2+6s+13$ looks *like* s^2+k^2? Can we say that

and maybe use the First Shift Theorem. To do so we would have to write $s^2+6s+13 = (s+a)^2+k^2$. This is called *completing the square.* We must learn a little about this first

Completing the Square

If you have a quadratic s^2+bs+c that can't be factorised because $b^2-4ac<0$, then $\boxed{\text{complete the square}}$ — that is write

$$s^2+bs+c = (s+a)^2+k^2$$