MATH7016: 20% Written Assessment Questions

Ordinary Differential Equations

1. (D) Consider an initial value problem

$$\frac{dy}{dx} = F(x, y)\,;\;\; y(x_0) = y_0.$$

Give a reason why a numerical method might have to be used to approximate values of $y(x)$ for $x > x_0$.

[5 Marks]

2. (M) Formulate the initial value problem(s) for the following:

   (a) the displacement of a body, $s(t)$, subject to a constant force F; given an initial displacement of 0 and initial speed of u.

   (b) the displacement of a spring, $x(t)$, after a time t; given an initial displacement of A and initial speed of u.

   (c) the displacement of a damped harmonic oscillator, $x(t)$, after a time t; given an initial displacement of A and initial speed of u.

   (d) the displacement, $x(t)$, of a of a free-falling body subject to a drag force proportional to either the speed or the speed-squared. The initial displacement and initial speed are both zero.

[7 Marks Each]

Taylor Series

3. (P) Calculate the first four terms of the Maclaurin Series of:

   (a) $f(x) = e^x$.

   (b) $f(x) = \sin x$.

   (c) $f(x) = \cos x$.

   (d) $f(x) = \ln(1 + x)$.

[4 Marks Each]

4. (D) Given the Taylor Series Formula for $y = y(x)$ near $x = a$:

$$y(x) = \sum_{i=0}^{\infty} \frac{y^{(i)}(a)}{i!}(x - a)^i,$$

derive the first three terms for $x = x_1 = x_0 + h$.

[5 Marks]

Euler's Method

5. (M) Consider an initial value problem

$$\frac{dy}{dx} = F(x, y)\,;\;\; y(x_0) = y_0.$$

   (a) What two things do we know about the graph $y = y(x)$?

   Euler's Method uses the tangent to the curve at $x = x_0$ to approximate the y-value at $x = x_1 = x_0 + h$.

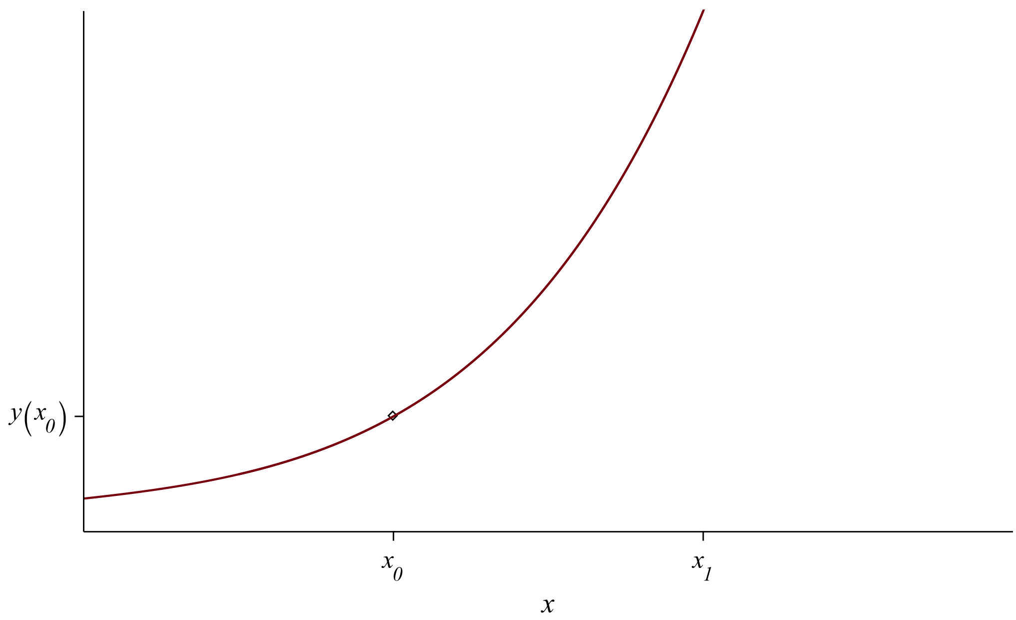
   (b) Use

$$y - y_1 = m(x - x_1) \tag{1}$$

   to find the equation of the tangent at $x = x_0$.

   (c) Find the value of y at $x = x_0 + h$ in terms of $y(x_0)$, h, and $F(x_0, y_0)$.

   (d) Draw the tangent at $(x_0, y(x_0))$, $x_1 = x_0 + h$ as well as y_1. Also show the error, $\Delta y = |y(x_1) - y_1|$.



   (e) Using the above graph, use the fact that

$$\frac{dy}{dx} = \text{slope} = \frac{\uparrow}{\rightarrow},$$

   to derive:

$$y_1 = y_0 + h \cdot F(x_0, y_0).$$

[12 Marks]

6. (M) Show, with the aid of a diagram, how a large second derivative causes problems for Euler's Method.

[3 Marks]

7. (P) Consider an initial value problem

$$\frac{dy}{dx} = F(x,y)\,;\;\; y(x_0) = y_0.$$

(P.27, Q. 1-10, use Euler's Method. P.33, Q. 1(a))

(a) Use Euler's Method with a step-size of h to approximate $y_3 = y(x_0 + 3h)$.

(b) If the exact solution is given by $y(x)$, find the error in the approximation.

[7 Marks per IVP]

8. (M) Consider an initial value problem

$$\frac{dy}{dx} = F(x,y)\,;\;\; y(x_0) = y_0.$$

Using Euler's Method with a step-size of h, it can be shown that an upper bound for the *local error*:

$$\varepsilon = \frac{\|y''\|_{\max} h^2}{2},$$

where $\|y''\|_{\max}$ is the maximum of the second derivative between $x = x_0$ and $x = x_1$.

(a) What does it mean to say that the local error is $\mathcal{O}(h^2)$?

(b) Show that if we use Euler's Method to approximate $y(x_n) = y(x_0 + n \cdot h)$, that the *global error* is $\mathcal{O}(h)$.

(c) What is the effect on the local error if we quarter the step-size?

(d) What is the effect on the global error if we half the step-size?

[7 Marks]

9. (D) Where m and c are constants, consider the initial value problem

$$\frac{dy}{dx} = m\,;\;\; y(0) = c.. \tag{2}$$

(a) Use anti-differentiation to solve the initial value problem.

(b) What does the graph of $y(x)$ look like?

(c) Use Euler's Method to approximate $y_1 = y(h)$.

(d) What is the error?

(e) Either using the exact solution $y(x)$, or the differential equation, calculate the second derivative of $y(x)$. Why does this mean that the error is zero?

[9 Marks]

Three Term Taylor Method

10. (D) Consider an initial value problem

$$\frac{dy}{dx} = F(x, y)\,; \;\; y(x_0) = y_0.$$

The Three-Term-Taylor Method uses the 'tangent-parabola' to the curve at $a = x_0$ to approximate the y-value at $x = x_1 = x_0 + h$.

(a) Use the first three terms of the Taylor Series

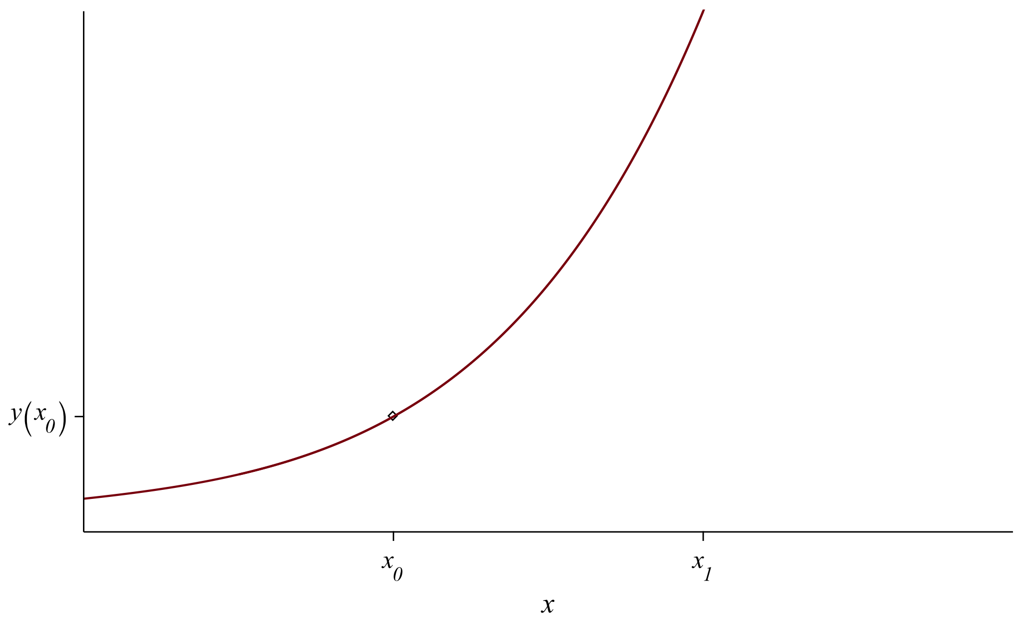
$$f(x) \approx y(a) + y'(a)(x - a) + \frac{y''(a)}{2}(x - a)^2 \tag{3}$$

to find the equation of the 'tangent-parabola' near $a = x_0$.

(b) Describe how to find the second derivative, $y''(x)$.

(c) Find the value of y at $x = x_1 = x_0 + h$ in terms of $y(x_0)$, h, $y'_0 := F(x_0, y_0)$ and $y''_0 = y''(0)$.

(d) Draw a rough diagram showing how the Three Term Taylor Method approximates $y(x_1)$.



[8 Marks]

11. (M) Consider an initial value problem

$$\frac{dy}{dx} = F(x, y)\,; \;\; y(x_0) = y_0.$$

(P.27, Q. 1-10, use the Three Term Taylor Method.)

(a) Use the Three Term Taylor Method with a step-size of h to approximate $y_3 = y(x_0+3h)$.

(b) If the exact solution is given by $y(x)$, find the error in the approximation.

[17 Marks per IVP]

12. (M) Consider an initial value problem

$$\frac{dy}{dx} = F(x, y)\,; \;\; y(x_0) = y_0.$$

Using the Three Term Taylor Method with a step-size of h, it can be shown that an upper bound for the *local error*:

$$\varepsilon = \frac{\|y'''\|_{\max} h^3}{3!},$$

where $\|y'''\|_{\max}$ is the maximum of the third derivative between $x = x_0$ and $x = x_1$.

(a) What does it mean to say that the local error is $\mathcal{O}(h^3)$?

(b) Show that if we use the Three Term Taylor Method to approximate $y(x_n) = y(x_0+n\cdot h)$, that the *global error* is $\mathcal{O}(h^2)$.

(c) What is the effect on the local error if we half the step-size?

(d) What is the effect on the global error if we quarter the step-size?

[7 Marks]

13. (D) Where $a \neq 0$, b and c are constants, consider the initial value problem

$$\frac{dy}{dx} = 2ax + b\,; \;\; y(0) = c, \tag{4}$$

(a) Use anti-differentiation to solve the initial value problem.

(b) What does the graph of $y(x)$ look like?

(c) Use the Three Term Taylor Method to approximate $y_1 = y(h)$.

(d) What is the error?

(e) Either using the exact solution $y(x)$, or the differential equation, calculate the third derivative of $y(x)$. Why does this mean that the error is zero?

[12 Marks]

Heun's Method

14. (P) Consider an initial value problem

$$\frac{dy}{dx} = F(x,y)\,;\;\; y(x_0) = y_0.$$

Heun's Method Method uses two slopes — one calculated from (x_0, y_0) and another calculated from $(x_1, y_1^0) \approx (x_1, y(x_1))$.

(a) i. How is y_1^0 found?

ii. How is y_1 found?

iii. How is y_2^0 found?

(b) Consider the formula:

$$y_{i+1} = y_i + h \cdot \frac{F(x_i, y_i) + F(x_{i+1}, y_{i+1}^0)}{2}.$$

i. What does $F(x_i, y_i)$ represent?

ii. What does $F(x_{i+1}, y_{i+1}^0)$ represent?

iii. Why is the second term divided by two?

[8 Marks]

15. (P) Consider an initial value problem

$$\frac{dy}{dx} = F(x,y)\,;\;\; y(x_0) = y_0.$$

(P.27, Q. 1-10, use Heun's Method. P.33, Q.1 (b))

(a) Use Heun's Method with a step-size of h to approximate $y_3 = y(x_0 + 3h)$.

(b) If the exact solution is given by $y(x)$, find the error in the approximation.

[14 Marks per IVP]

16. (M) Consider an initial value problem

$$\frac{dy}{dx} = F(x,y)\,;\;\; y(x_0) = y_0.$$

Using Heun's Method Method with a step-size of h, it can be shown that the local error is $\mathcal{O}(h^3)$.

(a) What does this mean?

(b) Show that if we use Heun's Method to approximate $y(x_n) = y(x_0 + n \cdot h)$, that the *global error* is $\mathcal{O}(h^2)$.

(c) Therefore the global error for Heun's Method has the same order error as the Three Term Taylor Method. Why would you prefer to use Heun's Method?

(d) What is the effect on the local error if we quarter the step-size?

(e) What is the effect on the global error if we half the step-size?

[9 Marks]

17. (D) Where $a \neq 0$, b and c are constants, consider the initial value problem

$$\frac{dy}{dx} = 2ax + b\,; \;\; y(0) = c, \tag{5}$$

(a) Use anti-differentiation to solve the initial value problem.

(b) What does the graph of $y(x)$ look like?

(c) Use Heun's Method to approximate $y_1 = y(h)$.

(d) What is the error?

(e) Either using the exact solution $y(x)$, or the differential equation, calculate the third derivative of $y(x)$. Why does this mean that the error is zero?

[13 Marks]

Second Order Differential Equations

18. (P) Write the following second order ode as two first order problems. (P. 39, Q. 1)

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \;;\;\; y(x_0) = y_0\,,\;\; y'(x_0) = y_0'.$$

[7 Marks]

19. (M) Consider an initial value problem

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \;;\;\; y(x_0) = y_0\,,\;\; y'(x_0) = y_0'.$$

Use Euler's Method with a step-size of h to approximate $y_3 = y(x_0 + 3h)$.

(a) P.39, Q. 1(a) estimate $y(0.6)$ using $h = 0.2$.

(b) P.39, Q. 1(b) let $m = 2$ and $a = 0.5$. Find $s(3)$ using $h = 1$.

(c) P.39, Q. 1(c) let $m = 1$, $\lambda = 5$, $k = 6$. Find $x(0.3)$ using $h = 0.1$.

(d) P.39, Q. 1(d) estimate $y(1.5)$ using $h = 0.5$.

[16 Marks Each]

20. (D) Consider an initial value problem

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \;;\;\; y(x_0) = y_0\,,\;\; y'(x_0) = y_0'.$$

Use Heun's Method with a step-size of h to approximate $y_2 = y(x_0 + 2h)$.

(a) P.39, Q. 1(a) estimate $y(0.5)$ using $h = 0.25$.

(b) P.39, Q. 1(b) let $m = 2$ and $a = 0.5$. Find $s(2)$ using $h = 1$.

(c) P.39, Q. 1(c) let $m = 1$, $\lambda = 5$, $k = 6$. Find $x(0.4)$ using $h = 0.2$.

(d) P.39, Q. 1(d) estimate $y(0.2)$ using $h = 0.1$.

[20 Marks Each]

Useful Formulae

A tables page will also be provided.

$$y(x) = \sum_{i=0}^{\infty} \frac{y^{(i)}(0)}{i!} x^i$$

$$y(x) = \sum_{i=0}^{\infty} \frac{y^{(i)}(a)}{i!} (x-a)^i$$

$$y_{i+1} = y_i + h \cdot F(x_i, y_i)$$

$$y_{i+1} = y_i + h \cdot y_i + \frac{h^2}{2} \cdot y_i''$$

$$y_{i+1}^0 = y_i + h \cdot F(x_i, y_i)$$
$$y_{i+1} = y_i + h \cdot \frac{F(x_i, y_i) + F(x_{i+1}, y_{i+1}^0)}{2}$$