MATH7021: Assignment 2 — Differential Equations [68 Marks]

April 2, 2015

- I want you to read the next page carefully. When you start getting down to actually doing the assignment, read the problems carefully
- You should start right away: you have only 25 days to complete the assignment. The hand in date is 17:30 Monday 27 April 2015. You can submit the assignment in any MATH7021 class otherwise drop it in A283 (I will be in the office from 16:30 to 17:30 on that Monday).
- We have all but covered everything in class that you need for this assignment. Some of the remaining material will be covered in the first week back after Easter.
- Yes you can ask me questions about the assignment before, after and sometimes during class (if ye are working on a problem). Also you can ask me questions via email.
- If you are having problems using Wolfram Alpha then you are to email me ASAP and arrange to visit me in my office.
- The assignment is worth 15%. The relevant questions on the final paper will be worth 50% of the 70% — which is 35% — on offer on the final paper. Do a good job with this assignment and you should do well questions two and three on the paper and you might just have 50% in the bag.
- MOST IMPORTANTLY — you are welcome to work together. All of you have different constants so none of you will have the same answers. Therefore you cannot copy each other. The best way to learn is by teaching.
- Work submitted after 17:30 on Monday 27 April will be assigned a mark of ZERO. Hand up whatever you have on time.

Instructions

When you open *MATH7021A2 - Student Data*, you should see a list of numbers that you are supposed to use in the questions. All of the c_i are to be taken as these constants. Everyone has different constants and this is to personalise each of your assignments and allow you to collaborate without copying.

I have also programmed some of the constants for particular questions into this file.

Undetermined Coefficients refers to the methods of Chapter Two and *Laplace* refers to the methods of Chapter Three.

Anyone who has problems with Wolfram Alpha is to contact me ASAP by email so they can be straightened out.

There should be absolutely no rounding done at any stage of this project unless you are explicitly asked to give solutions correct to a prescribed number of decimal places or significant figures.

I advise that you do the questions out roughly first because small mistakes are inevitable.

1 Background Mathematics [9 Marks]

1.1 Quadratics

Either by doing it in your head or using the method that I have been using in class, factorise if possible:

(a) $x^2 - (c_1 + c_2 + 2)x + (c_1 + 1)(c_2 + 1)$.

[1 Mark]

(b) $r^2 - (2c_3 + 2)r + (c_3 + 1)^2$.

[1 Mark]

It is not possible to factor these in your head or using the method that I did in class:

(c) $s^2 + (c_4 + 1)s - (c_4^2 + 1)$.

(d) $s^2 + (c_4 + 1)s + (4c_4^2 + 5)$.

Parts (c) and (d) are not factorisable for different reasons. Use the *Factor Theorem* and the Quadratic Formula $(-b \pm \sqrt{\cdots})$ to

(c)' factorise (c) using *surds*.

[1 Mark]

(d)' explain why (d) is not factorisable *over the real numbers*[1].

[1 Mark]

[1] it *can* be factorised using complex numbers

1.2 Exponential Decay

Consider the following functions:

- $e(t) = e^{-(c_1+1)t}$.
- $r(t) = t \cdot e^{-(c_2+1)t}$.
- $s(t) = e^{-(c_3+1)t} + e^{-(2c_3+2)t}$.
- $d(t) = e^{-(c_1+1)t}\left[\sin((c_2+1)t) + \cos((c_2+1)t)\right]$.

(a) Explain why $e(t) \to 0$ as t gets large. You can use the fact that $e \approx 2.718$.

[1 Mark]

(b) Note that as t gets big, we have that

$$e^{(c_2+1)t} \gg t.$$

Hence explain why this shows that $r(t)$ goes to zero as t gets large.

[1 Mark]

(c) Note that by part 1. both of the terms in $s(t)$ go to zero separately. Which one 'tends to zero'/decays faster?

[1 Mark]

(d) Use *Wolfram Alpha* to help you determine whether the following statement is true:

As time, t, gets large, $d(t)$ goes to zero.

[2 Marks]

HELP: To plot a function $y = f(x)$ on *Wolfram Alpha* between $x = a$ and $x = b$ use the code

```
Plot[f(x),{x,a,b}]
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To input $f(x) = e^x$ use exp(x); $\sin x$ use Sin[x]; $\cos x$ use Cos[x]

2 Abstract Example [4 Marks]

Using **only** the *Method of Undetermined Coefficients* solve the differential equation:

$$\frac{d^2r}{d\theta^2} - (7 - c_1 - c_2)\frac{dr}{d\theta} + (c_1c_2 + c_1 - 8c_2 - 8)r(\theta) = e^{-(c_2+1)\theta},$$

where $r(0) = 2$ and $r'(0) = -3$.

[4 Marks]

3 Harmonic Oscillators [21 Marks]

A *harmonic oscillator* consists of an object of mass m subject to a single *spring force*[2]. The spring force is proportional to the object's distance from an *equilibrium position* denoted by $x = 0$:

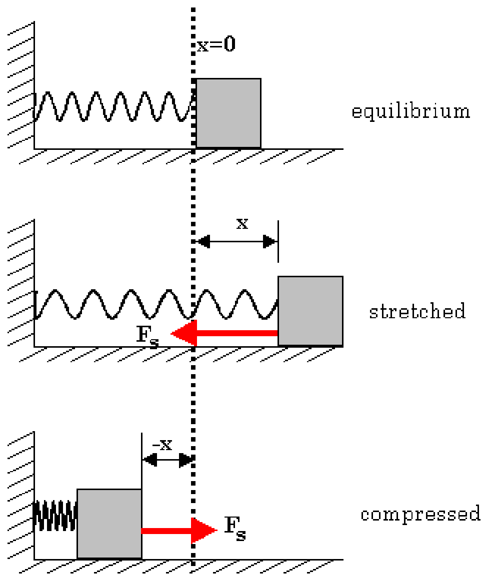

Figure 1: A spring force [credit: www.phys.ttu.edu]

Fix a time $t = 0$. Then the spring force is given by:

$$F_S(t) = -k \cdot x(t), \tag{1}$$

for a *spring constant* k.

Using Newton's Second Law:

$$F = ma, \tag{2}$$

we have

$$-k \cdot x(t) = m \cdot a(t) \Rightarrow m \cdot a(t) + k \cdot x(t) = 0.$$

[2]this force doesn't have to come from a spring — perhaps a *restorative* force might be a better term

Now we know from MATH6015 that

$$a(t) = \frac{d^2x}{dt^2},$$

so we have the *equation of motion*

$$m \cdot \frac{d^2x}{dt^2} + k \cdot x(t) = 0. \tag{3}$$

Note any system satisfying the equation of motion

$$a \cdot \frac{d^2x}{dt^2} + b \cdot x(t) = 0,$$

is termed a *harmonic oscillator* — there need not be an actual spring involved. Any function which satisfies this equation is said to undergo *simple harmonic motion*.

An example of simple harmonic motion is the bouncing motion of a mass hanging on a spring. If we write down the equation of motion (Newton's Second Law), it appears that we don't have an equation of the form (3), but here we show that actually we do.

We start the example with the mass and spring in a horizontal orientation with a frictionless surface as per Figure 1. Now we extend the spring to a length x_0 such that there is no spring force. Now the only forces in the x-direction is the spring force and if the mass is moved away from x_0 it will exhibit what is called simple harmonic motion.

(a) Using the *method of undetermined coefficients*, find the general solution of the differential equation

$$m \cdot \frac{d^2x}{dt^2} + k \cdot x(t) = 0,$$

with $m = c_1 + 1$ and $k = c_2 + 1$.

[2 Marks]

(b) Find the solution of differential equation in the following scenarios:

i. $x(0) = 0$ and $x'(0) = 0$. Explain this result.

[2 Marks]

ii. $x(0) = 1$ and $x'(0) = 0$. Show that we have $|x(t)| \leq 1$ for all t. Explain this result.

[2 Marks]

Now suppose that we put the mass and spring in the vertical position:

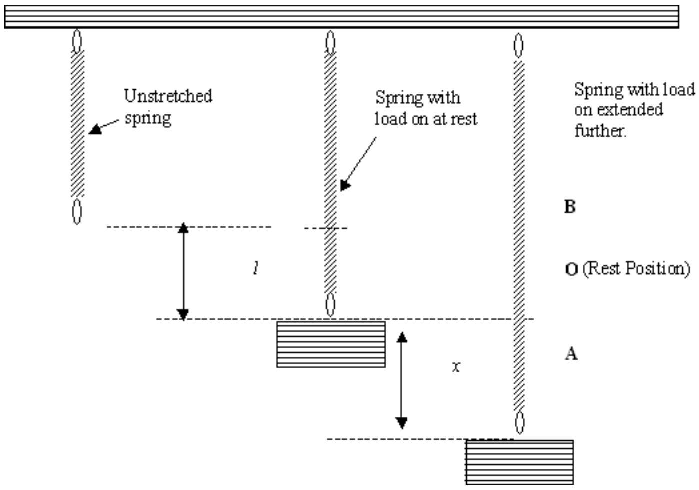

Figure 2: A system with the spring and gravity forces [credit: www.antonine-education.co.uk]

Now there is an additional force, *gravity*, so the equation of motion reads

$$ma = F$$
$$m \cdot \frac{d^2x}{dt^2} = -k \cdot x(t) + mg$$
$$m \cdot \frac{d^2x}{dt^2} + k \cdot x(t) = mg.$$

(c) Use **only** *Laplace methods* to solve this differential equation if $x(0) = 1$ and $x'(0) = 0$. Take[3] $g = 10$, $m = c_2 + 1$ and $k = c_3 + 1$.

[3 Marks]

(d) Show that where $x(t)$ is the solution of the previous problem, the function

$$\tilde{x}(t) = x(t) - \frac{mg}{k}$$

satisfies

$$m \cdot \frac{d^2\tilde{x}}{dt^2} + k \cdot \tilde{x}(t) = 0.$$

[2 Marks]

Therefore the bouncing motion of a mass hanging on a spring *is* an example of simple harmonic motion. It amounts to changing the equilibrium position of the spring from x_0 to $x_0 - mg/k$. In Figure 2, this mg/k is denoted by l.

[3] we should really take $g = 9.8$ but this is just to aid your calculations

(e) Another example of a(n approximate) harmonic oscillator is the angle, θ, that a pendulum of length ℓ with an object of mass m, makes with the horizontal:

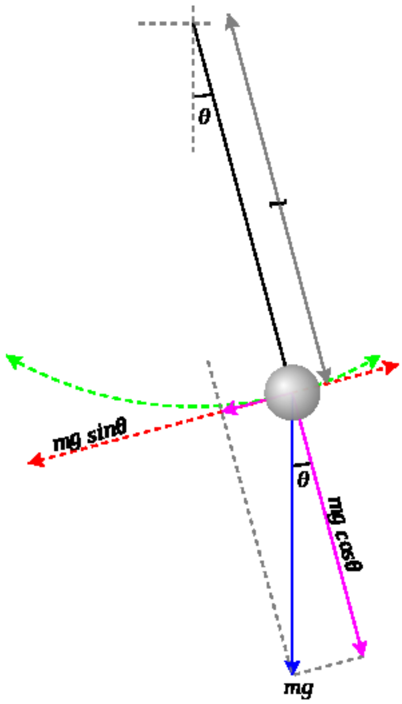

Figure 3: The forces on a pendulum... this diagram is missing the tension force but that is balanced by $mg\cos\theta$ so is irrelevent here [diagram credit: wikipedia.org]

The only forces on the object are the tension in the string and gravity $w = mg$. Working in a different coordinate system ($\hat{r}$ and $\hat{\theta}$ rather than $\hat{\imath}$ and $\hat{\jmath}$) and writing down Newton's Second Law we can show that the *angular* acceleration, $\alpha(t)$, satisfies

$$m\ell \cdot \alpha(t) = -mg\sin(\theta(t)),$$

and from here we can show that

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin(\theta(t)) = 0.$$

This is not an example of Simple Harmonic Motion as it is not of the form $ay''(x) + by(x) = 0$ for constants a and b.

However, the simple pendulum undergoes motion that looks terribly like Simple Harmonic Motion... especially if the angle is kept 'small'.

i. By finding the first two non-zero terms of the Maclaurin Series of $\sin\theta$, show for angles near $\theta = 0$ that

$$\sin\theta \approx \theta - \frac{\theta^3}{6}.$$

[2 Marks]

Recall that the Maclaurin Series for $y = f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$$

This *is* MATH7019 material rather than MATH7021 .

ii. Show that for $\theta = 5° = 0.08727$ radians we have

$$\theta - \frac{\theta^3}{6} \approx \theta,$$

in the sense that they agree to three decimal places.

[1 Mark]

ii. Hence for *small angles* approximately less than 5°, an approximate solution of $\theta(t)$ is found by replacing $\sin(\theta(t))$ with $\theta(t)$ and solving the *perturbed* differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta(t) = 0.$$

Using Laplace Methods **only**, solve the differential equation if $\ell = 10(c_3 + 1)$, $g = 10$, $\theta(0) = c_3 + 1$ and $\theta'(0) = 0$.

[3 Marks]

(f) Suppose that you have a harmonic oscillator of the form

$$a \cdot \frac{d^2y}{dt^2} + (c_1 + 1)^2 y(t) = 0.$$

'Find a value of a'/'design an oscillator' such that the oscillator undergoes simple harmonic motion with a *period* of 1 s.

[4 Marks]

You are free to use either the *method of undetermined coefficients*, *Laplace methods* or any other method you know to solve this problem.

HINT: A function of the form

$$f(t) = A\sin(\omega t) + B\cos(\omega t)$$

has a *period* of

$$T = \frac{2\pi}{\omega}. \tag{4}$$

The period is how long the function takes to make one full oscillation.

4 Damped Oscillators [26 Marks]

A *damped harmonic oscillator* consists of an object of mass m subject to two forces:

- a spring force
- a damping force

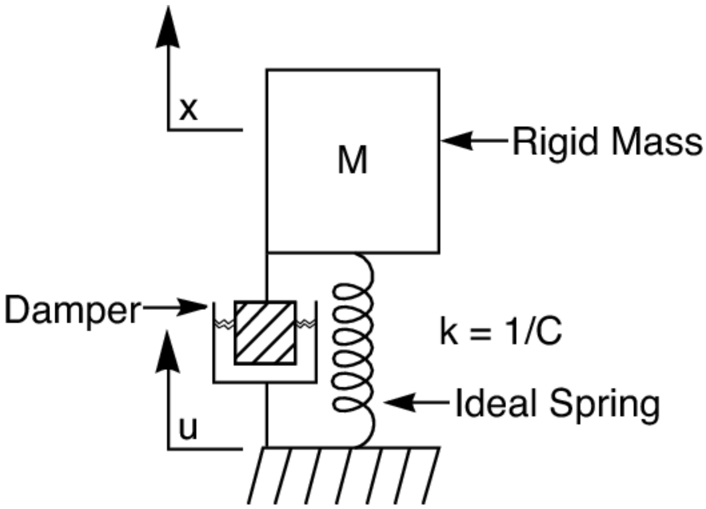

Figure 4: A damped harmonic oscillator... ignore the u and the $1/C$ [diagram credit: newport.com]

The spring force pulls the object back to an equilibrium position where $x(t)$ — and so the spring force — is zero. The damping force then in turn has the action of slowing the object down using a force proportional to the velocity. Hence we have

$$F_D(t) = -\lambda \cdot v(t), \tag{5}$$

for a *damping constant* λ.

Using Newton's Second Law:

$$F = ma, \tag{6}$$

we have

$$-k \cdot x(t) - \lambda \cdot v(t) = m \cdot a(t) \Rightarrow m \cdot a(t) + \lambda \cdot v(t) + k \cdot x(t) = 0.$$

Now we know from MATH6015 that

$$v(t) = \frac{dx}{dt},$$

so we have the *equation of motion*

$$m \cdot \frac{d^2x}{dt^2} + \lambda \cdot \frac{dx}{dt} + k \cdot x(t) = 0. \tag{7}$$

Note any object satisfying the equation of motion

$$a \cdot \frac{d^2x}{dt^2} + b \cdot \frac{dx}{dt} + c \cdot x(t) = 0,$$

is termed a *damped harmonic oscillator*.

There are three possible behaviours:

- underdamped — the damping force is not strong enough in comparison to the restoring force and the object undergoes oscillations before eventually tending to equilibrium
- overdamped — the damping is too strong in comparison to the restoring force and the object takes a long time to approach equilibrium
- critically damped — the damping and restoring forces are related in such a way that the object approaches equilibrium as fast as possible *without* oscillations.

See pages 114-120, particularly p.120 for more information. Further information in an appendix.

(a) When the object on the bottom of a pendulum is a magnet, and the motion of the pendulum is 'close' to a ferromagnetic material, the magnetic force between the magnet and material has the effect of damping the motion of the pendulum and careful analysis of the electromagnetism involved (and keeping $\theta(t) < 5°$) leads to an equation of motion of the form:

$$m \cdot \frac{d^2\theta^2}{dt^2} + b \cdot \frac{d\theta}{dt} + k \cdot \theta(t) = 0,$$

for a constant b. This damping can be made stronger by bringing the ferromagnetic material closer to the pendulum.

Watch all of https://www.youtube.com/watch?v=99ZE2RGwqSM and <u>write down:</u>

i. <u>for what range of distances do we have underdamping</u>

[1 Mark]

ii. <u>for what distance do we have critical damping</u>

[1 Mark]

iii. <u>for what range of distances do we have overdamping?</u>

[1 Mark]

(b) Consider the 'Wacoma' Suspension Bridge[4] that spans an East-West Valley. Occasionally, northerly gusts of wind blow the bridge off center. If the only force on the bridge was gravity then the displacement of the bridge[5], $y(t)$, would satisfy an equation of the form

$$a \cdot \frac{d^2y}{dt^2} + c \cdot y(t) = 0,$$

and so be an oscillator. Hence the designers fitted the bridge with dampers so that the equation of motion is that of a damped harmonic oscillator whenever there is wind:

$$a \cdot \frac{d^2y}{dt^2} + b \cdot \frac{dy}{dt} + c \cdot y(t) = 0.$$

[4] not Tacoma Bridge — that requires deeper analysis than this. You can *think* of Tacoma Bridge if you want but the oscillations occurred for different reasons

[5] this should probably be $\theta(t)$ but we have used that enough

i. If the bridge is subject to a constant northerly wind with a constant effective force, suppose that the equation of motion may be written as

$$\frac{d^2y}{dt^2} + b \cdot \frac{dy}{dt} + c \cdot y(t) = A.$$

- Solve the differential equation with $A = c_4 + 1$, $c = (c_4 + 1)^2$ and $b = 2(c_4 + 1)$, $y(0) = 2$ and $y'(0) = 0$. Use *Laplace Methods* **only**.

  [4 Marks]

- Assuming the wind stays constant, what is the behaviour of $y(t)$ for large t?

  [1 Mark]

- What type of damping will the bridge undergo if the wind suddenly stops?

  [2 Marks]

ii. Suppose the bridge is occasionally subject to very strange periodic winds. Suppose in fact that these winds may be modelled by sine waves:

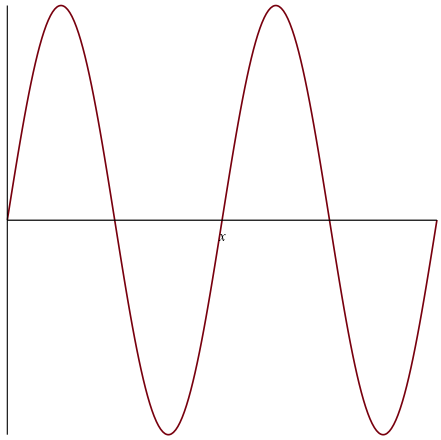

Figure 5: Initially the wind isn't blowing but begins to blow in the north direction. The wind strength increases up to a maximum and then begins to decrease to calm again before blowing in the south direction — again steadily rising to a max intensity and dying again momentarily. Once again the winds begin to blow in the northern direction and the behaviour repeats itself.

Suppose further under these conditions that the restorative and damping forces are adjusted so that the equation of motion is given by:

$$\frac{d^2y}{dt^2} + b \cdot \frac{dy}{dt} + c \cdot y(t) = \sin(\omega t).$$

- Solve the differential equation with $\omega = c_4 + 50$, $b = 2(c_2 + 1)$ and $c = (c_2 + 1)^2 + 4(c_3 + 1)^2$ if $y(0) = 0$ and $y'(0) = 0$. Use *Laplace Methods* **only**.

[4 Marks]

- What is the behaviour of $y(t)$ for large t?

[2 Marks]

- What is the period of the oscillations for large t? Does this match the period of the wind? If yes the bridge is said to oscillate with the *driving frequency* of the wind.

[2 Marks]

HINT: A function of the form

$$f(t) = A\sin(\omega t) + B\cos(\omega t)$$

has a *period* of

$$T = \frac{2\pi}{\omega}. \tag{8}$$

iii. Note in all cases the bridge survives quite easily and having not observed anything untoward the engineers declared the bridge safe. Furthermore they believed that there was no need to put on the dampers until the wind reached a particular speed. Until one day when the periodic wind blew not stronger but with a lower frequency[6]. This light wind did not cause the damper alarms to be set off and so the equation of motion looked like:

$$\frac{d^2y}{dt^2} + c \cdot y(t) = \sin((c_3 + 1)t).$$

- Solve the differential equation with $c = (c_3 + 1)^2$ if $y(0) = 0$ and $y'(0) = 0$. Use the *method of undetermined coefficients* **only**[7].

[4 Marks]

- The bridge behaves like a driven pendulum and the structural and material tests all assumed that the total angle the bridge moves between is less than 10° in total. If the angle gets big all the above analysis breaks down and the the bridge is likely to fail.

  Use *Wolfram Alpha* to plot the solution to the previous part. Sketch this into your report.

[2 Marks]

- Hence comment on the stability of the bridge in this scenario.

[2 Marks]

This phenomenon is known as *resonance* and also occurs for underdamped systems. If you don't understand this please watch
http://www.acoustics.salford.ac.uk/feschools/waves/shm3.php

[6] with different numbers above it could have been a higher frequency

[7] this is possible using Laplace but we haven't seen the techniques required to solve it yet... and we won't in MATH7021 neither.

5 A Simply Supported Beam [8 Marks]

We can also (and you will see this next year in DSE3) solve beam equations using Laplace Methods.

Consider a simply supported beam of length $L = c_1 + 1$ carrying a constant load per unit length of $w_0 = 12(c_2 + 1)$. Now we have boundary conditions:

- $M(0) = 0 = M(L)$.
- $y(0) = 0 = y(L)$.

(a) Using **only** *Laplace Methods*, solve the differential equation:

$$\frac{d^2M}{dx^2} = -w_0,$$

with $M(0) = 0$ and $M'(0) = V_A$, an unknown constant at this point. Your answer should be of the form

$$M = M(x, V_A),$$

that is M depends on x and V_A only.

[3 Marks]

(b) Now that you have $M = M(x, V_A)$, apply the boundary condition $M(L) = 0$ to find V_A and hence $M(x)$.

[1 Mark]

(c) Using your $M(x)$ from the previous part, use **only** *Laplace Methods* to solve the differential equation:

$$EI \cdot \frac{d^2y}{dx^2} = M(x),$$

with $y(0) = 0$ and $y'(0) = B$, an unknown constant at this point. Your answer should be of the form

$$y = y(x, V_A),$$

that is y depends on x and B only.

[3 Marks]

(d) Now that you have $y = y(x, B)$, apply the boundary condition $y(L) = 0$ to find B and hence $y(x)$.

[1 Mark]

Appendix: Analysing Damped Harmonic Oscillators

P.120 of your notes describes one way of analysing damped harmonic oscillators. There are others. First write the differential equation in this form:

$$\frac{d^2x}{dt^2} + b \cdot \frac{dx}{dt} + c \cdot x(t) = 0.$$

Now find the Laplace Transform of the solution. It will look like

$$X(s) = \frac{As + B}{s^2 + bs + c}.$$

Now there are three cases depending on whether $s^2 + bs + c$ has two distinct real roots, equal real roots, or complex roots. Note in all cases $a = 1$.

5.0.1 Underdamping: $b^2 - 4ac < 0$

In this case the roots are complex: no real roots implying no real factors hence we must complete the square

$$X(s) = \frac{As + B}{(s + a)^2 + k^2},$$

which is composed of shifted sines and cosines when we transform it back

$$x(t) = Ce^{-at} \cos kt + De^{-at} \sin kt.$$

5.0.2 Overdamping: $b^2 - 4ac > 0$

In this case the roots are real and distinct so we have two factors and hence a partial fraction expansion like this:

$$X(s) = \frac{As + B}{s^2 + bs + c} = \frac{As + B}{(s + \alpha)(s + \beta)} = \frac{C}{s + \alpha} + \frac{D}{s + \beta},$$

which is composed of two exponentially decaying terms when we transform back.

$$x(t) = Ce^{-\alpha t} + De^{-\beta t}.$$

5.0.3 Critical Damping $b^2 - 4ac = 0$

In this case the root are real and repeated hence we have repeated real factors and hence a partial fraction expansion like this:

$$X(s) = \frac{As + B}{s^2 + bs + c} = \frac{As + B}{(s + \alpha)^2} = \frac{C}{s + \alpha} + \frac{D}{(s + \alpha)^2},$$

which is composed of an exponentially decaying term and (before transforming) a shifted $\frac{1}{s^2}$ which will need the First Shift Theorem when we transform back:

$$x(t) = Ce^{-\alpha t} + Dte^{-\alpha t}.$$

In conclusion, if you are asked to analyse a damped harmonic oscillator of the form

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx(t) = 0,$$

then you have three options:

1. Calculate $b^2 - 4ac$. Over-zero = Over-damping, Under-zero = Under-damping and Equal Zero = Critical-damping

2. Calculate γ and ω_0 as described on page 120 of the notes and compare. It is actually equivalent to method 1.

3. Solve the differential equation using Laplace Methods and see which behaviour the solution corresponds to.