Problems with Parameters

The scenario is that we have a system of linear equations and we get down to the reduced row form, with leading 1s (pivots), and zeroes underneath pivots. At this point we have to ask are there solutions. If there is a row of the form:

$$[0 \quad 0 \quad 0 \quad 0 \mid k],$$

with $k \neq 0$ then there are *no* solutions and we are done. This is because this row corresponds to

$$\underbrace{0x_1 + 0x_2 + 0x_3 + 0x_4}_{=0} = k,$$

but we can't have k equal to zero as it is non-zero (e.g. if $k = 2$ you can't have $0 = 2$).

If you have no such row then their are solutions. Now the next thing to look at is the number of parameters and we have

$$\#\text{ parameters } = \#\text{ variables } - \#\text{ non-zero rows in reduced row form.} \tag{1}$$

Some Possibilities with Solutions

1. Often, as with many of the examples of 4×4 systems there are just as many variables as non-zero rows. This looks like:

$$\left[\begin{array}{cccc|c} 1 & \star & \star & \star & \star \\ 0 & 1 & \star & \star & \star \\ 0 & 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 & \star \end{array}\right].$$

   So

$$\begin{aligned} \#\text{ parameters } &= \#\text{ variables } - \#\text{ non-zero rows in reduced row form} \\ &= 4 - 4 = 0, \end{aligned}$$

   so there are *no* parameters and the solution is unique.

2. Consider a linear system in reduced row form:

$$\left[\begin{array}{cccc|c} 1 & \star & \star & \star & \star \\ 0 & 1 & \star & \star & \star \\ 0 & 0 & 1 & \star & \star \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

   In this case, we have

$$\begin{aligned} \#\text{ parameters } &= \#\text{ variables } - \#\text{ non-zero rows in reduced row form} \\ &= 4 - 3 \\ &= 1. \end{aligned}$$

   and so there is one parameter. There is nothing that says that x_4 has to equal anything in particular so we let $x_4 = t$, the parameter. In this scenario, there are an infinite number of solutions: one for each value of the parameter t.

3. Consider a linear system in reduced row form:

$$\left[\begin{array}{cccc|c} 1 & \star & \star & \star & \star \\ 0 & 1 & \star & \star & \star \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

In this case, once again we have

$$\begin{aligned} \#\text{ parameters } &= \#\text{ variables } - \#\text{ non-zero rows in reduced row form} \\ &= 4 - 3 \\ &= 1. \end{aligned}$$

and so there is one parameter.

This time there *is* something that says that x_4 has to equal something: the third rows says $x_4 = 3$. Now there is nothing that says that x_3 can't be a parameter so we let $x_3 = t$, the parameter and then find $x_1 \,\&\, x_2$ in terms of t.

This analysis, with only one parameter, isn't too bad. If x_N has to equal something just move onto x_{N-1} and repeat until you can let *some* variable equal the parameter. This analysis breaks down somewhat if you need two parameters.

4. Consider the linear system in reduced row form

$$\left[\begin{array}{cccc|c} 1 & \star & \star & \star & \star \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

In this case, we have

$$\begin{aligned} \#\text{ parameters } &= \#\text{ variables } - \#\text{ non-zero rows in reduced row form} \\ &= 4 - 2 \\ &= 2. \end{aligned}$$

and so there are two parameters.

Now this is what I thought the analysis broke down into:

> *There is nothing that says x_4 has to equal anything and there is nothing that says x_3 has to equal anything so I will let $x_4 = t$ and $x_3 = s$ be parameters.*

This means that $x_4 = t$ and $x_3 = s$ can be whatever they want... and each pair of values of (s, t) gives a different solution.

Unfortunately the case isn't so simple. The second row says $x_3 + x_4 = 3$ so x_3 and x_4 can't be whatever they want... you can't have $x_3 = 42$ and $x_4 = 2015$... once you have one of them nailed down... say $x_4 = 2015$, in turn then x_3 depends on it... e.g. $x_3 = -2012$. So this analysis fails.

The proper analysis breaks down like this

- There is nothing saying x_4 has to equal anything, so we let $x_4 = t$, a parameter.
- Now, once we let x_4 be a parameter, there *is* something saying that x_3 has to equal something: the second row. So we solve for x_3. We have

$$\begin{aligned} x_3 + x_4 &= 3 \\ \Rightarrow x_3 &= 3 - x_4 \\ &= 3 - t. \end{aligned}$$

- There is nothing saying that x_2 has to equal anything, so we let $x_2 = s$, a parameter.
- From here we can solve for x_1 in terms of t and s.

Note that you could also let $x_3 = t$ and then $x_2 = s$ but you can't have x_4 *AND* x_3 be parameters because they aren't free to be whatever they want because they have to add up to three! I see know that it can be done with pivots[1] but it is too late to do that now.

This is a level of analysis that I think is a bit on the tricky side and I don't want us to have to go through it. I only discovered it last week to be honest with you and I am teaching this linear systems material since 2010!

To be fair on you, I will set up your questions such that

- if there is one parameter, the *last* variable will be the parameter, $x_N = t$
- if there are two parameters, the *last two* variables will be the parameters, $x_N = t$ and $x_{N-1} = s$.

The two times when we had problems with two parameters in the notes... well I have those reproduced here:

Replacement Examples

1. **Page 10, 'Hard' Example (iii)** The augmented matrix for the linear systems has been brought into reduced row-echelon form. Find the solutions:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & -2 & 3 \\ 0 & 1 & 0 & 0 & 2 & 6 \\ 0 & 0 & 1 & 0 & 4 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Solution: We have

$$\begin{aligned} \#\text{ parameters } &= \#\text{ variables } - \#\text{ non-zero rows in reduced row form} \\ &= 5 - 3 \\ &= 2, \end{aligned}$$

so two parameters. Let $x_5 = t$ and $x_4 = s$. The third row says

$$\begin{aligned} x_3 + 4x_5 &= -5 \\ \Rightarrow x_3 &= -5 - 4x_5 \\ &= -5 - 4t. \end{aligned}$$

[1] # parameters = # variables − # pivots and variables without pivots are parameters... next year!!

The second row says

$$\begin{aligned} x_2 + 2x_5 &= 6 \\ \Rightarrow x_2 &= 6 - 2x_5 \\ &= 6 - 2t. \end{aligned}$$

The first row says

$$\begin{aligned} x_1 + 5x_4 - 2x_5 &= 3 \\ \Rightarrow x_1 &= 3 - 5x_4 + 2x_5 \\ &= 3 - 5s + 2t, \end{aligned}$$

so we have, for each pair (s, t) of numbers, a solution

$$(x_1, x_2, x_3, x_4, x_5) = (3 - 5s + 2t, 6 - 2t, -5 - 4t, s, t).$$

2. **Page 26** Consider the following pipe network

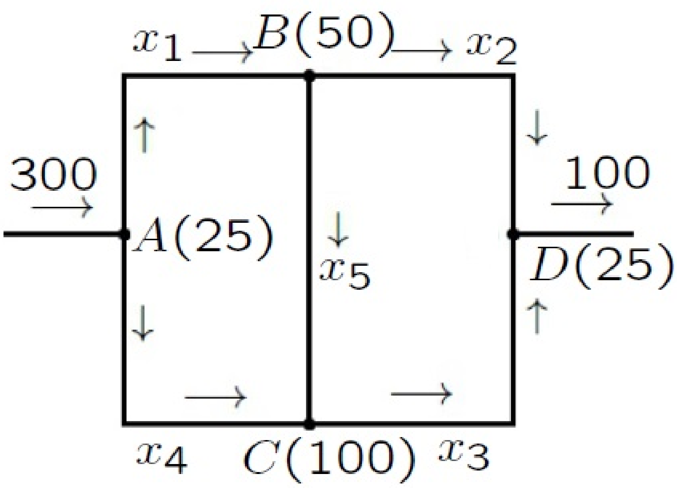

Figure 1: Recall 25 is used at A, 50 at B, 100 at C and 25 at D.

(a) Write down the linear system governing the flow.

(b) Solve the linear system identifying the parameters, if any.

Solution:

(a) The total flow into the system equals the total flow out but doesn't concern any of the variables. Looking at Junctions A, B, C & D we have the following equations:

$$\begin{array}{llllll} A: & 300 & = x_1 + x_4 + 25 & \Rightarrow & x_1 + x_4 & = 275 \\ B: & x_1 & = x_2 + x_5 + 50 & \Rightarrow & x_1 - x_2 - x_5 & = 50 \\ C: & x_4 + x_5 & = 100 + x_3 & \Rightarrow & x_3 - x_4 - x_5 & = -100 \\ D: & x_2 + x_3 & = 100 + 25 & \Rightarrow & x_2 + x_3 & = 125. \end{array}$$

(b) We now write this in augmented matrix form and apply Gaussian Elimination. Note there are five variables but only four equations so there will be at least one parameter.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 275 \\ 1 & -1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 1 & -1 & -1 & -100 \\ 0 & 1 & 1 & 0 & 0 & 125 \end{array}\right] \underset{r_2 \to r_2 - r_1}{\longrightarrow} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 275 \\ 0 & -1 & 0 & -1 & -1 & -225 \\ 0 & 0 & 1 & -1 & -1 & -100 \\ 0 & 1 & 1 & 0 & 0 & 125 \end{array}\right]$$

$$\underset{r_2 \to -r_2,\, r_4 \to r_4 - r_2'}{\longrightarrow} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 275 \\ 0 & 1 & 0 & 1 & 1 & 225 \\ 0 & 0 & 1 & -1 & -1 & -100 \\ 0 & 0 & 1 & -1 & -1 & -100 \end{array}\right]$$

$$\underset{r_4 \to r_4 - r_3}{\longrightarrow} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 275 \\ 0 & 1 & 0 & 1 & 1 & 225 \\ 0 & 0 & 1 & -1 & -1 & -100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We have

$$\begin{aligned} \#\text{ parameters } &= \#\text{ variables } - \#\text{ non-zero rows in reduced row form} \\ &= 5 - 3 \\ &= 2, \end{aligned}$$

so two parameters. Let $x_5 = t$ and $x_4 = s$. The third row says

$$\begin{aligned} x_3 - x_4 - x_5 &= -100 \\ \Rightarrow x_3 &= -100 + x_4 - x_5 \\ &= -100 + s + t. \end{aligned}$$

The second row says

$$\begin{aligned} x_2 + x_4 + x_5 &= 225 \\ \Rightarrow x_2 &= 225 - x_4 - x_5 \\ &= 225 - s - t. \end{aligned}$$

The first row says

$$\begin{aligned} x_1 + x_4 &= 275 \\ \Rightarrow x_1 &= 275 - x_4 \\ &= 275 - s \end{aligned}$$

so we have, for each pair (s, t) of numbers, a solution

$$(x_1, x_2, x_3, x_4, x_5) = (275 - s, 225 - s - t, s + t - 100, s, t).$$