MS 3011 Exercises

December 11, 2013

The exercises are divided into (A) easy (B) medium and (C) hard. If you are particularly interested I also have some projects at the end which will deepen your understanding further again but are again beyond exam questions. How many of these questions that you do is entirely up to you. Doing *all* of them might be just as foolish as doing none of them.

1. (A) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x + 1$. Find the first 4 iterates of $x_0 = 0$ under f. Find a formula for the nth iterate $x_n = f^n(0)$. Use induction to prove that your formula holds.

2. (C) Let $P \in \mathbb{R}$ be considered an initial value, $A = \mathbb{R}$ be considered a 'payment' and $i \in \mathbb{R}$ an interest rate. Now let $S = \mathbb{R}$ be the set of states. What financial products are described by the following iterator functions:

   (a) $d(x) = (1 + i)x$

   (b) $s(x) = (1 + i)x + A$

   (c) $m(x) = (1 + i)x - A$

3. (A) Let x_0 be an initial state. Find expressions for x_n where

   (a) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n^2$

   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$

   (c) $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(x) = \sqrt[n]{x}$

   (d) $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$, $f(x) = \dfrac{1}{x}$

4. (B) Find a set S and a function $f : S \to S$ which induces the orbits:

   (a) $\{1, 1, 1, 1, 1, \dots\}$

   (b) $\{a, b, c, c, \dots\}$

   (c) $\{a, b, c, a, b, c, a, b, c, \dots\}$

   (d) $\{1, 2, 3, 4, \dots\}$

   (e) $\left\{2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\}$

   (f) $\{2, 0, -2, -4, -6, -8, \dots\}$

   (g) $\{2, 1, 0, 0, 0, 0, 0, \dots\}$

5. (C) Write down a real-valued function f for which the orbit of 3 is given by the sequence:
$$x_n = n^3 + 3.$$

6. (A) Does there exist a real-valued function f such that the orbit of 1 is : $\{1, 2, 3, 4, 2, 3, 2, 3, \dots\}$? Justify your answer.

7. (A) **Autumn 2013** Does there exist an iterator function $f : \mathbb{N} \to \mathbb{N}$ such that the orbit of 1 is
$$\{1, 2, 4, 3, 5, 6, 8, 7, 3, 5, 6, 1, 2, 4, \dots\}?$$
Justify your answer.

8. (B) Find a set S and a function $f : S \to S$ which induces the orbits:

   (a) $\{1, i, -1, -i, 1, i, -1, -i, 1, \dots\}$

   (b) $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a^2 + bc \\ ac + cd \end{pmatrix}, \begin{pmatrix} a^3 + 2abc + bcd \\ a^2c + bc^2 + acd + cd^2 \end{pmatrix}, \dots \right\}$

   (c) $\{\cos x, -\sin x, -\cos x, \sin x, \cos x, \dots\}$

   (d) $\left\{ C, Cx + C_1, C\frac{x^2}{2} + C_1 x + C_2, C\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3, \dots \right\}$

9. (B) Prove that a polynomial of degree n has at most n fixed points.

10. (B) **Autumn 2013** Prove that a polynomial $p : \mathbb{R} \to \mathbb{R}$ of degree three has at most nine period-2 points.

11. (A) Write down a set S and a function $f : S \to S$ with no fixed points.

12. (C) Prove that if f has four period-3 points then f necessarily has a fixed point.

13. (C) **Autumn 2012** Suppose that $g : [0, 1] \to [0, 1]$ is a continuous and strictly increasing function such that $g(0) = 0$ and $g(1) = 1$. Under these hypotheses, $g(x)$ has an *inverse function* $g^{-1} : [0, 1] \to [0, 1]$ such that
$$g^{-1}(g(x)) = x \text{ and } g(g^{-1}(x)) = x \quad \text{for all } x \in [0, 1].$$

   (a) Show that if x_f is a fixed point of $g(x)$ then x_f is also a fixed point of $g^{-1}(x)$.

   (b) Suppose that the first n iterates of $x_0 \in (0, 1)$ under $g(x)$ are given by $x_0, x_1, x_2, \dots, x_n$. Write down the first n iterates of x_n under $g^{-1}(x)$.

   (c) What does it mean to say that $y \in [0, 1]$ is an eventually fixed point of $g(x)$?

   Hence show that all of the eventually fixed points of $g(x)$ are also fixed points of $g(x)$.

14. (B) Let x be a period-n point of a mapping $f : S \to S$. Suppose that p is the prime period of $x \in S$. Prove that n is an integer multiple of p.

15. (B) Let x be a period-n point of a mapping $f : S \to S$. Prove that each of the points on the orbit of x also has period n.

16. (B) **Summer 2013** Let $S = \{a, b, c, o\}$ be a finite set and suppose we know $f : S \to S$ has the property that
$$f(a) = f(b) = f(c) = o.$$
Prove that all the orbits of (S, f) are either eventually fixed or eventually prime-period-2. Recall that (S, f) denotes the dynamical system generated by an iterator function $f : S \to S$.

17. (C) **Autumn 2013** Suppose that S is a set containing four elements together with an iterator function $f : S \to S$ such that all of the elements of S are period-3. Prove that f has a fixed point.

18. (C) Let S be a non-empty *finite* set $S = \{a_1, a_2, \ldots, a_n\}$ with $n \geq 2$, and $f : S \to S$. Consider the dynamical system (S, f).
    (a) Prove that all elements of S are eventually periodic.
    (b) Suppose that all orbits are fixed. Describe f in this case.
    (c) Suppose that S contains just three elements and that $f : S \to S$ has the property that $f^2(x) = (f \circ f)(x) = x$ (i.e. every point is period[1] 2). Prove that f has a fixed point.

19. (B) **Summer 2012** Let $0 < r < 1$ and $a > 0$ and consider the mapping $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = rx + a$.
    (a) Draw a sketch of $y = f(x)$ and $y = x$ on the same diagram.
    (b) Starting at $x_0 = 0$, produce a cobweb diagram to illustrate the dynamics of $x_0 = 0$ under f.
    (c) Find, in terms of a and r, the first four iterates of $x_0 = 0$ under f.
    (d) Hence find $\displaystyle\sum_{i=0}^{\infty} ar^i$, in terms of a and r.

20. (B) Use the Factor Theorem to carefully prove our *Fixed-Point Factor-Theorem*:

    Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a polynomial. Then for all $p \in \mathbb{N}$ and $n \in \mathbb{N}$ $f^p(x) - x$ is a factor of $f^{np}(x) - x$.

[1] such a function is known as an *involution*

21. (A) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 3x^2 - 2$; defined on the interval $[-1, 1]$.

    (a) Sketch the graph of $f(x)$ and the graph of the function $d(x) = x$ on the same diagram.
    (b) Find the period-1 points of $f(x)$.
    (c) Find the period-2 points of $f(x)$.
    (d) Which of the period-2 points are prime period-2?

22. (A) **Summer 2009** Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 - x$, defined on the interval $[0, 2]$.

    (a) Sketch the graphs of $f(x)$ and the graph of the function $d(x) = x$ on the same diagram.
    (b) Find the period-1 points of $f(x)$.
    (c) Find the period-2 points of $f(x)$.
    (d) Are there points of prime period-2?

23. (A) **Summer 2011** Given the real valued function

$$f(x) = x^2 - 1,$$

    (a) Find the fixed points of the function f.
    (b) Find the period-2 points of the function f.
    (c) Determine the nature of the fixed points.

24. (B) **Autumn 2012** Consider the real valued function

$$f(x) = x^2 - 3x + 3.$$

    (a) Find the fixed points of $f(x)$ and classify them as either attracting, repelling or indifferent.
    (b) Show that $f(x)$ has no prime-period-2 points.
    (c) Find an eventually periodic point of $f(x)$ which is not periodic.

25. (B) **Summer 2010** Given the real valued function

$$f(x) = x^2 - 4x + 2,$$

    (a) Find the fixed points of the function f.
    (b) Find the period-2 points of the function f.
    (c) Are the fixed points attracting, repelling or indifferent?
    (d) Find two eventually periodic points of f which are not periodic.

26. (C) **Summer 2012** Suppose that x_0 is a prime-period-p point of a polynomial mapping $f : \mathbb{R} \to \mathbb{R}$.
    (a) Show that that x_0 is a period-n point of f **if and only if** $n = pq$ for some $q \in \mathbb{N}$.
    (b) Using the factor theorem, deduce that $f(x) - x$ is a factor of $f^2(x) - x$. Hence find the fixed points and prime-period-2 points of $g(x) = x^2 + x - 2$.
    (c) Prove that if x is a period n point of f, then each of the points in the orbit of x also has period n.
    (d) Hence, describe separately the orbits of each of the period-2 points of g.
    (e) Use parts (a) and/or (d) to show that the prime-period-2 points of g are period 50 but not period 51.

27. (B) **Summer 2008** Find all fixed points of the real valued function

$$f(x) = x^3 - \sqrt{5}\,x^2,$$

and determine whether they are attracting, repelling or neutral.

28. (B) **Autumn 2013** Consider the real-valued function

$$f(x) = x^3 - 2x^2 + 2.$$

    (a) Find the fixed points of $f(x)$ and classify them as either attracting, repelling or indifferent.

[7 Marks]

    (b) Find a non-periodic-point of $f(x)$ that is eventually fixed at $x = 2$.

29. (B) **Summer 2010**
    (a) Find the local minimum and local maximum of the real valued function

$$g(x) = x^3 + x^2 - x + 1$$

    and sketch its graph.
    (b) How many fixed points does the real valued function

$$f(x) = x^3 + x^2 + 1$$

    have? Are they repelling or attracting or indifferent points?

30. (B) **Summer 2013** Consider the real-valued function

$$g(x) = \ln(x^2 - 1) + x.$$

Find and classify the fixed points of $g(x)$.

31. (B) **Autumn 2012** Consider the real valued function

$$f(x) = x + \sin x.$$

Show that $f(x)$ has an infinite number of fixed points. Which of these fixed points are attracting and which are repelling?

32. (B) **Autumn 2013** Consider the real-valued function

$$g(x) = \frac{1}{2}xe^x.$$

Find and classify the fixed points of $g(x)$.

33. (B) **Autumn 2012** Consider the function $s : [0, \infty) \to \mathbb{R}$ given by

$$s(x) = \sqrt{x}.$$

Describe the behaviour of the dynamical system $([0, \infty), s(x))$. Use the terms *seed*, *orbit*, *fixed point*, *attracting* and *repelling* in your explanation. Recall that (S, f) denotes the dynamical system generated by an iterator function $f : S \to S$.

34. (C) **Summer 2012** Consider the sequence

$$0, \sqrt{7}, \sqrt{7-\sqrt{7}}, \sqrt{7-\sqrt{7-\sqrt{7}}}, \sqrt{7-\sqrt{7-\sqrt{7-\sqrt{7}}}}, \ldots$$

It can be shown that this sequence converges to $\frac{1}{2}(\sqrt{29}-1)$. Explain this result using your knowledge of dynamical systems. Use the terms *iterator function*, *orbit* and *attracting fixed point* in your explanation.

35. (C) **Summer 2013**

(a) Use the Intermediate Value Theorem to show that the function

$$g(x) = \cos x - x$$

has a root in $[0, \pi/4]$. This root is in fact the only root in all of $\mathbb{R}$.

(b) Consider the sequence

$$\cos 0, \cos(\cos 0), \cos(\cos(\cos 0)), \cos(\cos(\cos(\cos 0))), \ldots$$

It can be shown that this sequence converges to a point $\alpha \approx 0.7391$. Explain this result using part (i) and your knowledge of dynamical systems. Use the terms *iterator function*, *orbit* and *attracting fixed point* in your explanation.

36. (C) **Summer 2008** Newton's iterative procedure to find the roots of a given function $P(x)$ is given by

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}.$$

(a) Show that a root x of $P(x)$ for which $P'(x) \neq 0$ is a fixed point for Newton iteration.

(b) Suppose $P(x) = x^3 - 3x^2 + 2x$. Determine whether the fixed points for the corresponding Newton iteration are attracting or repelling.

37. (C) **Summer 2011** Newton's method for solving the roots of a function $g(x)$ is given by the iterative procedure

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

where $g'(x)$ is the derivative of $g(x)$. Newton's method is convergent if $x_{n+1} \to x_r$, where x_r is a root of $g(x)$, as $n \to \infty$.

(a) Rewrite Newton's method in the form of an iterated function.

(b) Show that if $g'(x) \neq 0$ then a root of $g(x)$ is a fixed point of the iterated function.

(c) Find a condition on $g(x)$ and its derivatives guaranteeing convergence of the method.

38. (A) **Summer 2009** Consider the family of functions given by

$$F_\mu(x) = x^2 + \mu \text{ for } x \in [-2, 2].$$

(a) Sketch the graphs of $F_\mu(x)$ for $\mu = \frac{1}{4}$, 0 and -2.

(b) Find the fixed points of $F_{1/4}$, F_0 and F_{-2}.

39. (A) If $Q_{3.2}(x) = 3.2x(1 - x)$, show that the fixed points are $x = 0$ and $x = 0.6875$. What is the nature of these fixed points?

40. (B) Show that the Tent Map T maps $[0, 1]$ to itself; i.e. show that $T(x) \in [0, 1]$.

41. (B) Show that the Logistic Map for $0 \leq \mu \leq 4$) and the Tent Map (both have the following properties (ie. for $f = T$ or Q_μ):

(a) the mapping satisfies $f(1/2 - x) = f(1/2 + x)$ for all $x \in [0, 1/2]$ so the mapping is symmetric about the line $x = 1/2$.

(b) The values of f increase steadily from $f(0) = 0$ at the left to the maximum value at $x = 1/2$ and decrease steadily to $f(1) = 0$. So the maps are *unimodal*.

42. (B) **Autumn 2012** The dynamical system $([0, 1], Q_\mu(x))$ is a model of population growth where x is the proportion of the maximum population and μ is a growth rate.

(a) Argue that for small populations the population growth is approximately geometric. That is for $x_0 \approx 0$ we have that $x_1 \approx \mu x_0$.

(b) Show that if the maximum population is reached then extinction will follow.

(c) Suppose that $\{x_0, x_1, x_2, \dots\}$ is the orbit of $x_0 \in (0, 1)$ under $Q_2(x) = 2x(1 - x)$.

(d) By writing $x_{n+1} = Q_2(x_n)$, show that for $n = 0, 1, 2, \dots$

$$(1 - 2x_{n+1}) = (1 - 2x_n)^2.$$

(e) Hence using induction, or otherwise, show that

$$(1 - 2x_n) = (1 - 2x_0)^{2^n}$$

for $n \in \mathbb{N}$.

(f) Hence, solve for x_n in terms of x_0 and compute

$$\lim_{n\to\infty} x_n$$

to find the limiting behaviour for any $x_0 \in (0, 1)$.

(g) Consider the statement:

> *This result shows that all seeds $x_0 \in (0, 1)$ converge to the the attracting fixed point of $Q_2(x)$.*

Do you agree with this statement? Justify your answer.

43. (C) **Summer 2011** The Logistic family of mappings is given by

$$Q_\mu(x) = \mu x(1 - x),$$

where $0 \leq \mu \leq 4$ and $0 \leq x \leq 1$.

(a) Motivate the use of the logistic equation as a model for population growth explaining the reasoning behind each of the terms, μ, x and $(1 - x)$.

(b) Find the fixed points of the system in terms of μ.

(c) Find the nature of the fixed points (that is, classify them as attracting, repelling or indifferent points).

(d) Suppose an infection enters the population introducing a rate of mortality that is proportional to the population at any time. Find the fixed points of the amended system in terms of μ and the new mortality rate.

44. (B) **Summer 2012** Consider the *Doubling Mapping* given by

$$D(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2x - 1 & \text{if } 1/2 < x \leq 1 \end{cases}$$

(a) Where x has the binary representation

$$x = 0.a_1a_2a_3a_3a_4a_5a_6a_7a_8\ldots,$$

find expressions for $D(x)$ and $D^5(x)$. Hence find points $y, z \in [0, 1]$ such that y and z agree to 5 binary digits but $D^N(y)$ and $D^N(z)$ differ in the first binary digit for some $N \in \mathbb{N}$.

(b) Describe the period-5 points of D. Let $w \in [0, 1]$ have a binary representation beginning $w = 0.01001\ldots.$ Find a period-5 point γ of D such that w and γ agree to five binary digits.

(c) Find a $\delta \in [0,1]$ such that there are iterates of δ, $D^{n_1}(\delta), D^{n_2}(\delta), D^{n_3}(\delta)$, with $n_1, n_2, n_3 \in \mathbb{N}_0$, that agree with $0.111\ldots$, $0.101\ldots$ and $0.010\ldots$ to three binary digits.

(d) Consider the statement:

These arguments could be generalised to show that $D(x)$ is a chaotic mapping.

Do you agree or disagree with this statement? Justify your answer.

45. (C) **Autumn 2013** Consider the *Doubling Mapping* given by

$$D(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2x - 1 & \text{if } 1/2 < x \leq 1 \end{cases}$$

(a) Where x has the binary representation

$$x = 0.a_1a_2a_3\ldots a_na_{n+1}\ldots,$$

write down expressions for $D(x)$ and $D^n(x)$.

[4 Marks]

(b) Describe the period-n points of $D(x)$.

[4 Marks]

(c) Let $w \in [0,1]$ have a binary representation

$$w = 0.b_1b_2b_3\ldots b_n\ldots.$$

Find a period-n point γ of D such that w and γ agree to n binary digits.

[4 Marks]

(d) Write down the binary representation of a $\delta \in [0,1]$ such that the orbit of δ is dense in $[0,1]$.

[5 Marks]

(e) Where y has the binary representation

$$y = 0.c_1c_2c_3\ldots c_n000\ldots,$$

write down a point $z \neq y$ such that

$$|z - y| \leq \frac{1}{2^n}$$

but

$$|D^n(z) - D^n(y)| \geq \frac{1}{2}.$$

[4 Marks]

(f) Consider the statement:

Parts (c), (d) and (e) show that $D(x)$ is a chaotic mapping.

Do you agree or disagree with this statement? Justify your answer.

46. (A) **Summer 2009** Find the first six iterates of the point $x_0 = 1/7$ under the action of the function

$$T(x) = \begin{cases} 3x & \text{if } x \in [0, 1/3) \\ 3x - 1 & \text{if } x \in [1/3, 2/3) \\ 3x - 2 & \text{if } x \in [2/3, 1] \end{cases} .$$

47. (A) **Autumn 2010** Let

$$T(x) = \begin{cases} 3x & \text{if } x \in [0, 1/3) \\ 3x - 1 & \text{if } x \in [1/3, 2/3) \\ 3x - 2 & \text{if } x \in [2/3, 1] \end{cases}$$

be the *Tripling Mapping*.

(a) If x has the *ternary representation*

$$x = \frac{1}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{1}{3^4},$$

compute $T(x)$.

(b) Show that all rational points in $[0, 1]$ are eventually periodic.

48. (A) **Summer 2010** Consider the function

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 2 - 2x & \text{if } x \in (1/2, 1] \end{cases} .$$

(a) Find the first five iterates of the point $x_0 = \frac{2}{11}$ under T.

(b) Show that any positive rational number smaller than one is an eventually periodic point.

49. (C) **Summer 2013** Consider the *Tent Mapping* $T : [0,1] \to [0,1]$ given by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 2 - 2x & \text{if } x \in (1/2, 1] \end{cases} .$$

(a) Sketch a graph of $T(x)$ and $y = x$ on the same graph.

(b) Show that $2/3$ is a fixed point of $T(x)$. How many other fixed points does $T(x)$ have? Write them down.

(c) Using a result from the lectures, or otherwise, prove that all fractions $q = m/n \in [0, 1]$ are eventually periodic points of T.

(d) Sketch graphs of $T^2(x)$ and $T^3(x)$. In the case of $T^2(x)$, explain how you sketched the graph.

(e) How many period-n points does $T(x)$ have? Hence prove that the periodic points of $T(x)$ are dense in $[0, 1]$.

50. (A) Show that $2\ \mathrm{Re}(z) = z + \bar{z}$ and $2\ \mathrm{Im}(z) = z - \bar{z}$ for all $z \in \mathbb{C}$.

51. (B) Show that $\overline{(w + z)} = \bar{w} + \bar{z}$ and $\overline{(wz)} = \bar{w}\bar{z}$ for all $w,\ z \in \mathbb{C}$. Hence prove the Conjugate Root Theorem — which exhibits the symmetry between i and $-i$ with respect to the real numbers. In particular, $\sqrt{-1} = i$ is imprecise: $i^2 = -1$ is better.

52. (A) **Autumn 2009** Iterate the complex valued function $f(z) = z^2 - i$ starting at the seed $z_0 = -i$. Describe the behaviour of the orbit. Do you think it tends to infinity or remains bounded? Explain your answer.

53. (A) **Summer 2009** Find the first four iterates of the point $z_0 = 1+i$ under the iterator function $g(z) = z^2 + 2i$.

54. (A) **Summer 2010** Show that $e^{2\pi i/5}$ is a periodic point of the function $f(z) = z^2$ and find its prime period.

55. (A) **Autumn 2010** Show that $e^{2\pi i/5}$ is a periodic point for the function $f(z) = z^3$ and find its prime period.

56. (B) **Summer 2010** Let

$$Q(z) = z^2 + 0.5i + 0.25.$$

Find the fixed points of Q and determine if they are attracting, repelling or indifferent.

57. (B) **Summer 2009** Find all the fixed points of the complex valued function

$$f(z) = 3iz(2 - z),$$

and determine whether they are attracting, repelling or neutral.

58. (A) **Summer 2013** Consider the *complex*-valued function

$$f(z) = z^2 - 2z + 2.$$

(a) Find the fixed points of $f(z)$ and classify them as either attracting, repelling or indifferent.

(b) Find the prime-period-2 points of $f(z)$.

(c) Denote one of the prime-period-2 points by α. Prove that α is period-8 but not period-9.

59. (A) **Autumn 2009** Find all the fixed points of the complex valued function

$$f(z) = z^3 + z - 1,$$

and determine whether they are attracting, repelling or neutral.

60. (B) **Summer 2008** Find all complex c-values for which

$$Q_c(z) = z^2 + c$$

has a fixed point z_0 with $Q_c'(z_0) = 1$.

61. (B) **Summer 2013** Plot the complex number $z = \sqrt{3} + i$ on an Argand Diagram. Hence, or otherwise, write z in exponential form $z = r\,e^{i\,\theta}$.

[4 Marks]

Similarly all complex numbers z can be written in exponential form. In the sequel we write $e^z = \exp(z)$. Hence we can define the *principal square root function* $s : \mathbb{C} \to \mathbb{C}$ by

$$s(z) = s(r\,\exp(i\theta)) = \sqrt{r}\,\exp(i\,\theta/2).$$

If z is a positive real number we have

$$s(z) = \sqrt{z}.$$

(a) Show that 0 and 1 are fixed points of $s(z)$.

[2 Marks]

(b) Show using an inductive argument, or otherwise, that if $z_0 = r\,\exp(i\,\theta)$ then

$$s^n(z_0) = r^{1/2^n}\,\exp(i\,\theta/2^n).$$

[3 Marks]

(c) Hence show that for $z_0 \neq 0$ we have

$$\lim_{n\to\infty} s^n(z_0) = 1.$$

[3 Marks]

This proves that $z = 0$ is a repelling fixed point of $(\mathbb{C}, s(z))$ while $z = 1$ is an attracting fixed point.

(d) Consider now the following subsets of the complex plane

$$\mathbb{P} = [0, \infty) = \text{ positive real numbers}$$
$$\mathbb{T} = \{z \in \mathbb{C} : |z = 1|\}$$
$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

Plot each of these subsets of $\mathbb{C}$ on a single Argand Diagram.

[HINT: $\mathbb{P} \cap \mathbb{D} = [0, 1)$, $\mathbb{P} \cap \mathbb{T} = \{1\}$, $\mathbb{D} \cap \mathbb{T} = \emptyset = \{\}$]

[3 Marks]

(e) Show geometrically and/or algebraically that the following hold

i. There exists an $x_0 \in \mathbb{R}$ such that $s(x_0) \notin \mathbb{R}$.

[1 Mark]

ii. For all $p \in \mathbb{P}$, $s(p) \in \mathbb{P}$.

iii. For all $t \in \mathbb{T}$, $s(t) \in \mathbb{T}$.

iv. For all $d \in \mathbb{D}$, $s(d) \in \mathbb{D}$.

[All 2 Marks each]

[HINT: $a \in A$ means that a is an element of the set A and $b \notin B$ means that b is not an element of the set B]

(f) Hence we may consider the dynamical systems $(\mathbb{P}, s(z))$, $(\mathbb{T}, s(z))$ and $(\mathbb{D}, s(z))$. Considering the behaviour of the dynamical system $(\mathbb{C}, s(z))$, or otherwise, justify the following true statements:

i. $(\mathbb{P}, s(z))$ has one attracting fixed point and one repelling fixed point.

ii. $(\mathbb{T}, s(z))$ has one attracting fixed point and no repelling fixed point.

iii. $(\mathbb{D}, s(z))$ has one repelling fixed point and no attracting fixed point.

[All 1 Mark each]

Recall that (S, f) denotes the dynamical system generated by an iterator function $f : S \to S$.

62. (B) **Summer 2012** Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane and consider the function $f : \mathbb{T} \to \mathbb{C}$ given by $f(z) = z^2$.

(a) Suppose that z has polar representation

$$z = e^{i\theta} = \cos\theta + i \sin\theta.$$

Find an expression for z^2 in terms of θ. For a $0 < \theta < \pi/2$, plot z and z^2 on the same Argand diagram.

(b) Hence, or otherwise, explain why f maps points on the unit circle to other points on the unit circle; i.e. $f : \mathbb{T} \to \mathbb{T}$.

(c) Find the fixed points of f and determine whether they are attracting, repelling or indifferent.

(d) Plot the the period-2 points of f on an Argand Diagram.

(e) Show that all points of the form $z_0 = e^{2i\pi/2^q}$ are eventually fixed points for $q \in \mathbb{N}$.

(f) *The orbit of $e^{2\sqrt{2}\pi i}$ is eventually fixed.*

Do you agree with this statement? Justify your answer.

63. (B) **Autumn 2012** Consider the function $c : \mathbb{C} \to \mathbb{C}$ given by $c(z) = z^3$.

(a) Suppose that z has polar representation

$$z = e^{i\theta} = \cos\theta + i\sin\theta.$$

Write down an expression for z^3 in terms of θ. For a $0 < \theta < \pi/2$, plot z and z^3 on the same Argand diagram.

(b) Find the fixed points of $c(z)$ and determine whether they are attracting, repelling or indifferent.

(c) Plot the the period-2 points of $c(z)$ on an Argand Diagram.

(d) Show that all points of the form $z_0 = \exp\left(\frac{2i\pi}{3^q}\right)$ are eventually fixed points of $c(z)$ for $q \in \mathbb{N}$. Note that $\exp(x) := e^x$.

(e) Consider the statement:

The behaviour of the dynamical system $(\mathbb{C}, c(z))$ is very simple with only two types of orbits. A seed will either converge to zero or diverge to infinity in magnitude.

Do you agree with this statement? Justify your answer. Recall that (S, f) denotes the dynamical system generated by an iterator function $f : S \to S$.

Here are some harder applied problems.

1. **Abstract Algebra** Suppose that $f : S \to S$ is a function on a finite set such that all of the points in S are periodic. Define a relation on S:

   x is related to y if $x \in orb(y)$.

   Prove that this is an equivalence relation on S.

2. **Number Theory**
   Let $\mathbb{N}$ be the set of natural numbers and $f : \mathbb{N} \to \mathbb{N}$ and consider the dynamical system $(\mathbb{N}, f)$.

   Find a function $f : \mathbb{N} \to \mathbb{N}$ such that f has *no* eventually periodic points.

   Let $g : \mathbb{N} \to \mathbb{N}$ be a function with the property that $g(n) > 1$ and suppose that $f(n) = g(n)n$. Let $\mathbb{P}$ be the set of primes. Prove that under the iterator function f, for all composite $n_0 > 1$, the orbit of n_0 contains no prime numbers.

3. **Probability**
   Consider a set $X = \{a, b\}$. Suppose there is a particle at the point a at the time $t = 0$. Suppose at times $t = 1, 2, \ldots$ the particle either stays where it is or moves to the other point 'at random'. Let ξ_k be the *random variable* describing the position of the particle at time $k \in \mathbb{N}$. Hence the distribution of ξ_0 is given by $\mathbb{P}[\xi_0 = a] = 1$ (i.e the particle is at a at time $t = 0$ with probability 1). For p, $q \in [0, 1]$, suppose we have the following probabilities:

   $$\mathbb{P}[\xi_{k+1} = b|\xi_k = a] = p; \quad \text{i.e. the probability of moving from } a \to b \text{ is given by } p$$
   $$\mathbb{P}[\xi_{k+1} = a|\xi_k = b] = q; \quad \text{i.e. the probability of moving from } b \to a \text{ is given by } q$$

   (a) Show that $\mathbb{P}[\xi_{k+1} = a|\xi_k = a] = 1 - p$ and $\mathbb{P}[\xi_{k+1} = b|\xi_k = b] = 1 - q$.

   The motion of the particle is an example of a *Markov chain*.

   (b) Draw a labeled diagram describing the Markov chain.

   Note that for $(p, q) = (1, 1), (0, 0), (1, 0), (0, 1)$, there is an iterator function $f_{(p,q)} : X \to X$ such that $(X, f_{(p,q)})$ is a dynamical system such that $f^k_{(p,q)}(a) = \xi_k$.

   For example, consider $(p, q) = (1, 0)$. In this case

   $$\text{probability of moving from } a \to b = 1$$
   $$\text{probability of moving from } b \to a = 0$$

   We always begin at a so we have

   $$\{\xi_0, \xi_1, \xi_2, \ldots\} = \{a, b, b, \ldots\}.$$

   This is exactly the orbit of a under the iterator function:

   $$\begin{cases} f_{(1,0)}(a) = b \\ f_{(1,0)}(b) = b \end{cases}$$

   (c) Explain why, for $p \in (0, 1)$, there is no function $g : X \to X$ such that (X, g) is a dynamical system such that $g^k(a) = \xi_k$.

   Let

   $$\theta_k = (\mathbb{P}[\xi_k = a] \ \ \mathbb{P}[\xi_k = b])$$

   be the 1×2 matrix describing the distribution of ξ_k, i.e. the probabilities that the particle is in position a or b when $t = k$.

(d) Hence show that $\theta_0 = (1 \;\; 0)$.

Define

$$S := \{(x \;\; y) : x \in [0,1] \text{ and } y \in [0,1]\},$$

i.e. the set of probability distributions on X, so that $\theta_k \in S$. Let P be a 2×2 matrix defined as

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}. \tag{1}$$

(e) Show that if $f : S \to S$ is given by (matrix multiplication)

$$\begin{aligned} f((\mu_1 \;\; \mu_2)) &= (\mu_1 \;\; \mu_2) \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \\ &= (\mu_1 \;\; \mu_2)P, \end{aligned}$$

then $\theta_1 = f(\theta_0)$.

(f) Also show that if $\theta_k = (\mu_1 \;\; \mu_2)$, then $\theta_{k+1} = f(\theta_k)$.

(g) Hence show that (S, f) is a dynamical system such that $f^k(\theta_0) = \theta_k$.

4. **Integral Calculus**
Consider the initial value problem:

$$\frac{dy}{dx} = 2xy, \;\; y(0) = 1. \tag{2}$$

(a) Solve this differential equation for $y = f(x)$ using a separation of variables.

Our analysis leads to a unique solution but it is not clear that there are not other solutions. The answer to this question is answered in the language of dynamical systems. Let us consider a general initial value problem

$$\frac{dy}{dx} = F(x, y(x)), \tag{3}$$

where $y(x_0) = y_0$. Here we can take $F(x, y(x))$ as some nice[2] formula in x and y; $F(x, y(x)) = 2xy$ as above for example. We just want to find solutions in some non-empty closed interval $[a, b]$ for example $I = [0, 1]$. Then the boundary condition $y(x_0) = y_0$ will be described in terms of some $x_0 \in [a, b]$. Suppose we take as an approximation to a solution the constant function $\varphi_0(x) = y_0$.

(b) Show that $\varphi_0(x)$ agrees with the solution at at least one point $x \in [a, b]$.

(c) Integrate both sides of (3) from $x_0 \to x$ to show that the differential equation can be transformed/rewritten as an *integral equation*:

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt. \tag{4}$$

[2]what it takes for a formula to be 'nice' involves some technical stuff that need not concern us.

To generate a better approximation to the solution $y(x)$ we plug $\varphi_0(x) = y_0$ into (4) to generate a second approximation $\varphi_1(x)$:

$$\varphi_1(x) = y_0 + \int_{x_0}^{x} F(t, \varphi_0(t))\, dt \tag{5}$$

This second approximation is then put into (4) to generate a third approximation:

$$\varphi_2(x) = y_0 + \int_{x_0}^{x} F(t, \varphi_1(t))\, dt \tag{6}$$

and the process may be iterated.

(d) Consider the initial value problem (2). Find the approximations $\varphi_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ to the solution.

(e) Now let $S = \{\text{Integrable Functions} : [a, b] \to \mathbb{R}\}$. Taking inspiration from (5) and (6), find an expression for an iterator function $\Gamma : S \to S$ such that $\Gamma(\varphi_n(x)) = \varphi_{n+1}(x)$.

(f) Suppose that $\varphi(x)$ is a fixed point of the dynamical system (S, Γ). Explain why φ solves the initial value problem (3).

5. **Linear Algebra** Consider the set $S = M_2(\mathbb{C})$ of 2×2 matrices with complex valued entries. Let A be an element of S and consider the function $\mathcal{M}_A : S \to S$ given by

$$\mathcal{M}_A(X) = AX, \tag{7}$$

where the product is matrix multiplication (the $\mathcal{M}$ stands form $\mathcal{M}$ultiplication). We consider the family of dynamical systems $\{(\mathcal{M}_A, S) : A \in M_2(\mathbb{C})\}$ generated by the family of mappings $A \mapsto \mathcal{M}_A$.

(a) Find a matrix X_0 that is a fixed point for all of these dynamical systems.

(b) Find a matrix A such that all matrices are fixed points under the iterator function $\mathcal{M}_A$.

(c) Show that all $X_0 \in M_2(\mathbb{C})$ are period 2 points for $\mathcal{M}_A$ where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find an example of an X_0 such that X_0 is prime period 2.

(d) Prove that all matrices are period 4 points for

$$A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(e)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The *adjoint* or *conjugate transpose* of A is given by

$$A^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}.$$

If a matrix A has the property that $A^* = A$ and all of the *eigenvalues* of A are positive real numbers then A is said to be *positive*. In this case there is a unique *positive* matrix $\sqrt{A}$ such that $(\sqrt{A})^2 = A$. Show that if X_0 is a fixed point of the iterator function $\mathcal{M}_A$ (A positive), that X_0 is a period 4 point of $\mathcal{M}_{\sqrt{\sqrt{A}}}$.

(f) Show that all matrices are points are eventually zero for $\mathcal{M}_A$ for

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

and eventually fixed for

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

(g) Suppose that A is a diagonal matrix with diagonal entries μ and λ such that $|\mu| < 1$ and $|\lambda| < 1$. Describe the limiting behaviour for any $X_0 \in M_2(\mathbb{C})$.

(h) Show that if A is invertible and X_0 is invertible then the orbit of X_0 contains invertible matrices only.

(i) Suppose that A is invertible and suppose that

$$Y = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

is the 100th iterate of $\mathcal{M}_A$, $\mathcal{M}_A^{100}(X_0)$. For how many seeds $X_0 \in M_{2\times 2}(\mathbb{C})$ is $Y = \mathcal{M}_A^{100}(X_0)$?

(j) Suppose that $P^N = I_2$ for some $N \in \mathbb{N}$ and let $A = XPX^{-1}$. Show that X is a period N point of $\mathcal{M}_{XPX^{-1}}$.

6. **Differential Calculus**
Consider the set of real valued functions $f(x)$ for which the nth derivative, $f^{(n)}(x)$, exists at all points in $\mathbb{R}$; i.e. the set of infinitely differentiable functions $S = C^\infty(\mathbb{R})$. Define a function $D : S \to S$ by $D(f(x)) = f'(x)$ and consider the dynamical system $(S, f) = (C^\infty(\mathbb{R}), D)$ where the points are infinitely differentiable functions and the iterator function is differentiation.

(a) Solve the differential equation $D(f(x)) = f(x)$ to find all the fixed points of D.

(b) Show that $\exp(-x) = e^{-x}$ is a prime period 2 point of D.

(c) Show that $\sin(x)$ is a prime period 4 point of D.

(d) Let $p_0 : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree N. Show that $D^{N+1}(p_0(x)) = 0$ and hence explain why $p_0(x)$ is eventually fixed.

(e) Let $f_0(x) = \sin(x/2)$. We can show, for $k \in \mathbb{N}$, that

$$D^n(f_0(x)) = \begin{cases} \frac{1}{2^n}\sin(x/2) & \text{if } n = 4k \\ \frac{1}{2^n}\cos(x/2) & \text{if } n = 4k+1 \\ -\frac{1}{2^n}\sin(x/2) & \text{if } n = 4k+2 \\ -\frac{1}{2^n}\cos(x/2) & \text{if } n = 4k+3 \end{cases}.$$

Hence show that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} D^n(f_0(x)) = 0,$$

for all $x \in \mathbb{R}$. Is $f_0(x)$ eventually fixed?

(f) Let $g_0(x) = xe^x$ and $h_0(x) = xe^{-x}$. Find expressions for $g_n(x) = D^n(g_0(x))$ and $h_n(x) = D^n(h_0(x))$ and prove that they are correct using induction or otherwise.

(g) Let $q : \mathbb{R} \to \mathbb{R}$ be any polynomial and define $r_0(x) = q(x) + e^x$. Prove that $r_0(x)$ is eventually fixed.

(h) We can define the derivative of a *complex valued function* $F : \mathbb{C} \to \mathbb{C}$. Let S_z be the set of infinitely differentiable complex valued functions and, $D_z : S \to S$ the function given by

$$D_z(F(z)) = \frac{dF}{dz}.$$

- If we define $\exp(z) = e^z$ in a natural way, it turns out that for $a \in \mathbb{C}$ we have that
$$\frac{de^{az}}{dz} = ae^{az}.$$
- jth *roots of unity* are the complex roots of the polynomial/solutions of the equation
$$z^j - 1 \Leftrightarrow z^j = 1.$$
There are j such roots of unity by the *Fundamental Theorem of Algebra* and we can show that they are all powers of a *primitive root* ζ_j:
$$j\text{th roots of unity} = \{1, \zeta_j, \zeta_j^2, \zeta_j^3, \ldots, \zeta_j^{j-1}\}. \tag{8}$$
$\zeta \in \mathbb{C}$ is a primitive jth root if $\zeta_j^m \neq 1$ for $m < j$.

Use these two pieces of information to construct prime period p elements in the dynamical system (S_z, D_z) for $p = 1, 2, 3, \ldots$.