We obtain:

0 0 0	1 0 0	2 0 0	3 0 0
0 0 1	1 0 1	2 0 1	3 0 1
0 1 0	1 1 0	2 1 0	3 1 0
0 1 1	1 1 1	2 1 1	3 1 1
0 2 0	1 2 0	2 2 0	3 2 0
0 2 1	1 2 1	2 2 1	3 2 1

TABLE 1

We wish to associate each reduced choice-code in Table 1 with a unique integer in the range 0 to 23. At this stage the entries in Table 1 seem to form a familiar pattern; they remind us of numbers written to a fairly small base. However, this idea, although useful, is clearly not correct. After some reflection we realise that these entries may be taken as representing the numbers 0 to 23 in the form $a.3! + b.2! + c.1!$ where $0 \leqslant a \leqslant 3$, $0 \leqslant b \leqslant 2$, $0 \leqslant c \leqslant 1$. Thus each reduced choice-code (and therefore each permutation) corresponds to a unique integer in the range 0 to 23. The permutations are therefore ordered!

Factorial-base numbers

We shall call numbers (i.e. integers) written in the form

$$a_1.1! + a_2.2! + a_3.3! + \ldots$$

factorial-base numbers (where $0 \leqslant a_i \leqslant i$). It does not take long to realise that every non-negative integer can be written in this form (see Appendix). The (easy) algorithm used to convert base-10 numbers to factorial-base form can be left to the reader. As we have seen above, there is a one-one correspondence between factorial-base numbers and permutations. We have only to write down factorial-base numbers in increasing order and we thereby list the corresponding permutations in a simple order. We have considered this in detail for permutations on {1 2 3 4 } but it clearly carries over to permutations on {1 2 3 4 5} and indeed to permutations on {1 2 … n}.

Example 1: The permutation (4 5 3 1 2) on {1 2 3 4 5} has choice-code [4 4 3 1] and reduced-choice code [3 3 2 0]. It therefore corresponds to the non-negative integer $3.4! + 3.3! + 2.2! + 0.1! = 94$. Hence, this permutation stands at no. 94 in the order which we have imposed.

Example 2: Permutation no. 115 on {1 2 3 4 5} corresponds to the factorial-base number $4.4! + 3.3! + 0.2! + 1.1! = 115$. The choice-code is therefore [5 4 1 2] and the permutation is (5 4 1 3 2).

Conclusion

By the use of choice-codes, reduced choice-codes and factorial-base numbers we have rendered it very easy to write a program for listing permutations in a definite order. We can, if we like, arrange that only particular permutations are found, e.g. those which are derangements.

Appendix

We prove that every non-negative integer N can be written uniquely in the form

$$N = a_1.1! + a_2.2! + \ldots + a_p.p! \qquad (1)$$

where $0 \leqslant a_i \leqslant i$ for $i = 1, 2, \ldots, p$. This result depends on the known summation $\sum_{i=1}^{p} i.i! + 1 = (p + 1)!$.

Let us say that any a_i in (1) is *maximal* if, and only if, it is equal to i. Clearly the right-hand side of (1) gives a *unique* non-negative integer. We must now show, conversely, how to write any given non-negative integer in the form (1). This is done by induction on N.

If N has been written uniquely in the form (1) let us consider $N + 1$. In order to add 1 to the right-hand side of (1), we proceed as follows. First, if $a_1 = 0$, we change it to 1 leaving the other a_i unaltered. If a_1 is non-zero, we locate the first non-maximal a_i ($i = 2, 3, \ldots, p$). If this is a_s, then replace it by $a_s + 1$ and set all a_i for $i = 1, \ldots, s - 1$ to zero. If all the a_s ($s = 1, \ldots, p$) are maximal, add $a_{p+1}.(p + 1)!$, with $a_{p+1} = 1$, to the right-hand side of (1) and set all a_i for $i = 1, \ldots, p$ to zero. This process leads to a unique representation of $N + 1$ in the form (1) and the required result easily follows by induction. (The method used here is familiar in simple arithmetic to base 10. For instance $299 + 1 = 300$, since 9 is maximal in this arithmetic.)

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92.47 The converse of the theorem of Pythagoras

The theorem of Pythagoras – that if ABC is a right-angled triangle, with side-lengths a, b and c, where c is the hypotenuse, then $a^2 + b^2 = c^2$ – is one of the best-known and arguably one of the most useful results in mathematics. The converse of this theorem is also true. This states that, if the lengths of the sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is right-angled at C. In all of the textbooks I have consulted, the converse theorem is proved as follows, where the theorem of Pythagoras is assumed to be true:

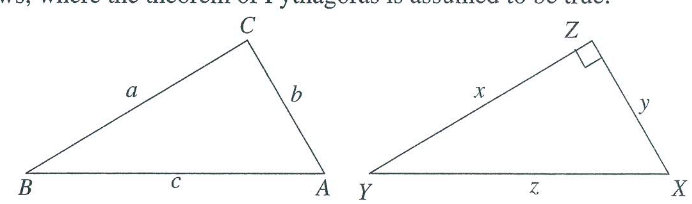

FIGURE 1

Let ABC be a triangle such that $a^2 + b^2 = c^2$. Construct a triangle XYZ such that $x = a$, $y = b$ and $\angle XZY = 90°$. Then, by Pythagoras' theorem,

$$z^2 = x^2 + y^2 = a^2 + b^2 = c^2,$$

so $z = c$. Thus the triangles ABC and XYZ are congruent (SSS) so $\angle ACB = \angle XZY = 90°$, as claimed.

This note is motivated by the following question:

> 'Is it possible to prove the converse of the theorem of Pythagoras without assuming the theorem of Pythagoras or any result which depends on it?'

Once this is achieved, we show that the theorem of Pythagoras can be deduced from its converse, thus producing a new proof of the theorem of Pythagoras, admittedly in a rather roundabout way.

We remark that mathematical folklore has it that the converse theorem predates the theorem, in that Egyptian mathematicians in the pre-Greek era produced a right angle by knotting a string at points corresponding to lengths of 3, 4 and 5 units. By experience, they knew that this always gave a right angle, but they may not have realised that this was a consequence of the fact that $3^2 + 4^2 = 5^2$.

Theorem: Let ABC be a triangle in which $a^2 + b^2 = c^2$. Then $\angle ACB$ is a right angle.

Proof: Let X be a point on AB such that $\angle CXA = \angle ACB$. We use contradiction to prove that X is an internal point of AB.

Suppose first that A lies between X and B as shown in Figure 2:

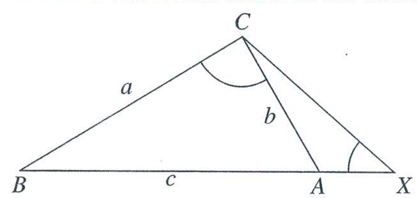

FIGURE 2

Since $a^2 + b^2 = c^2$, it follows that c is the longest side of the triangle. [1, Proposition 19] tells us that the largest angle in a triangle is opposite the longest side. Therefore, $\angle BCA$ is the largest angle in the triangle above and so

$$\angle BCA > \angle CAB.$$

The angle $\angle CAB$ is an external angle of the triangle CAX. [1, Proposition 16] tells us that

$$\angle CAB > \angle CXA.$$

It follows that

$$\angle BCA > \angle CXA$$

which is a contradiction, as these angles were assumed equal. Hence, it follows that A cannot lie between X and B.

By the same method, we show that B cannot lie between X and A and so X is an internal point of AB.

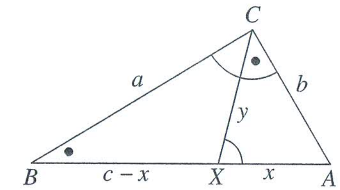

FIGURE 3

Consider Figure 3. Triangle ACB is similar to triangle CXB as they have two angles in common and hence, by [1, Proposition 32], have three angles in common.

Therefore, it follows from [2, Proposition 4] that

$$\frac{x}{b} = \frac{y}{a} = \frac{b}{c}. \tag{1}$$

By [3, Proposition 15], we find that

$$\frac{x}{b} = \frac{cx}{cb} \text{ and } \frac{b}{c} = \frac{b^2}{bc}.$$

Hence

$$\frac{b^2}{bc} = \frac{cx}{bc}.$$

By [3, Proposition 9] it follows that

$$b^2 = cx.$$

So, by assumption

$$c^2 - a^2 = cx$$

and

$$c^2 - cx = a^2.$$

By [3, Proposition 7]

$$\frac{c^2 - cx}{ac} = \frac{a^2}{ac}.$$

Hence, by [3, Proposition 15], it follows that

$$\frac{c - x}{a} = \frac{a}{c}. \tag{2}$$

Also, from (1) and by [3, Proposition 16],

$$\frac{y}{b} = \frac{a}{c}. \tag{3}$$

From (2) and (3) it follows that

$$\frac{c - x}{a} = \frac{a}{c} = \frac{y}{b},$$

and so, by [2, Proposition 5], the triangles BXC and ABC are similar. Hence

$$\angle BXC = \angle BCA = \angle CXA.$$

But, by [1, Proposition 13],

$$\angle BXC + \angle CXA = 180°$$

so

$$\angle BCA = \angle BXC = \angle CXA = 90°,$$

as claimed.

Corollary: The converse theorem implies the theorem of Pythagoras.

Proof: Suppose that $a^2 + b^2 = c^2$ implies that C is a right angle. Given a triangle XYZ with $\angle XZY = 90°$, we wish to show that $x^2 + y^2 = z^2$.

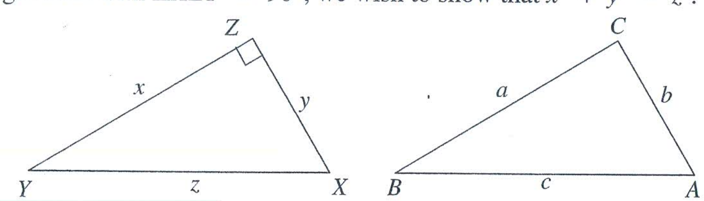

FIGURE 4

We construct a triangle ABC with $a = x$, $b = y$ and $c = \sqrt{a^2 + b^2}$. It is worth describing here how the length c is constructed. Firstly, to construct a length a^2 we start with a line segment OP of unit length and produce it to OA of length a (or, if $a < 1$, find the point A between O and P). We then draw a line through O so as to make an acute angle with OP and find the point Y where it meets the circle with centre O which passes through A, so that $OY = OA$. We then draw the parallel to PY through A. Let Z be the point where this parallel meets the line OY. Then the length of OZ is a^2.

In a similar manner, we can construct the length b^2 and, consequently, the length $a^2 + b^2$. Then $\sqrt{a^2 + b^2}$ may be constructed using [2, Proposition 13].

Thus $c^2 = a^2 + b^2$ so $\angle ACB = 90°$. Thus the triangles XYZ and ABC are congruent (SAS) so

$$z^2 = c^2 = a^2 + b^2 = x^2 + y^2,$$

as claimed.

Note: It is worth noting that Pythagoras' theorem occurs in *Euclid's Elements of Geometry, Book I* and so, in Euclidean terms, the proof does need more advanced material than occurs in Book I.

Acknowledgement: I wish to thank Professor Des MacHale of UCC for posing this question at a problem-solving session and the referee for various suggestions.

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92.48 The upside-down Pythagorean theorem

Introduction

A triple of positive integers (a, b, c) is a Pythagorean triple if, and only if, $a^2 + b^2 = c^2$. Positive integers a and b will also be the lengths of the sides of a right-angled triangle and integer c will be the length of the hypotenuse. Let d equal the length of the segment that is perpendicular to the hypotenuse and that passes through the vertex of the right angle. It can then be proved that $a^{-2} + b^{-2} = d^{-2}$. We shall call the triple (a, b, d) an *upside-down Pythagorean triple*. The objective is to determine the upside down Pythagorean triples that are integers and present a visual representation of the result.

Theorem 1: (Upside-down Pythagorean theorem): In ΔABC, assume $\angle ACB$ is a right angle, the point D is on AB such that $CD = d$ and $CD \perp AB$. Then, $a^{-2} + b^{-2} = d^{-2}$.

Proof: Calculating the area of ΔABC in two different ways gives $\frac{1}{2}ab = \frac{1}{2}cd$, i.e. $ab/d = c$. Since ΔABC is a right-angled triangle, its sides satisfy the Pythagorean theorem, so that

$$b^2 + a^2 = c^2$$

i.e. $$b^2 + a^2 = \left(\frac{ab}{d}\right)^2$$

i.e. $$\frac{b^2}{a^2b^2} + \frac{a^2}{a^2b^2} = \frac{1}{d^2}$$

i.e. $$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{d^2},$$ as required.

A triple (a, b, c) is *primitive* if the three integers are coprime. In the following, the abbreviation PPT stands for 'primitive Pythagorean triple' and PUPT stands for 'primitive upside-down Pythagorean triple'.