1 The Substitution Method

Examples

1.

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$$

Solution: The first thing to note here is that this is a definite integral (with limits): the answer is a number. The Fundamental Theorem of Calculus tells us how to calculate this. We find the *anti-derivative* $I = \int \frac{x}{\sqrt{x^2+1}}\,dx$ and write

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx = [\textit{anti-derivative}]_0^{\sqrt{3}} = \text{ plug in top } - \text{ plug in bottom.}$$

Therefore we can forget about the limits for now and just find $\frac{x}{\sqrt{x^2+1}}\,dx$.

Confronted with an integral like this we have a strategy. First we see if we can do it directly — is it in the tables? A quick look in the tables shows that $\frac{x}{\sqrt{x^2+1}}$ is not there. Second we try a manipulation — a rewriting of $\frac{x}{\sqrt{x^2+1}}$; nothing obvious springs to mind. Thus we do a substitution. There are three ways to pick the 'u':

(a) The 'u' will always be inside another function; i.e.

$$\sin(u),\ e^u,\ \sqrt{u},\ \frac{1}{u}, \log(u),\ \frac{1}{u^2},\ u^5, \text{ etc.}$$

Here we have that x^2+1 is inside the square-root and we choose $u = x^2+1$.

(b) We look for the function-(multiple of)derivative pattern and let u =function. Here we have a function — x^2+1 — whose derivative is $2x$; a multiple of x (which is in the numerator). We should once again choose $u = x^2+1$.

(c) We pick the most complicated thing on the LIATE hierarchy. There are no log nor inverse trig. There are algebraic: x and x^2+1. Most complicated of these means higher power. We should pick $u = x^2+1$.

Now what we are going to do is take the integral, take it apart and reassemble it in terms of u. The $\sqrt{x^2+1} = \sqrt{u}$. We also want to write dx in terms of du. To get a relationship between dx and du we find $\frac{du}{dx}$ by differentiating $u = x^2+1$:

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}.$$

Now put the integral back together, not worrying about the limits. If we choose our u properly the other stuff, the x in this case, will disappear:

$$I = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2x} = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du.$$

Now this is a new integral. It is not in the tables but yields to a manipulation

$$\frac{1}{\sqrt{u}} = \frac{1}{u^{1/2}} = u^{-1/2},$$

so

$$I = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} = u^{1/2}.$$

Now put this back in terms of x and we have

$$I = \sqrt{x^2+1}.$$

This is the anti-derivative so to calculate the original integral we plug in the top limit minus the bottom limit

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx = \left[\sqrt{x^2+1}\right]_0^{\sqrt{3}} = \sqrt{3+1} - \sqrt{0+1} = 2-1 = 1.$$

The following two are left as an exercise.

2. Integrate:

$$I = \int x^2 \sec^2(x^3+1)\,dx$$

Hint: Let $u = x^3 + 1$.

3. Evaluate:

$$\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

Hint: Let $u = \sqrt{x}$.

2 Integration by Parts

The chain rule for differentiation leads to the substitution method for integration. The product rule for differentiation leads to a new integration technique: *integration by parts*:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}.$$

Now integrate both sides with respect to x:

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

Integrating, we get the integration by parts formula:

$$\int u\,dv = uv - \int v\,du.$$

To use this formula to evaluate an integral, you must make a double substitution: choose u and dv so that $u\,dv$ equals the integrand, then apply the formula. (Clearly this is more complicated than the ordinary substitution method, which you would normally try first.)

It can help if one follows the LIATE guideline in choosing u. The reason for this is once you choose u, dv is determined and the LIATE u.

2.0.1 Examples

Evaluate each of the following:

1. $I = \int x \ln x \, dx$. You may assume the derivative of $\ln x$ is $1/x$:

   Solution: Choose $u = \ln x$ by LIATE and therefore $dv = x\,dx$. Now we need du and v. We get du by differentiating and v by integrating:

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x},$$

   and

$$v = \int dv = \int x\,dx = \frac{x^2}{2} = \frac{1}{2}x^2.$$

   Now we use the formula

$$\begin{aligned}\int x \ln x\,dx &= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x} \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{x^2}{2} \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C.\end{aligned}$$

   Check this answer by differentiating it—observe that this needs the product rule, as one would expect. This example is typical: one does not need to insert "$+C$" at the early stage of going from dv to v, but it is needed when the final integral is evaluated.

   Sometimes one needs to apply integration by parts more than once, as the next example illustrates.

2. $I = \int x^2 e^x \, dx$.

   Solution: Let $u = x^2$, $dv = e^x\,dx$ by LIATE. We get du by differentiating u and v by integrating dv:

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx,$$

   and

$$v = \int dv = \int e^x\,dx = e^x.$$

   Therefore we have

$$\begin{aligned}I &= x^2 e^x - \int e^x (2x\,dx) \\ &= x^2 e^x - 2\int x e^x\,dx.\end{aligned}$$

   We do not know immediately how to evaluate $\int x e^x\,dx$. Has the integration by parts failed? No, because we have replaced $\int x^2 e^x\,dx$ by the simpler integral $\int x e^x\,dx$. Thus continue down the same road by applying integration by parts to this new integral, hoping to simplify it still further: set $J = \int xe^x \, dx$. Let $u = x, \ dv = e^x \, dx$ so we have $du = dx$ and $v = e^x$ for a second integration by parts:

$$\begin{aligned} J &= xe^x - \int e^x \, dx \\ &= xe^x - e^x. \end{aligned}$$

Substituting this formula into I gives

$$\begin{aligned} I &= x^2 e^x - 2(xe^x - e^x) \\ &= x^2 e^2 - 2xe^x + 2e^x + C. \end{aligned}$$

Integration by parts, which comes from the product rule, is usually applied to integrands that are products of different types of functions. Our three examples above are such products: a polynomial times a log function, a polynomial times a trigonometric function, and a polynomial times an exponential function. But sometimes the product nature of the integrand is not immediately apparent, as in the next example.

3. $I = \displaystyle\int \arccos x \, dx$. Use the fact that $\dfrac{d}{dx} \arccos(x) = -\dfrac{1}{\sqrt{1-x^2}}$.

Solution: LIATE tells us to choose $u = \arccos(x)$ and let $dv = dx$. Hence we find du and v:

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}} \Rightarrow du = -\frac{1}{\sqrt{1-x^2}} \, dx,$$

and $v = x$. Now apply the formula

$$\begin{aligned} I &= \arccos(x) \cdot x - \int x \left(-\frac{1}{\sqrt{1-x^2}} \, dx \right) \\ &= x \arccos x + \underbrace{\int \frac{x}{\sqrt{1-x^2}} \, dx}_{=:J} \end{aligned}$$

Now for J, let $w = 1 - x^2$ (function-derivative) and do a substitution:

$$\frac{dw}{dx} = -2x \Rightarrow dw = -2x \, dx \Rightarrow dx = -\frac{dx}{2x}.$$

Now putting J back together

$$\begin{aligned} J &= \int \frac{x}{\sqrt{w}} \left(-\frac{dx}{2x} \right) \\ &= -\frac{1}{2} \int \frac{1}{\sqrt{w}} \, dw \\ &= -\frac{1}{2} \int w^{-1/2} \, dw \\ &= -\frac{1}{2} \frac{w^{1/2}}{1/2} \\ &= -w^{1/2} = -(x^2+1)^{1/2} = -\sqrt{1-x^2} \end{aligned}$$

Hence we have

$$I = x \arccos x + J = x \arccos x - \sqrt{1-x^2} + C.$$

One has to choose u and dv correctly for the method to work. In the last example, if we had taken $u = 1$ and $dv = \arccos(x)\,\mathrm{dx}$, we would have been stuck because to continue we need to know v, and finding this is the same problem as evaluating the original integral, so we cannot proceed further.

Similarly, a poor choice of u and dv can make things worse instead of better. Consider $J = \int xe^x\,dx$. If we set $u = e^x,\ dv = x\,dx$, then $du = e^x\,dx,\ v = x^2/2$, so

$$J = uv - \int v\,du = \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2e^x\,dx.$$

While this equation is true, it is of no help to us since we have replaced the original integral by a more difficult one. If instead we had started by choosing $u = x$ and $dv = e^x$, then integration by parts works.

Winter 2012: Question 1 (e)

Use *integration by parts* to evaluate $\displaystyle\int_0^{\pi/2} x \sin 2x\,dx.$

[10 Marks]

Solution: Firstly we will find an anti-derivative by finding

$$I = \int x \sin 2x\,dx,$$

and worry about the limits later. Prompted to use integration by parts we choose $u = x$ by LIATE. Hence we have $dv = \sin 2x\,dx$. We want to use $\displaystyle\int u\,dv = uv - \int v\,du$ so will need v and du. Differentiating u and integrating dv does this for us:

$$\frac{du}{dx} = 1 \Rightarrow du = dx.$$

or just note that if $u = x$ then $du = dx$. Also

$$v = \int dv = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x.$$

Now we use the formula:

$$\begin{aligned} I = \int x \sin 2x\,dx &= x\left(-\frac{1}{2}\cos 2x\right) - \int \left(-\frac{1}{2}\cos 2x\right) dx \\ &= -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx. \end{aligned}$$

We have that $\displaystyle\int \cos 2x\,dx = \frac{1}{2}\sin 2x$ so we have

$$I = -x\frac{\cos 2x}{2} + \frac{1}{4}\sin 2x.$$

Now we must plug in the limits to evaluate the integral:

$$\begin{aligned}\int_0^{\pi/2} x\sin 2x\,dx &= \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} \\ &= \left(-\frac{1}{2}\,(\pi/2)\cos\pi + \frac{1}{4}\sin\pi\right) - \left(-0(\cdot) + \frac{1}{4}\sin 0\right) \\ &= \frac{\pi}{4} + 0 - (0+0) = \frac{\pi}{4}.\end{aligned}$$

where we used the fact that $\cos\pi = -1$, $\sin\pi = 0$ and $\sin 0 = 0$.