MS 3011: Homework

Please complete *one* of the following homeworks.

The final date for submission is 2 p.m. Friday 12 April 2013. You will have full freedom in which one you want to do and can hand up early if you want. You will be submitting to the big box at the School of Mathematical Science. If I were you I would aim to get it done and dusted early as this is creeping into your study time and is very close to the summer examinations.

Note that you are will be free to collaborate with each other and use references but this must be indicated on your hand-up in a declaration. Evidence of copying or plagiarism will result in divided marks or no marks respectively. You will not receive diminished marks for declared collaboration or referencing although I demand originality of presentation. If you have a problem interpreting any question feel free to approach me, comment on the webpage or email.

Ensure to put your name, student number, module code (MS 3011), and your declaration on your homework.

You have seven weeks notice: <u>work submitted even a minute late will be awarded a mark of zero</u>.

General Theory
Let S be a set and $f : S \to S$. Define inductively

$$x_{n+1} = f(x_n), n \in \mathbb{N} \tag{1}$$

for some choice of $x_0 \in S$. Then (S, f) is a *dynamical system* with state space S, initial state or *seed* x_0 and *iterator function* f. The state of the dynamical system at time n is given by x_n. The *orbit* of a point $x \in S$ is the set

$$\text{orb}(x) = \{f^k(x) : k \in \mathbb{N}\}, \tag{2}$$

where k is the kth iterate of f, $f^k = f \circ f \circ \cdots \circ f$ (k times). A point $x_f \in S$ is said to be *fixed* if $f(x_f) = x_f$. A point x_p is said to be a *period n point* if $f^n(x_p) = x_p$. A point has *prime period* n if x is not periodic for any $m < n$. A point $x \in S$ is said to be *eventually* 'P' if there exists a $N \in \mathbb{N}$ such that

$$\{f^N(x), f^{N+1}(x), \dots\}$$

has the property 'P'.

1. **Discrete Mathematics, Number Theory & Abstract Algebra**

(a) Let S be a non-empty *finite* set $S = \{a_1, a_2, \ldots, a_n\}$ with $n \geq 2$, and $f : S \to S$. Consider the dynamical system (S, f).

i. Prove that all elements of S are eventually periodic.

ii. Suppose that all orbits are fixed. Describe f in this case.

iii. Suppose that S contains just three elements and that $f : S \to S$ has the property that $f^2(x) = (f \circ f)(x) = x$ (i.e. every point is period[1] 2). Prove that f has a fixed point.

iv. Suppose that $f : S \to S$ is a function such that all of the points in S are periodic. Define a relation on S:

x is related to y if $x \in orb(y)$.

Prove that this is an equivalence relation on S.

(b) Let $\mathbb{N}$ be the set of natural numbers and $f : \mathbb{N} \to \mathbb{N}$ and consider the dynamical system $(\mathbb{N}, f)$.

Find a function $f : \mathbb{N} \to \mathbb{N}$ such that f has *no* eventually periodic points.

Let $g : \mathbb{N} \to \mathbb{N}$ be a function with the property that $g(n) > 1$ and suppose that $f(n) = g(n)n$. Let $\mathbb{P}$ be the set of primes. Prove that under the iterator function f, for all $n_0 > 1$, the orbit of n_0 contains no prime numbers.

(c) Use the Factor Theorem to carefully prove our *Fixed-Point Factor-Theorem*:

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a polynomial. Then for all $p \in \mathbb{N}$ and $n \in \mathbb{N}$ $f^p(x) - x$ is a factor of $f^{np}(x) - x$.

[1] such a function is known as an *involution*

2. **Probability**
Consider a set $X = \{a, b\}$. Suppose there is a particle at the point a at the time $t = 0$. Suppose at times $t = 1, 2, \ldots$ the particle either stays where it is or moves to the other point 'at random'. Let ξ_k be the *random variable* describing the position of the particle at time $k \in \mathbb{N}$. Hence the distribution of ξ_0 is given by $\mathbb{P}[\xi_0 = a] = 1$ (i.e the particle is at a at time $t = 0$ with probability 1). For p, $q \in [0, 1]$, suppose we have the following probabilities:

$$\mathbb{P}[\xi_{k+1} = b|\xi_k = a] = p; \quad \text{i.e. the probability of moving from } a \to b \text{ is given by } p$$
$$\mathbb{P}[\xi_{k+1} = a|\xi_k = b] = q; \quad \text{i.e. the probability of moving from } b \to a \text{ is given by } q$$

(a) Show that $\mathbb{P}[\xi_{k+1} = a|\xi_k = a] = 1 - p$ and $\mathbb{P}[\xi_{k+1} = b|\xi_k = b] = 1 - q$.

The motion of the particle is an example of a *Markov chain*.

(b) Draw a labeled diagram describing the Markov chain.

Note that for $(p, q) = (1, 1), (0, 0), (1, 0), (0, 1)$, there is an iterator function $f_{(p,q)} : X \to X$ such that $(X, f_{(p,q)})$ is a dynamical system such that $f^k_{(p,q)}(a) = \xi_k$.

For example, consider $(p, q) = (1, 0)$. In this case

$$\text{probability of moving from } a \to b = 1$$
$$\text{probability of moving from } b \to a = 0$$

We always begin at a so we have

$$\{\xi_0, \xi_1, \xi_2, \ldots\} = \{a, b, b, \ldots\}.$$

This is exactly the orbit of a under the iterator function:

$$\begin{cases} f_{(1,0)}(a) = b \\ f_{(1,0)}(b) = b \end{cases}$$

(c) Explain why, for $p \in (0, 1)$, there is no function $g : X \to X$ such that (X, g) is a dynamical system such that $g^k(a) = \xi_k$.

Let

$$\theta_k = (\mathbb{P}[\xi_k = a] \quad \mathbb{P}[\xi_k = b])$$

be the 1×2 matrix describing the distribution of ξ_k, i.e. the probabilities that the particle is in position a or b when $t = k$.

(d) Hence show that $\theta_0 = (1 \quad 0)$.

Define

$$S := \{(x \quad y) : x \in [0, 1] \text{ and } y \in [0, 1]\},$$

i.e. the set of probability distributions on X, so that $\theta_k \in S$. Let P be a 2×2 matrix defined as

$$P = \begin{pmatrix} 1 - p & p \\ q & 1 - q \end{pmatrix}. \tag{3}$$

(e) Show that if $f : S \to S$ is given by (matrix multiplication)

$$\begin{aligned} f((\mu_1 \;\; \mu_2)) &= (\mu_1 \;\; \mu_2) \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \\ &= (\mu_1 \;\; \mu_2) P, \end{aligned}$$

then $\theta_1 = f(\theta_0)$.

(f) Also show that if $\theta_k = (\mu_1 \;\; \mu_2)$, then $\theta_{k+1} = f(\theta_k)$.

(g) Hence show that (S, f) is a dynamical system such that $f^k(\theta_0) = \theta_k$.

3. **Differential Calculus**
Consider the set of real valued functions $f(x)$ for which the nth derivative, $f^{(n)}(x)$, exists at all points in $\mathbb{R}$; i.e. the set of infinitely differentiable functions $S = C^\infty(\mathbb{R})$. Define a function $D : S \to S$ by $D(f(x)) = f'(x)$ and consider the dynamical system $(S, f) = (C^\infty(\mathbb{R}), D)$ where the points are infinitely differentiable functions and the iterator function is differentiation.

(a) Solve the differential equation $D(f(x)) = f(x)$ to find all the fixed points of D.

(b) Show that $\exp(-x) = e^{-x}$ is a prime period 2 point of D.

(c) Show that $\sin(x)$ is a prime period 4 point of D.

(d) Let $p_0 : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree N. Show that $D^{N+1}(p_0(x)) = 0$ and hence explain why $p_0(x)$ is eventually fixed.

(e) Let $f_0(x) = \sin(x/2)$. We can show, for $k \in \mathbb{N}$, that

$$D^n(f_0(x)) = \begin{cases} \frac{1}{2^n}\sin(x/2) & \text{if } n = 4k \\ \frac{1}{2^n}\cos(x/2) & \text{if } n = 4k+1 \\ -\frac{1}{2^n}\sin(x/2) & \text{if } n = 4k+2 \\ -\frac{1}{2^n}\cos(x/2) & \text{if } n = 4k+3 \end{cases}.$$

Hence show that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} D^n(f_0(x)) = 0,$$

for all $x \in \mathbb{R}$. Is $f_0(x)$ eventually fixed?

(f) Let $g_0(x) = xe^x$ and $h_0(x) = xe^{-x}$. Find expressions for $g_n(x) = D^n(g_0(x))$ and $h_n(x) = D^n(h_0(x))$ and prove that they are correct using induction or otherwise.

(g) Let $q : \mathbb{R} \to \mathbb{R}$ be any polynomial and define $r_0(x) = q(x) + e^x$. Prove that $r_0(x)$ is eventually fixed.

(h) We can define the derivative of a *complex valued function* $F : \mathbb{C} \to \mathbb{C}$. Let S_z be the set of infinitely differentiable complex valued functions and, $D_z : S \to S$ the function given by

$$D_z(F(z)) = \frac{dF}{dz}.$$

- If we define $\exp(z) = e^z$ in a natural way, it turns out that for $a \in \mathbb{C}$ we have that

$$\frac{de^{az}}{dz} = ae^{az}.$$

- jth *roots of unity* are the complex roots of the polynomial/solutions of the equation

$$z^j - 1 \Leftrightarrow z^j = 1.$$

  There are j such roots of unity by the *Fundamental Theorem of Algebra* and we can show that they are all powers of a *primitive root* ζ_j:

$$j\text{th roots of unity} = \{1, \zeta_j, \zeta_j^2, \zeta_j^3, \dots, \zeta_j^{j-1}\}. \tag{4}$$

  $\zeta \in \mathbb{C}$ is a primitive jth root if $\zeta_j^m \neq 1$ for $m < j$.

Use these two pieces of information to construct prime period p elements in the dynamical system (S_z, D_z) for $p = 1, 2, 3, \dots$.

4. **Integral Calculus**

Consider the initial value problem:

$$\frac{dy}{dx} = 2xy, \ y(0) = 1. \tag{5}$$

(a) Solve this differential equation for $y = f(x)$ using a separation of variables.

Our analysis leads to a unique solution but it is not clear that there are not other solutions. The answer to this question is answered in the language of dynamical systems. Let us consider a general initial value problem

$$\frac{dy}{dx} = F(x, y(x)), \tag{6}$$

where $y(x_0) = y_0$. Here we can take $F(x, y(x))$ as some nice[2] formula in x and y; $F(x, y(x)) = 2xy$ as above for example. We just want to find solutions in some non-empty closed interval $[a, b]$ for example $I = [0, 1]$. Then the boundary condition $y(x_0) = y_0$ will be described in terms of some $x_0 \in [a, b]$. Suppose we take as an approximation to a solution the constant function $\varphi_0(x) = y_0$.

(b) Show that $\varphi_0(x)$ agrees with the solution at at least one point $x \in [a, b]$.

[2]what it takes for a formula to be 'nice' involves some technical stuff that need not concern us.

(c) Integrate both sides of (6) from $x_0 \to x$ to show that the differential equation can be transformed/rewritten as an *integral equation*:

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt. \tag{7}$$

To generate a better approximation to the solution $y(x)$ we plug $\varphi_0(x) = y_0$ into (7) to generate a second approximation $\varphi_1(x)$:

$$\varphi_1(x) = y_0 + \int_{x_0}^{x} F(t, \varphi_0(t))\, dt \tag{8}$$

This second approximation is then put into (7) to generate a third approximation:

$$\varphi_2(x) = y_0 + \int_{x_0}^{x} F(t, \varphi_1(t))\, dt \tag{9}$$

and the process may be iterated.

(d) Consider the initial value problem (5). Find the approximations $\varphi_0(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ to the solution.

(e) Now let $S = \{\text{Integrable Functions} : [a, b] \to \mathbb{R}\}$. Taking inspiration from (8) and (9), find an expression for an iterator function $\Gamma : S \to S$ such that $\Gamma(\varphi_n(x)) = \varphi_{n+1}(x)$.

(f) Suppose that $\varphi(x)$ is a fixed point of the dynamical system (S, Γ). Explain why φ solves the initial value problem (6).

We can show that in fact if $F(x, y)$ is 'nice' enough that Γ has fixed points so that (6) has at least one solution (existence). If $F(x, y)$ is 'nicer' again then we can show that Γ has a *unique* fixed point and so (6) has a unique solution. This has probably been the case in all the differential equations which you looked at in MS2002.

5. **Linear Algebra** Consider the set $S = M_2(\mathbb{C})$ of 2×2 matrices with complex valued entries. Let A be an element of S and consider the function $\mathcal{M}_A : S \to S$ given by

$$\mathcal{M}_A(X) = AX, \tag{10}$$

where the product is matrix multiplication (the $\mathcal{M}$ stands form $\mathcal{M}$ultiplication). We consider the family of dynamical systems $\{(\mathcal{M}_A, S) : A \in M_2(\mathbb{C})\}$ generated by the family of mappings $A \mapsto \mathcal{M}_A$.

(a) Find a matrix X_0 that is a fixed point for all of these dynamical systems.

(b) Find a matrix A such that all matrices are fixed points under the iterator function $\mathcal{M}_A$.

(c) Show that all $X_0 \in M_2(\mathbb{C})$ are period 2 points for $\mathcal{M}_A$ where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find an example of an X_0 such that X_0 is prime period 2.

(d) Prove that all matrices are period 4 points for

$$A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(e)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The *adjoint* or *conjugate transpose* of A is given by

$$A^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}.$$

If a matrix A has the property that $A^* = A$ and all of the *eigenvalues* of A are positive real numbers then A is said to be *positive*. In this case there is a unique *positive* matrix $\sqrt{A}$ such that $(\sqrt{A})^2 = A$. Show that if X_0 is a fixed point of the iterator function $\mathcal{M}_A$ (A positive), that X_0 is a period 4 point of $\mathcal{M}_{\sqrt{\sqrt{A}}}$.

(f) Show that all matrices are points are eventually zero for $\mathcal{M}_A$ for

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

and eventually fixed for

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

(g) Suppose that A is a diagonal matrix with diagonal entries μ and λ such that $|\mu| < 1$ and $|\lambda| < 1$. Describe the limiting behaviour for any $X_0 \in M_2(\mathbb{C})$.

(h) Show that if A is invertible and X_0 is invertible then the orbit of X_0 contains invertible matrices only.

(i) Suppose that A is invertible and suppose that

$$Y = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

is the 100th iterate of $\mathcal{M}_A$, $\mathcal{M}_A^{100}(X_0)$. For how many seeds $X_0 \in M_{2\times 2}(\mathbb{C})$ is $Y = \mathcal{M}_A^{100}(X_0)$?

(j) Suppose that $P^N = I_2$ for some $N \in \mathbb{N}$ and let $A = XPX^{-1}$. Show that X is a period N point of $\mathcal{M}_{XPX^{-1}}$.

6. **Complex Numbers**
The aim of this homework is to answer past Leaving Cert Higher Level complex number questions using the geometric techniques developed in MS3011. This means understanding how to add complex numbers geometrically, multiply complex numbers geometrically, take roots geometrically, do conjugates geometrically, etc. The Marking Schemes and hence solutions of these questions are to be found online at examinations.ie. However these solutions are *algebraic solutions* — without pictures. In this homework, you may present the algebraic solutions if you want but *you must describe the solutions geometrically*.

All of these may be found at examinations.ie. All are Higher Level.

(a) Question 3 from 2012 Project Maths Paper I
http://examinations.ie/archive/exampapers/2012/LC003ALP130EV.pdf

(b) Question 3 (c) from 2012 Maths Paper I
http://examinations.ie/archive/exampapers/2012/LC003ALP100EV.pdf

(c) Question 2 from 2011 Project Maths Paper I
http://examinations.ie/archive/exampapers/2011/LC003ALP130EV.pdf

(d) Question 3 (b), (c) from 2010 Maths Paper I
http://examinations.ie/archive/exampapers/2010/LC003ALP100EV.pdf

(e) Question 3 (a) from 2009 Maths Paper I
http://examinations.ie/archive/exampapers/2009/LC003ALP100EV.pdf

(f) Question 3 (c) (without induction) from 2008 Maths Paper I
http://examinations.ie/archive/exampapers/2008/LC003ALP100EV.pdf

(g) Question 3 (b) (i) from 2007 Maths Paper I
http://examinations.ie/archive/exampapers/2007/LC003ALP100EV.pdf

(h) Question 3 (c) from 2006 Maths Paper I
http://examinations.ie/archive/exampapers/2006/LC003ALP100EV.pdf

(i) Question 3 (b) from 2004 Maths Paper I[3]
http://examinations.ie/archive/exampapers/2004/LC003ALP100EV.pdf

[3] This is the paper I sat... I got 100% ☺

Example: Show that w^7 is a real number where $w = \cos \pi/7 + i \sin \pi/7$.

Solution: w is a complex number of modulus one and hence lies on the unit circle. Its argument is $\pi/7$ which is an argument between 0 and $\pi/2$ as shown:

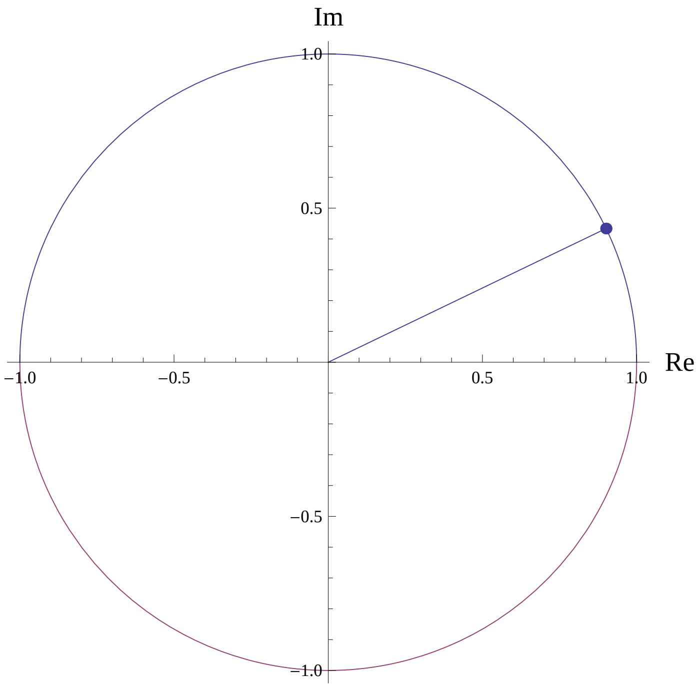

Figure 1: z on the complex plane.

Now when we multiply z_1 by z_2 we start with z_1 and stretch it by the modulus of z_2. Then we rotate anti-clockwise through the argument of z_2 and we end up with

$$|z_1 z_2| = |z_1||z_2| \tag{11}$$

$$\arg\, z_1 z_2 = \arg\, z_1 + \arg\, z_2 \tag{12}$$

Now if we do w^7 be are multiplying w by itself seven times. The modulus of w is one so there is no stretching (multiplying by one means no stretch). However when we multiply by w seven times be will rotate through an angle of $\pi/7$ seven times so that

$$\arg\, w^7 = \pi.$$

Now an argument of π puts w^7 on the negative real line. Hence w^7 is real.

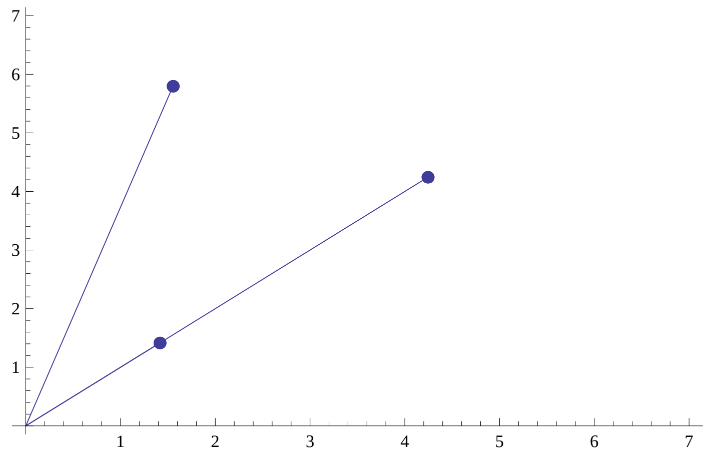

Figure 2: Starting with z_1, stretch by multiplying by $|z_2|$ and rotate through the argument of z_2

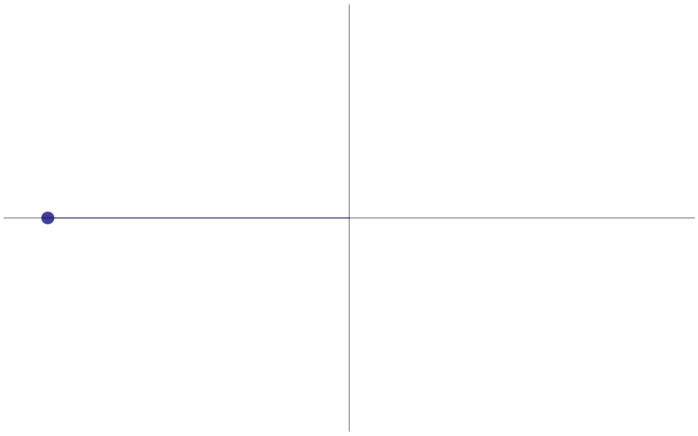

Figure 3: If $\arg z = \pi$ then z lies on the negative real line.