4.7 Applied Optimisation Problems

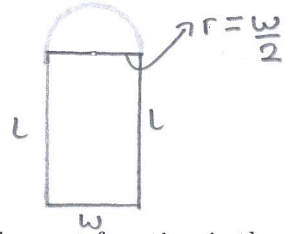
The techniques we have learned in this course can be used to solve problems that involve some sort of optimisation. The basic strategy for such a problem is the following:

1. Identify the *cost function* to be maximised/minimised, and all of the variables it depends on. Usually you will want to draw a diagram to figure this out.
2. Use (hopefully obvious) constraints to eliminate all but one of the variables, and determine the range of values this variable should take to be meaningful.
3. Find the absolute maximum and/or minimum of the cost function by using either the closed interval method, the second derivative test or the first derivative test — ensuring that we check the endpoints of the domain.

Example (Harder)

A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 10 m, find the dimensions of the window so that the greatest possible amount of light is admitted. What is the area of the window when this achieved?

Solution: First things first we need a diagram:



$$A_\Delta = \frac{\pi r^2}{2}$$

$$l_\Delta = \pi r$$

Now the cost function is the area. As area is additive, the cost function is given by:

$$C(l,w) = lw + \frac{\pi}{2}\frac{w^2}{4} = lw + \frac{\pi}{8}w^2$$

Now we use the constraint on the variables to eliminate all but one:

$$2l + w + \frac{\pi w}{2} = 10 \Rightarrow 2l = 10 - w\left(1+\frac{\pi}{2}\right)$$

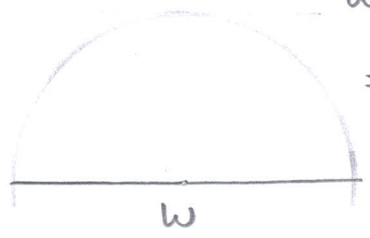
$$\Rightarrow l = 5 - \frac{w}{2}(1+\pi/2)$$

So now our cost function is:

$$C(l,w) = \left(5 - \frac{w}{2}(1+\pi/2)\right)w + \frac{\pi}{8}w^2$$

$$= 5w - \frac{w^2}{2}(1+\pi/2) + \frac{\pi}{8}w^2$$

Finding a domain for w is not that simple: it can certainly tend to 0 but not equal 0 as we wouldn't have a window then. For $l = 0$:



$$w + \frac{\pi w}{2} = 10$$

$$\Rightarrow w(1+\pi/2) = 10$$

$$\Rightarrow w_{max} = \frac{10}{1+\pi/2}$$

So we are looking for solutions with $w \in (0, 10/(1+\pi/2)] \approx (0,4]$. As $C(w)$ is differentiable I will try and use the second derivative test:

$$\frac{dC}{dw} = 5 - \frac{\cancel{2}w(1+\pi/2)}{\cancel{2}} + \frac{\cancel{2}\pi w}{\cancel{8}4} = 0$$

$$\Rightarrow 20 - w(4+2\pi) + \pi w = 0$$

$$\Rightarrow w(4+\pi) = 20 \Rightarrow w = \frac{20}{4+\pi}$$

Note however that $C(w)$ is an '$\bigcap$' quadratic and hence has a unique maximum and sure enough $20/(4+\pi) \in (0, 10/(1+\pi/2)]$. Hence these dimensions give the maximum area:

$$w = \frac{20}{\pi+4}$$

$$\Rightarrow l = 5 - \frac{10(1+\pi/2)}{\pi+4} = \frac{1}{\pi+4}(20+5\pi-10-5\pi)$$

$$= \frac{10}{\pi+4}$$

All that remains is to find this maximum area. [Ex:] — simplify and leave in terms of π; units? •

Example

Find the point on the parabola $y = x^2$ that is closest to the point $(3,0)$.
Solution: First off a diagram:

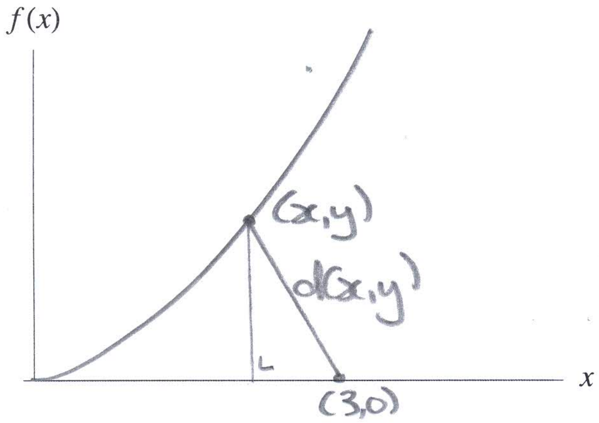


Figure 4.11: We want to minimise the distance $d(x,y)$.

We can use a distance formula to find $d(x, y)$ — or equivalently (?), use Pythagoras:

$$d^2(x,y) = (x-3)^2 + y^2$$

Now we want to minimise $d(x, y)$ but luckily, as $q(x) = x^2$ (nothing to do with this problem (?)) is increasing, and $d(x, y)$ is positive, minimising $d^2(x, y)$ is the same as minimising $d(x, y)$ [Ex:]. Is there a relationship between x and y? What does it take to be on a curve?

$$y = x^2 \Rightarrow d^2(x) = (x-3)^2 + x^4$$

The domain here is pretty much the entire real line and as $x \to \pm\infty$, $d^2 \to \infty$ so we have no problems there. As d^2 is continuous, I might use the first derivative test[8]. First of the derivative of d^2:

$$\frac{dd^2}{dx} = 2(x-3) + 4x^3 = 2x - 6 + 4x^3 = 2(2x^3 + x - 3)$$

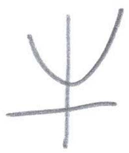
As d^2 is differentiable, as a polynomial, the only split points are where the derivative vanishes:

$$c(x) = 2x^3 + x - 3 = 0$$

How do we find the roots of cubics?

$c(1) = 2 + 1 - 3 = 0 \Rightarrow x - 1$ a root

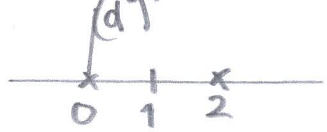
$$\begin{array}{r l} & 2x^2 + 2x + 3 \\ x-1 & \overline{)\,2x^3 + 0x^2 + x - 3} \\ & \underline{-2x^3 + 2x^2} \\ & 2x^2 + x \\ & \underline{-2x^2 + 2x} \\ & 3x - 3 \end{array} \quad \Rightarrow (d^2)' = (x-1)(2x^2 + 2x + 3)$$



There aren't obvious factors of $2x^2 + 2x + 3$ so we could consult the formula. However, this '⋃' quadratic looks a bit 'positive' to me — suggesting complex roots so I calculate $b^2 - 4ac$:

$$4 - 4(2)(3) < 0 \Rightarrow \text{no } \underline{\text{real}} \text{ roots.}$$

Sure enough $x = 1$ is the only split point. Hence

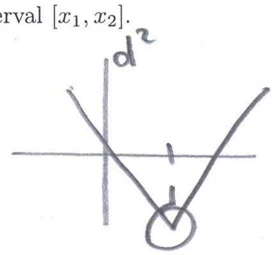


$(d^2)'(0) = (-)(+) < 0 \Rightarrow$ decr.

$(d^2)'(2) = (+)(+) > 0 \Rightarrow$ incr

Yielding schematic:

[8] we could also use the second derivative test — but not the closed interval method — unless we were able to justify restricting to some interval $[x_1, x_2]$.



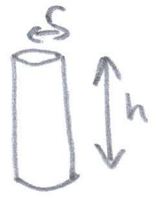
Hence the minimum is at $x = 1$: the closest point on the curve to $(3, 0)$ is $(1, 1)$ •

Summer 2011 Question 1 (d)

A cylinder is to be made such that the sum of its radius r, and its height, h, is 6 cm. Find, in terms of π, the maximum possible volume of such a cylinder.

Solution: The volume of a cylindar is given by $V(r, h) = \pi r^2 h$. However the radius and height are related by $r + h = 6$. Hence $h = 6 - r$ so we write:

$$V(r) = \pi r^2(6-r) = 6\pi r^2 - \pi r^3$$



$r \in (0,6)$

Using the Closed Interval Method.

V is differentiable so extrema at $0, 6$ and $V' = 0$

$$\frac{dV}{dr} = 12\pi r - 3\pi r^2 = 3\pi r(4-r) = 0 \Rightarrow r = 4$$

$$V(0) = 0, \quad V(6) = 0, \quad V(4) = 6\pi(16) - 64\pi = 32\pi$$

Autumn 2011 Question 1 (d)

The total length of the 12 edges of a rectangular box is 64 cm and the total surface area is 104 cm^2. The volume of the box is V cm^3. If x, y and z are the length, breadth and height show that:

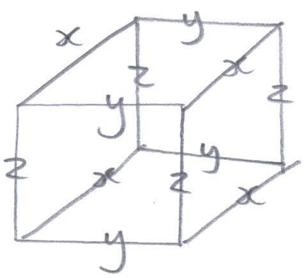
(i) $y + z = 16 - x$.

(ii) $yz = 52 - x(16 - x)$.

(iii) $V = 52x - 16x^2 + x^3$.

Hence find the maximum volume of the box.

Solution: First we draw a picture:



(i) We know that the total length of edges is 64 cm:

$$4x + 4y + 4z = 64 \text{ cm}$$
$$x + y + z = 16 \text{ cm}$$
$$\Rightarrow y + z = 16 - x \text{ cm}$$

(ii) We know that the total area is 104 cm^2:

$$2xy + 2xz + 2yz = 104 \text{ cm}^2$$
$$\Rightarrow xy + xz + yz = 52$$
$$\Rightarrow x(y+z) + yz = 52$$
$$\Rightarrow yz = 52 - x(16-x)$$

(iii) The volume, V, is the length by the width by the breadth:

$$V = xyz$$
$$= x(52 - 16x + x^2) = x^3 - 16x^2 + 52x$$

Hence we want to maximise the function $V(x)$ in the region $x \in (0, 16)$. To find the maxima/ minima, differentiate:

$$\frac{dV}{dx} = 3x^2 - 32x + 52$$

We are interested in the points where this is zero:

$$3x^2 - 32x + 52 = 0$$

This hasn't got factors so we use the quadratic formula:

$$x_\pm = \frac{32 \pm \sqrt{32^2 - 4(3)(52)}}{6} = \frac{32 \pm \sqrt{400}}{6}$$
$$= \frac{32 \pm 20}{6} = 2 \text{ or } \frac{52}{6} = \frac{26}{3}$$

To know which is which we use the second derivative test. The second derivative is:

$$\frac{d^2V}{dx^2} = 6x - 32$$

So testing the stationary points:

$$\left.\frac{d^2V}{dx^2}\right|_2 = 12 - 32 < 0 \Rightarrow \max \Rightarrow V_{max} = 8 - 64 + 104$$
$$= 48 \text{cm}^3$$

$$\left.\frac{d^2V}{dx^2}\right|_{\frac{26}{3}} = 52 - 32 > 0 \Rightarrow \min$$

$$V(0) = 0$$

Remark

How do we know that V doesn't have another max – which is not local

[Ex]