1. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial mapping.

6 (a) Suppose that x_0 is a prime-period-p point of f. Show that that x_0 is a period-n point of f **if and only if** $n = pq$ for some $q \in \mathbb{N}$.

8 (b) Using the factor theorem, deduce that $f(x) - x$ is a factor of $f^2(x) - x$.

5 (c) Prove that if x is a period-n point of f, then each of the points in the orbit of x also has period-n.

4 (d) Use parts (b) and (c) to find the fixed points and prime-period-2 points of $g(x) = x^2 + x - 2$. Hence, describe separately the orbits of each of the period-2 points of g.

2 (e) Use part (a) to show that the prime-period-2 points of g are period-50 but not period-51.

2. Consider the *Doubling Mapping* given by

$$D(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2x - 1 & \text{if } 1/2 < x \le 1 \end{cases}$$

6 (a) Where x has the binary representation

$$x = 0.a_1a_2a_3a_4a_5a_6a_7a_8 \cdots ,$$

find expressions for $D(x)$ and $D^5(x)$. Hence find points $y, z \in [0, 1]$ such that y and z agree to 5 binary digits but $D^N(y)$ and $D^N(z)$ differ in the first binary digit for some $N \in \mathbb{N}$.

3 (b) Describe the period-5 points of D. Let $w \in [0, 1]$ have a binary representation beginning $w = 0.01001\ldots$. Find a period-5 point γ of D such that w and γ agree to five binary digits.

3 (c) Find a $\delta \in [0, 1]$ such that there are iterates of δ, $D^{n_1}(\delta), D^{n_2}(\delta), D^{n_3}(\delta)$, with $n_1, n_2, n_3 \in \mathbb{N}_0$, that agree with $0.111\ldots$, $0.101\ldots$ and $0.010\ldots$ to three binary digits.

13 (d) Consider the statement:

> *These arguments could be generalised to show that $D(x)$ is a* chaotic mapping.

Do you agree or disagree with this statement? Justify your answer.

$$
\begin{array}{r}
x + 2 \\
x^2 + 0x - 2 \,\big)\, x^3 + 2x^2 - 2x - 4 \\
x^3 + 0x^2 - 2x \\
\hline
2x^2 - 4
\end{array}
$$ [1]

$\Rightarrow f^2(x) - x = (x^2-2)x(x+2)$ [1]

The prime period-2 points are 0 and -2 [1] 4

The fixed points are $\pm\sqrt{2}$. [1]

(c) [Bookwork]

5 Let x be period n and have orbit

$x_0, x_1, x_2, \ldots, x_j, \ldots, x_{n-1}, x_0, \ldots$ [1]

We claim that $f^n(x_j) = x_j$. [1] Write $n = (n-j)+j$ [1]

$\Rightarrow f^n(x_j) = f^{(n-j)+j}(x_j) = f^j(f^{n-j}(x_j))$ [1]

$= f^j(x_0) = x_j$ [1] $\square$

(d) [Familiar]

$\mathrm{orb}(\sqrt{2}) = \{\sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots\}$ [1]

$\mathrm{orb}(-\sqrt{2}) = \{-\sqrt{2}, -\sqrt{2}, -\sqrt{2}, \ldots\}$ [1]

$\mathrm{orb}(0) = \{0, -2, 0, -2, 0, -2, \ldots\}$ [1]

$\mathrm{orb}(-2) = \{-2, 0, -2, 0, -2, \ldots\}$ [1]

(e) [Unseen]

Using part (a). $50 = 25 \times 2$ so the prime-period 2 points are period 50. [1]

$51 \neq 2q$ so " " " " " " not period-51 [1]