1. (a) Find the solution set of the inequality

$$\left|\frac{x-3}{2x+1}\right| \leq 3;\ x \neq -1/2.$$ 7 6

(b) Use the Calculus of Limits to evaluate:

$$\lim_{x\to -2} \frac{x^3+3x^2+3x+2}{x+2}.$$ 4

(c) Differentiate each of the following functions:

$$f(x) = (x^3 - \cos x)^3 \text{ and } g(x) = e^{x^2+1}.$$

Which of the following functions are differentiable (everywhere)?

(i) $f(x) + g(x)$ (iv) $\dfrac{g(x)}{f(x)}$

(ii) $f(x)g(x)$ (v) $\dfrac{f(x)}{g(x)}$

(iii) $[g(x)]^{100}$ (vi) $f(g(x))$ 8 9

Justify your answer briefly in each case, stating clearly which rules/theorems you use. You may assume that both $f(x)$ and $g(x)$ are differentiable.

(d) A cylinder is to be made such that the sum of its radius r, and its height, h, is 10 cm. Find, in terms of π, the maximum possible volume of such a cylinder. 6

2. Using the Closed Interval Method, or otherwise, find a positive upper bound $M \in \mathbb{R}$ such that,

$$\left|\frac{x+4}{x+1}\right| < M.$$

for all $x \in [2, 4]$.

Hence use the ε-δ definition of a limit to prove that:

$$\lim_{x\to 3} \frac{x^2+6x-7}{x+1} = 5.$$

3. (a) Let a and $b \in \mathbb{R}$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by: 12

$$f(x) = \begin{cases} x + a & \text{if } x < 0, \\ b\,e^x & \text{if } x \geq 0. \end{cases}$$

For what value(s) of a and b is f continuous?

Suppose that $a = 0$ and $b = 1$. Is f differentiable at $x = 0$? Justify your answer.

(b) Consider the implicitly defined curve 13

$$\cos x + xy + y^2 = 15\pi^2/16.$$

Find the equation of the tangent to the curve at the point $(\pi/2, 3\pi/4)$.

4. (a) State the Intermediate Value Theorem.

   (b) Consider the function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2 - 2x & \text{if } 1/2 < x \leq 1 \end{cases}.$$

   Does f satisfy the hypotheses of Rolle's Theorem? If yes, please find all points that satisfy the conclusions of the theorem. Otherwise explain why f does not satisfy the hypotheses.

   (c) Consider the function $g : [1, 2] \to \mathbb{R}$

$$g(x) = x^3 + x^2 + x + 1.$$

   Use either the Mean Value Theorem or the Intermediate Value Theorem to prove that there there is a point $c \in (1, 2)$ such that

$$g'(c) = 11.$$

5. Let $f : [-3, 3] \to \mathbb{R}$ be defined **on the closed interval** $[-3, 3]$ by:

$$f(x) = \frac{4x}{x^2 + 4}.$$

   (i) For what values of $x \in [-3, 3]$ is this function defined?

   (ii) Find the roots of $f(x)$, if any.

   (iii) Find the value of $f(3)$. Determine if f is odd, even or neither. Hence write down the value of $f(-3)$.

   (iv) Using the method of split points or otherwise, find the intervals where f is increasing/decreasing. Hence or otherwise find the local maxima and minima of $f(x)$.

   (v) Find $f''(x)$. Hence or otherwise find the intervals of $[-3, 3]$ where f is concave up/concave down.

   Use **all** of this information to sketch the graph of $y = f(x)$.

END OF EXAM