Here are some standard indefinite integrals, each of which can be verified by differentiating the right-hand side. The ones of any use will be in mathematical tables (attached at the end of this set of notes) and don't need to be learnt off (note that we don't include the exponential and logarithmic functions — they are defined properly in a later chapter):

2.1.7 Proposition

$$\int \sin x\,dx = -\cos x + C.$$
$$\int \cos x\,dx = \sin x + C.$$
$$\int \sec^2 x\,dx = \tan x + C.$$
$$\int \csc^2 x\,dx = -\cot x + C.$$
$$\int \sec x \tan x\,dx = \sec x + C.$$
$$\int \csc x \cot x\,dx = -\csc x + C.$$
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin(x/a) + C.$$
$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan(x/a) + C.$$

Summer 2011 Question 3 (b) (i)

Evaluate

$$\int \left(\frac{1}{x^4} + \frac{1}{\sqrt[4]{x}}\right) dx.$$

Solution: First we rewrite the integrand so that we can use the power rule:

$$I = \int x^{-4}\,dx + \underbrace{\int \frac{1}{x^{1/4}}\,dx} \;\; = \int x^{-1/4}\,dx$$

Now we can use the power rule to write:

$$I = \frac{x^{-3}}{-3} + \frac{x^{3/4}}{3/4} + C$$

$$= -\frac{1}{3}x^{-3} + \frac{4}{3}x^{3/4} + C.$$

Exercises Evaluate the following indefinite integrals.

1. $\int \sqrt{3t}\,dt$ Ans: $\frac{2}{\sqrt{3}}t^{3/2} + C$

2. $\int (2-\sqrt{x})^2\,dx$. (Hint: multiply out) Ans: $4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2 + C$

3. $\int (2x^3 - 3x^2 + 4x)\,dx$ Ans: $\frac{1}{2}x^4 - x^3 + 2x^2 + C$

4. $\int \sqrt[3]{x^2}\,dx$ Ans: $\frac{3}{5}x^{5/3} + C$

5. $\int \left(\frac{1}{x^4} + \frac{1}{\sqrt[4]{x}}\right)dx$ Ans: $-\frac{1}{3x^3} + \frac{4}{3}x^{3/4} + C$

6. $\int (\sin\theta + \cos\theta)\,d\theta$ Ans: $-\cos\theta + \sin\theta + C$

7. $\int (s+1)^2\,ds$ Ans: $\frac{1}{3}s^3 + s^2 + s + C$

8. $\int \frac{x^2+1}{x^2}\,dx$ Ans: $x - \frac{1}{x} + C$

9. $\int \frac{t^2-1}{\sqrt{t}}\,dt$ Ans: $\frac{2}{5}t^{5/2} - 2\sqrt{t} + C$

2.2 The Substitution Method

A table of indefinite integrals is helpful but no table can cover all possible integrands $f(x)$. Nevertheless, the usefulness of such a table is greatly increased by the following technique, which can transform an unfamiliar integrand into a recognizable form.

The basic strategy for integration is as follows:

1. Direct — straight from the tables.
2. Manipulation — use trigonometric identities or rewrite the integrand.
3. Substitution — the method developed in this section.
4. Integration by Parts — a technique using the Product Rule for Differentiation.

The substitution method is a technique of integration that comes from the chain rule. Suppose that $f(x)$ is a function with anti-derivative $F(x)$; i.e. $F'(x) = f(x)$. Now consider, for some other function $F(g(x))$ and differentiate with the Chain Rule:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

$$\Rightarrow \int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

This seems to look like a particularly difficult pattern to spot. However, if we let $u = g(x)$ we can make the following (justified by the above comments) calculation, starting with:

$$\int f(g(x))g'(x)\,dx,$$

$$\frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)}$$

$$I = \int f(u)\cancel{g'(x)}\frac{du}{\cancel{g'(x)}} = \int f(u)\,du$$

Now u is just a dummy variable so hopefully we can integrate away with respect to u. The key here is that, starting from the complicated integral $\int f\,(g(x))\,g'(x)\,dx$, find the a function-(multiple of the)derivative pattern and *choose* the substitution $u =$ function. Then everything should hopefully work out. Note we have *not* evaluated the integral; we have replaced it by a simpler integral.

Oral Exercise

Spot the function-derivative pattern and state what the substitution should be:

$\int \sin 2x\,dx$	$\int \frac{\cos x}{1+\sin x}\,dx$	$\int \frac{\sin x}{1+\cos x}\,dx[1ex]$
$\int \sin x\sqrt{1+\cos x}\,dx$	$\int 3x^2\sin(x^3)\,dx$	$\int x\sqrt{x^2+9}\,dx$
$\int x(1+x^2)^3\,dx$	$\int \frac{2x+1}{x^2+x+1}\,dx$	$\int \sin x\cos^3 x\,dx[1ex]$
$\int \frac{2x}{\sqrt{1+x^2}}\,dx$	$\int \frac{2x-4}{x^2-4x+29}\,dx$	$\int \frac{x+4}{x^2+8x+1}\,dx[1ex]$
$\int \frac{x}{x^2+4}\,dx$	$\int (x+3)\sec^2(x(x+6))\,dx$	$\int \frac{x-2}{x^2-4x+5}\,dx$

Remarks

1. With indefinite integrals, always transform back to the original variable after the integration is over.

2. One usually cannot integrate a mixture of variables such as $\int x^2\,du$ or $\int \sin\theta\,dx$. Thus when using the substitution method, be careful to transform all variables from x to u — and do this using the equations

$$g(x) = u \text{ and } dx = \frac{du}{g'(x)}.$$

Sometimes after a substitution both an x and u are present — can you do a *back-substitution*: write x in terms of u?

3. When the integrand contains different types of functions, the selection of the expression to be substituted by u is often clarified by invoking the LIATE rule-of-thumb: in order of preference, the type of function to set equal to u is

   L ogarithmic — more on this later.

   I nverse Trigonometric — arcsin/arctan.

   A lgebraic — polynomials.

   T rigonometric — sin/cos/tan.

   E xponenetial — more on this later.

   The reason this seems to work shall be clarified in a later section (Integration by Parts).

4. When choosing your substitution $u = g(x)$, the only fixed rule is that $kg'(x)$ (some constant multiple of the derivative) must be a factor of integrand. Usually $g(x)$ appears "inside another a function".

5. If a sum or difference of terms (e.g., $x^2 + x - 6$) appears in the integrand, never break them up when choosing u (i.e., don't try $u = x^2$ or $u = x^2 - 6$; but $u = x^2 + x - 6$ may work).

Examples

Evaluate each of the following:

1. $\displaystyle\int 3x^2\sqrt{x^3+9}\,dx.$

   Solution: The integrand is not in the tables and has no obvious manipulation. We try a substitution. Function-Derivative pattern:

Let $u = x^3 + 9 \Rightarrow \dfrac{du}{dx} = 3x^2 \Rightarrow dx = \dfrac{du}{3x^2}$

$$\Rightarrow I = \int \cancel{3x^2}\sqrt{u}\,\frac{du}{\cancel{3x^2}} = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}(x^3+9)^{3/2} + C$$

2. $\displaystyle\int \sqrt{2x+1}\,dx.$

   Solution: The integrand is not in the tables and has no obvious manipulation. We try a substitution. Function-Derivative pattern:

Let $u = 2x+1 \Rightarrow \dfrac{du}{dx} = 2 \Rightarrow dx = \dfrac{du}{2}$

$$I = \int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{\cancel{2}}\,\frac{u^{3/2}}{3/\cancel{2}} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

3. $\int x^5\sqrt{1+x^2}\,dx$.

Solution: The integrand is not in the table and has no obvious manipulation. We try a substitution. Function-Derivative pattern... In this case our strategy has failed. According LIATE we should choose $u = x^5$ as this has the higher degree... Try $u = 1 + x^2$.

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$I = \int x^{\cancel{5}\,4}\sqrt{u}\,\frac{du}{2\cancel{x}} = \frac{1}{2}\int x^4 . u^{1/2}\,du$$

It seems as if all is lost but in fact we can do a *back-substitution*:

$$x^2 = u-1 \Rightarrow x^4 = (u-1)^2 = (u-1)(u-1)$$
$$= u^2 - 2u + 1$$

$$\Rightarrow I = \frac{1}{2}\int (u^2-2u+1)u^{1/2}\,du = \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$$

$$= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - \frac{2u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$$

4. $\int x\sin(x^2)\,dx$.

Solution: The integrand is not in the table and has no obvious manipulation. We try a substitution. Function-Derivative pattern:

$$\text{Let } u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$I = \int \cancel{x}\sin(u)\frac{du}{2\cancel{x}} = \frac{1}{2}\int \sin(u)\,du$$

$$= -\frac{1}{2}\cos(u) + C$$

$$= -\frac{1}{2}\cos(x^2) + C.$$

Summer 2011 Question 1(b)(ii)

Evaluate

$$\int \frac{t}{\sqrt{t+3}}\,dt.$$

Solution: There is no direct integration or manipulation. By LIATE choose the complicated $u = t+3$ and look out for a back-substitution:

$du = dt \qquad t = u-3$

$$I = \int \frac{u-3}{\sqrt{u}}\,du = \int \left(u^{1/2} - 3u^{-1/2}\right) du = \frac{u^{3/2}}{3/2} - \frac{3u^{1/2}}{1/2} + C$$

$$= \frac{2}{3}(t+3)^{3/2} - 6(t+3)^{1/2} + C.$$

Exercises

Evaluate the following integrals. Note that you can differentiate your answer for each indefinite integral to check its correctness.

1. $\int 2x^2\sqrt{x^3+1}\,dx$ Ans: $\frac{4}{9}(x^3+1)^{3/2} + C$
2. $\int \sqrt{3x+4}\,dx$ Ans: $\frac{2}{9}(3x+4)^{3/2} + C$
3. $\int t(5+3t^2)^8\,dt$ Ans: $\frac{1}{54}(5+3t^2)^9 + C$
4. $\int s^2\sqrt[5]{7-4s^3}\,ds$ Ans: $-\frac{5}{72}(7-4s^3)^{6/5} + C$
5. $\int x^2\sqrt{1+x}\,dx$ Ans: $\frac{2}{7}(1+x)^{7/2} - \frac{4}{5}(1+x)^{5/2} + \frac{2}{3}(1+x)^{3/2} + C$
6. $\int \frac{t}{\sqrt{t+3}}\,dt$ Ans: $2\left[\frac{1}{3}(t+3)^{3/2} - 3(t+3)^{1/2}\right] + C$
7. $\int \frac{27r^2-1}{\sqrt[3]{r}}\,dr$
8. $\int \sqrt{1+\frac{1}{3x}}\,\frac{dx}{x^2}$ Ans: $-2\left(1+\frac{1}{3x}\right)^{3/2} + C$
9. $\int \cos 5x\,dx$ Ans: $\frac{1}{5}\sin 5x + C$

10. $\int (x^2+1)\sin(x^3+3x)\,dx$ Ans: $-\frac{1}{3}\cos(x^3+3x)+C$

11. $\int x^2\sec^2(x^3+1)\,dx$ Ans: $\frac{1}{3}\tan(x^3+1)+C$

12. $\int \sin^2 x\cos x\,dx$ Ans: $\frac{1}{3}\sin^3 x+C$

2.2.1 The substitution method in definite integrals

When evaluating a definite integral by means of a substitution, you must transform the limits of integration — alternatively you can suppress the limits and when you have integrated with respect to u, and transformed back into x, use the original limits. Either method is correct. Personally I much prefer the latter but I'll do the first example by transforming the limits.

Examples

Evaluate each of the following:

1. $\displaystyle\int_4^9 \frac{\sqrt{x}}{(30-x^{3/2})^2}\,dx.$

Solution: No direct integration or manipulation:

Let $u = 30 - x^{3/2} \Rightarrow \frac{du}{dx} = -\frac{3}{2}x^{1/2} \Rightarrow dx = -\frac{2}{3}\frac{du}{x^{1/2}}$

We must also transform the limits: $4 \to 30-(\sqrt{4})^3 = 22$

$9 \to 30 - (\sqrt{9})^3 = 3$

$$I = \int_{22}^{3} \frac{\cancel{\sqrt{x}}}{u^2}\left(\frac{-2}{3}\frac{du}{\cancel{x^{1/2}}}\right) = -\frac{2}{3}\int_{22}^{3} u^{-2}\,du$$

$$= -\frac{2}{3}\left[\frac{u^{-1}}{-1}\right]_{22}^{3} = -\frac{2}{3}\left[-\frac{1}{3}-\left(-\frac{1}{22}\right)\right]$$

$$= \frac{2}{3}\left[\frac{22}{66}-\frac{3}{66}\right] = \frac{\cancel{2}}{3}\left[\frac{19}{\cancel{66}\,33}\right] = \frac{19}{99}$$

2. $\displaystyle\int_{-1}^{3} \frac{dy}{(y+2)^3}$.
Solution: No direct integration or manipulation:

Let $u = y+2 \Rightarrow du = dy$

$$I = \int \frac{du}{u^3} = \int u^{-3}\,du = \left[\frac{u^{-2}}{-2}\right] = \left[\frac{(y+2)^{-2}}{-2}\right]_{-1}^{3}$$

$$= -\frac{1}{2}\left(\frac{1}{5^2}\right) - \left(-\frac{1}{2}\frac{1}{1}\right) = \frac{1}{2} - \frac{1}{50} = \frac{24}{50} = \frac{12}{25}$$

3. $\displaystyle\int_{0}^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.
Solution: No direct integration or manipulation. Lookout for $\sqrt{x}$ in the denominator. Recall

$$y = \sqrt{x} = x^{1/2}$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

A $\sqrt{x}$ in the denominator is a (multiple of a) derivative of $\sqrt{x}$. Hence let $u = \sqrt{x}$:

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,du$$

$$I = \int \frac{\cos(u)\, 2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}\,du = 2\int \cos(u)\,du = 2\sin(u)$$

$$= \left[2\sin(\sqrt{x})\right]_0^{\pi^2/4} = 2\sin\left(\frac{\pi}{2}\right) - 2\sin(0) = 2.$$

4. $\displaystyle\int_{0}^{3} x\sqrt{1+x}\,dx$.
Solution: No direct integration or manipulation — and seemingly no substitution. Chance our arm with the more complicated $u = 1+x$ and maybe hope for a back-substitution:

$du = dx \qquad x = u - 1$

$$I = \int (u-1)\sqrt{u}\,du = \int (u-1)u^{1/2}\,du = \int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}$$

$$= \left(\frac{2}{5}(\sqrt{4})^5 - \frac{2}{3}(\sqrt{4})^3\right) - \left(\frac{2}{5}(1)^{5/2} - \frac{2}{3}(1)^{3/2}\right)$$

$$= \frac{2(32)}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3} = \frac{3\times 64}{15} - \frac{80}{15} - \frac{6}{15} + \frac{10}{15}$$

$$= \frac{192-76}{15} = \frac{116}{15}$$

Exercises Evaluate

1. $\displaystyle\int_0^1 x(1-x^2)^5\,dx$ Ans: $\dfrac{1}{12}$.

2. $\displaystyle\int_0^{\sqrt{\pi/2}} x\cos(x^2)\,dx$ Ans: $\dfrac{1}{2}$.

3. $\displaystyle\int_1^3 \frac{t^2+9}{t^2}\,dt$ Ans: 8.

2.3 Completing the Square

The substitution method replaces complicated indefinite integrals by simpler ones, but one must then be able to evaluate those simpler integrals. This is often done by using a table of standard integrals such as the list in the mathematical tables. It is not necessary to memorize all of these, but one should recognize each one of them if it arises when attempting to integrate some function.

In a table of standard integrals, quadratic expressions always appear in one of the forms x^2+a^2, x^2-a^2 or a^2-x^2, where $a \in \mathbb{R}$ is some constant.

If you have a quadratic that is not a sum or difference of two squares (i.e., if an x term appears also) then $\boxed{\text{complete the square}}$ — that is write

$$ax^2+bx+c = \pm(px+q)^2 \pm r^2$$

for some p, q, $r \in \mathbb{R}$, where the $\pm$ depend on whether we will be integrating an integrand with x^2+a^2, x^2-a^2 or a^2-x^2.