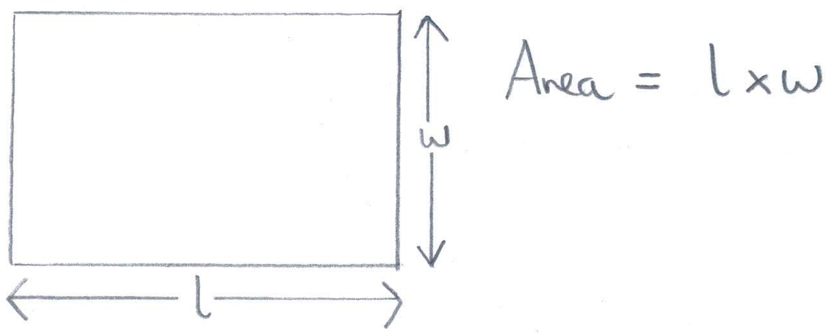
0.2 Motivation: The Problem of Measure

> *Mathematics is facts; just as houses are made of stones, so is mathematics made of facts; but a pile of stones is not a house and a collection of facts is not mathematics.*
>
> Henri Poincaré

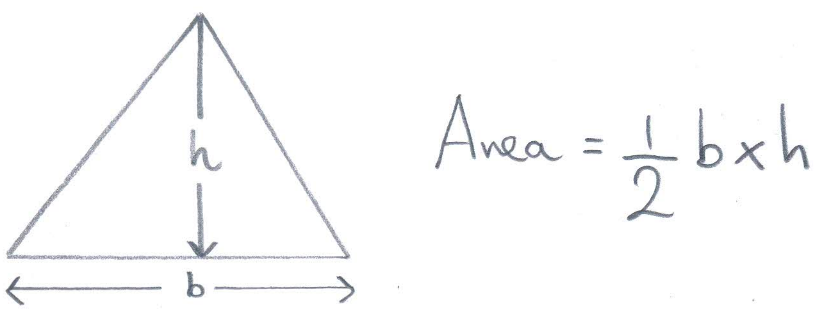
The theme of this module is arguably how to assign a size to certain sets — usually shapes and solids (you will probably disagree with this in time!). In everyday life this is usually pretty straightforward; we

- count: $\{a, b, c, \ldots, x, y, z\}$ has 26 letters.
- take measurements: length (in one dimension), area (in two dimensions), volume (in three measurements) or time;
- calculate: rates of radioactive decay.

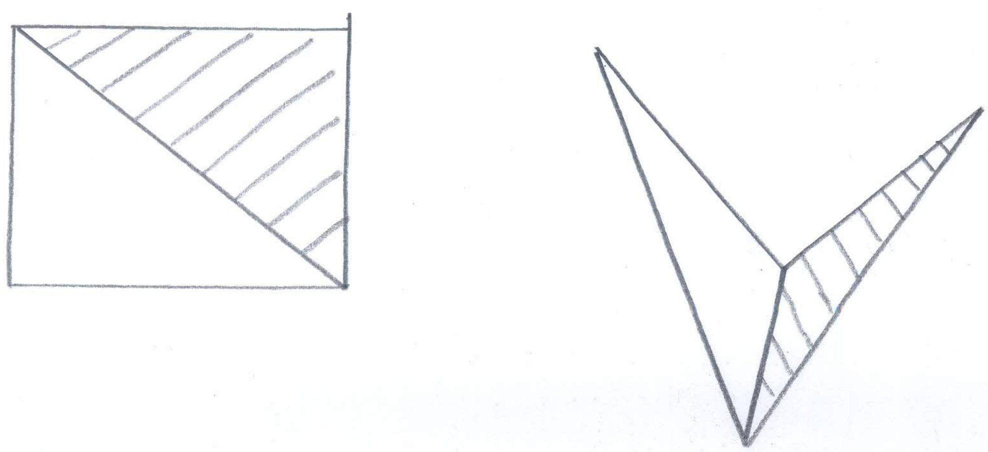
In each case we compare (and express the outcome) with respect to some base unit; most of the measurements just mentioned are supposed to be intuitively clear. Nevertheless, let's have a closer look at areas:



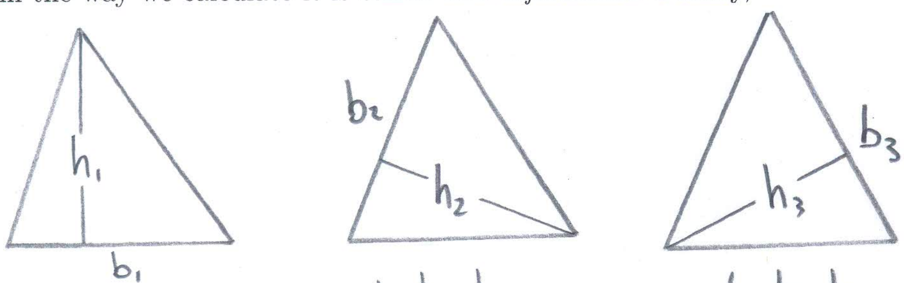
An even more flexible shape than the rectangle is the triangle:



Triangles are actually more basic than rectangles since we can represent every rectangle, and actually and odd-shaped quadrangle, as the 'sum' of two non-overlapping triangles:



In doing so we have *tacitly* assumed a few things. For the triangles we have chosen a *particular* base line and the corresponding height arbitrarily. But the concept of *area* should not depend on such a choice and the calculation this choice entails. Independence of the area from the way we calculate it is called *well-definedness.* Plainly,



$$\frac{1}{2} h_1 b_1 = \frac{1}{2} h_2 b_2 = \frac{1}{2} h_3 b_3$$

Notice that this allows us the most convenient method to work out in the area. In calculating the area of a quadrangle we actually used two assumptions:

- the area of non-overlapping (disjoint[1]) sets can be added, i.e.

$$A(X \cup Y) = A(X) + A(Y) \quad \text{if } X \cap Y = \emptyset$$

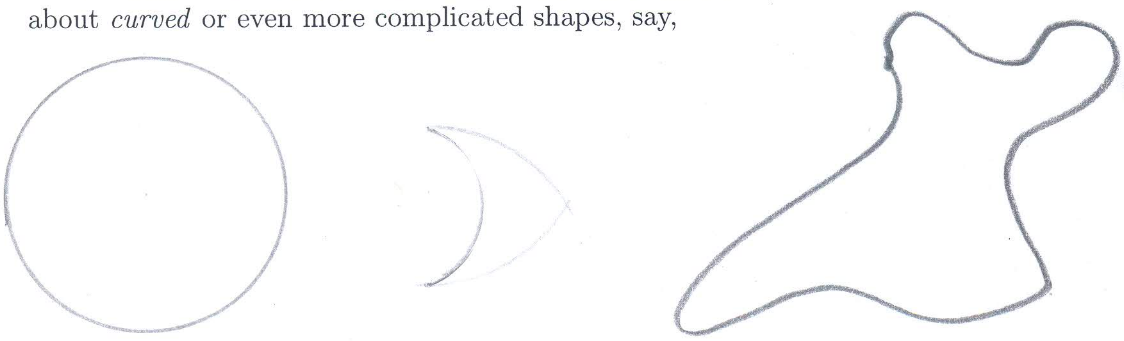
- congruent triangles have the same area[2].

This shows that the least we should expect from a reasonable area measure A is that it is

well-define, take values in $[0, \infty]$, and $A(\emptyset) = 0$;
additive, i.e. $A(X \cup Y) = A(X) + A(Y)$ whenever $X \cap Y = \emptyset$.

An additional property is that area is *invariant* under congruences.

The above rules allow us to measure arbitrarily odd-looking *polygons*[3] using the following recipe: dissect the polygon into non-overlapping triangles and add their areas. But what about *curved* or even more complicated shapes, say,



[1]empty intersection

[2][Ex:] argue using the idea of congruent triangles why the area *should* be half the base times the perpendicular height — this argument here takes the area of a triangle as fundamental

[3]a figure formed by three or more points in the plane joined by line segments