1 Non-Commutative Geometry

It is a theme of modern mathematics that geometry and algebra are dual:

\[ \text{Geometry} \leftrightarrow \text{Algebra} \]

Arguably this began when Descartes began to answer questions about synthetic geometry using the (largely) algebraic methods of coordinate geometry. Since then this duality has been extended and refined to consider:

\[ \text{Spaces} \leftrightarrow \text{Algebra of Functions on the Space} \]

Here a space is a set of points with some additional structure, and the idea is that for a given space, there will be a canonical algebra of functions on the space. For example, given a compact, Hausdorff topological space \( X \), the canonical algebra of functions is \( C(X) \) — the continuous functions on \( X \). The algebra of functions on a space encodes many of the properties of that space. In the example of a compact, Hausdorff space \( X \) and its algebra of functions \( C(X) \), the Banach-Stone Theorem says that the algebra of functions determines the topology on \( X \).

1.1 Examples

1. Cardinality

Let \( X = \{a_1, a_2, \ldots, a_n\} \) be a set. Now consider \( F(X) \) the space of complex valued functions \( X \to \mathbb{C} \). Now to define \( f \in F(X) \), all we must do is choose complex numbers:

\[ f(a_i) = \lambda_i \text{; for } i = 1, \ldots, n. \]

Hence every \( f \in F(X) \) may be uniquely written in the form:

\[ f = \sum_{i=1}^{n} \lambda_i 1_{\{a_i\}}. \]

That is \( \{1_{\{a_i\}} : 1 \leq i \leq n\} \) is a basis of \( F(X) \) so \( \dim F(X) = n \). Now I would argue that the only feature of this space is that \( |X| = n \). So for a finite set \( X \) such as this one, with no additional structure at all, the dimension of the algebra of functions \( F(X) \) can tell us everything about \( X \).
2. Connectedness

Consider the interval $X = [0, 1]$. In the norm topology it is connected which means that we cannot represent $X$ as a union of non-empty, open disjoint subsets. Consider the continuous functions on $X$, $C(X)$. Now call $p \in C(X)$ a projection if $p^2 = p$. That is

$$[p(x)]^2 = p(x), \text{ for all } x \in X \iff p(x) = 0 \text{ or } 1.$$ 

Now this means that $p$ either takes the value 0 or the value 1. Suppose $p$ is a non-zero projection and set

$$A = \{x \in X : p(x) = 1\}.$$ 

Now $p = 1_A$. Now it should be clear that either $A = \emptyset$ or $X$; otherwise $p$ is not continuous as it will have jump discontinuities on the boundary of $A$. Hence the only projections on the connected set $X$ are the trivial projections 0 and $1_X$.

Now consider $X = [0, 1] \cup [2, 3]$. Now $X$ is certainly disconnected but $1_{[0,1]}$ and $1_{[2,3]}$ are continuous non-trivial projections. If $X \subset \mathbb{R}$ and if $C(X)$ contains non-trivial projections, then $X$ is disconnected.

1.2 Non-Commutative Spaces

Note that in these examples, the algebra of functions has a common structure:

1. **Vector Space** — for any complex valued function we can define point-wise the functions $f + g$ and $\lambda f$ for $\lambda \in \mathbb{C}$.

2. **Normed Space** — there are various norms we could put on the algebra of functions; supremum norm, one norm, two norm, etc. In particular, we will want the algebra of functions to be a Banach space, that is a complete normed vector space.

3. **Algebra** — we can define pointwise a product on the algebra of functions.

4. ***-Algebra** — the algebra of functions takes on an involution $^*$, namely the conjugation: $f^*(x) = \overline{f(x)}$.

Any algebra $A$ which has these four features (with the C*-equation condition on how the norm interacts with the involution: $\|a^*a\| = \|a\|^2$ for all $a \in A$), is known as a C*-algebra and by and large, the canonical algebra of functions on a space will have this structure. As the complex numbers $\mathbb{C}$ are commutative, the algebra of functions on $X$ is commutative:

$$f(x)g(x) = g(x)f(x); \text{ for all } x \in X.$$ 

1.3 C*-Algebras

What is the nature of these seemingly inevitable algebras? We have outlined their basics features above but here we explore the second two features a little further. An associative algebra is a (complex) vector space together with a bilinear map

$$\nabla : A \times A \to A, \ (a, b) \mapsto ab,$$
such that \( a(bc) = (ab)c \). Of course it is natural to refer to this map as the multiplication on \( A \). If \( A \) admits a submultiplicative norm and a unit — an element \( 1_A \in A \) such that \( a1_A = a = 1_Aa \) for all \( a \in A \) — such that \( \|1_A\| = 1 \), then we say that \( A \) is a unital normed algebra. If, further, a unital normed algebra \( A \) is complete then we call \( A \) a unital Banach algebra. We say that \( a \in A \) is invertible if there is an element \( a^{-1} \in A \) such that \( aa^{-1} = 1_A = a^{-1}a \). The set

\[
G(A) = \{ a \in A : a \text{ is invertible} \}
\]
is an (open) group with the multiplication got from \( A \). We define the spectrum of an element \( a \) to be the set

\[
\sigma(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1_A \notin G(A) \}.
\]

A theorem of Gelfand states that if \( a \) is an element of a unital Banach algebra \( A \), then the spectrum of \( a \) is non-empty. As a corollary, Gelfand and Mazur proved that if \( A \) is a unital algebra in which every non-zero element is invertible, then \( A \) is isometrically isomorphic to \( \mathbb{C} \).

If an algebra is not unital, then we can adjoin a unit to it forming the unitisation of \( A \), \( \tilde{A} \) by defining \( \tilde{A} = A \oplus \mathbb{C} \) and

\[
(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu).
\]
The unit is \((0, 1)\). Often times we can reduce to the unital case via this construction — but not always. It can be shown that there is a (necessarily unique) C*-norm on the unitisation.

Now consider the non-zero linear functionals \( \rho : A \to \mathbb{C} \) that are also homomorphisms. We call set of all such functionals the character space of \( A \), \( \Phi(A) \). Suppose that \( A \) is an abelian Banach algebra for which the space \( \Phi(A) \) is non-empty. For \( a \in A \), define the evaluation map:

\[
\hat{a} : \Phi(A) \to \mathbb{C}, \; \rho \mapsto \rho(a).
\]

We may endow \( \Phi(A) \) with the weakest topology that makes all of the evaluation maps continuous: this coincides with the weak* topology. If \( A \) is a unital abelian Banach algebra, then \( \Phi(A) \) is a compact Hausdorff space. We can show that the set \( \{ \rho \in \Phi(A) : |\rho(a)| \geq \varepsilon \} \) is weak*-compact. Hence \( \hat{a} \in C_0(\Phi(A)) \). We call \( \hat{a} \) the Gelfand transformation of \( a \). At this point we can prove a deep theorem about the Gelfand transformation but it is better that we wait to learn more about the *-algebra structure.

An involution on an algebra is a conjugate-linear map \( a \mapsto a^* \) on \( A \) such that \( a^{**} = a \) and \( (ab)^* = b^*a^* \). The pair \( (A, *) \) is called a *-algebra. An element is said to be self-adjoint if \( a^* = a \). A C*-algebra is a Banach *-algebra such that

\[
\|a^*a\| = \|a\|^2, \text{ for all } a \in A.
\]

This seemingly mild condition is in fact very strong. In particular, it implies that there is at most one norm on a *-algebra making it a C*-algebra.
1.3.1 Example

Consider the Hilbert space $H = \mathbb{C}^n$ with the usual inner product on $\mathbb{C}^n$ and the set of bounded operators on $H$, $A = B(H) \cong M_n(\mathbb{C})$. Using the usual matrix addition and multiplication, $A$ becomes a *-algebra when we equip it with the involution of the conjugate-transpose $a_{ij} \mapsto a_{ji}^*$. Now which is the correct norm making $A$ into a C*-algebra? A quick calculation shows that the operator norm satisfies the C*-equation... We can also show that features of the matrix algebra above extend to general C*-algebras. For example, the fact that a self-adjoint matrix has real eigenvalues is in fact a more general result about C*-algebras — namely that the spectrum of a self-adjoint element of a C*-algebra $A$ is real. Note at this point that $F(X)$ arises as a diagonal (and hence commutative) subalgebra of $M_j X_j(\mathbb{C})$ via the map:

$$
\sum_{i=1}^{n} \lambda_i 1_{\{a_i\}} \mapsto \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_n
\end{pmatrix}.
$$

This analysis culminates in the beautiful theorem of Gelfand that states that every abelian C*-algebra is isomorphic to an algebra of functions on a space:

1.3.2 Theorem (Gelfand)

*If $A$ is a non-zero abelian C*-algebra, then the Gelfand representation

$$
\varphi : A \to C_0(\Phi(A))
$$

is an isometric *-isomorphism.*

But what about non-commutative C*-algebras? How about:

Commutative Space $\leftrightarrow$ Commutative Algebra of Functions

Non-Commutative Space $\leftrightarrow$ Non-Commutative Algebra of Functions?

Ordinarily we look at a space $X$, and then look at the induced algebra of functions $F(X)$. Our work above has shown that often it is equally valid to look at a C*-algebra, say $F$, and look at the induced space $X(F)$. Why not do this for a non-commutative C*-algebra $F$? We could then say things about $X(F)$ based on $F$. We have to realise that $X(F)$ is not going to look like our intuitive idea of space as a set of points. For example, some definitions we could decide to use could be:

$X(F)$ has cardinality $n$ if $F$ has dimension $n$,

$X(F)$ is connected if $F$ contains non-trivial projections,

From now on we will refer to non-commutative spaces as quantum spaces. This work is concerned with the/a quantisation of Markov chains and random walks on groups. So essentially, this means that you quantise objects, such as Markov chains, by replacing a commutative C*-algebra $F(X)$ with a not-necessarily commutative one. If a quantisation is valid in a certain sense, then morally we should be able to recover a classical space or process by restricting to a commutative subalgebra.
2 Markov Chains

Consider a particle in a finite space \( X = \{x_i : i = 1, 2, \ldots, N\} \). Suppose at time \( t = 0 \) the particle is at the point \( x \), and at times \( 1, 2, \ldots \) moves to another point in the space chosen ‘at random’. The probability that the particle moves to a certain point \( x_j \) at a time \( t \) is dependent only upon its position \( x_i \) at the previous time. This is the Markov property.

A Markov Chain is a mathematical process which models these dynamics. Such a Markov chain can be described by the transition probabilities \( p(x_i, x_j) \), which give the probability of the particle being at point \( x_j \) given that the particle is at the point \( x_i \) at the previous time.

To formulate, let \( X \) be a finite set. Denote by \( M_p(X) \) the probability measures on \( X \). Let \( \delta^x \) be the element of \( M_p(X) \) which puts a measure of 1 on \( x \) (and zero elsewhere). These Dirac measures, \( \{\delta^x : x \in X\} \), are the standard basis for \( \mathbb{R}^{|X|} \supseteq M_p(X) \). Denote by \( F(X) \) the complex functions on \( X \) and \( L(V) \) the linear operators on a vector space \( V \). The similarly defined Dirac functions, \( \{1_{\{x\}} : x \in X\} \), are the standard basis for \( F(X) \). With respect to this basis \( P \in L(F(X)) \) has a matrix representation \( [p(x,y)]_{xy} \). \( P \in L(F(X)) \) is a stochastic operator if:

\begin{enumerate}
  
  \item \( p(x,y) \geq 0, \forall x, y \in X \)
  
  \item \( \sum_{y \in X} p(x,y) = 1, \forall x \in X \)

\end{enumerate}

(row sum is unity)

Given \( \nu \in M_p(X) \), a stochastic operators \( P \) acts on \( \nu \) as \( \nu P(x) = \sum_y \nu(y)p(y,x) \). Stochastic operators are readily characterised without using matrix elements as being \( M_p(X) \)-stable in the sense that \( M_p(X)P \subset M_p(X) \) if and only if \( P \) is a stochastic operator. Equivalently, stochastic operators are positive, unital maps \( F(X) \to F(X) \). In this context, positive means that if \( F(X)^+ \) is the set of functions with \( f(x) \geq 0 \) for all \( x \in X \), then \( P(F(X)^+) \subset F(X)^+ \).

Unital means that \( P(1_X) = 1_X \).

2.1 Definition

Let \( X \) be a finite set and \( \nu \in M_p(X) \), \( P \) a stochastic operator on \( X \), and \((Y, \mu)\) a probability space. A sequence \( \{\xi_k\}_{k=0}^n \) of random variables \( \xi_k : Y \to X \) are a Markov Chain with initial distribution \( \nu \) and stochastic operator \( P \), if

\begin{enumerate}
  
  \item \( \mu(\xi_0 = x_0) = \nu(x_0) \).
  
  \item \( \mu(\xi_{k+1} = x_{k+1} | \xi_0 = x_0, \ldots, \xi_k = x_k) = p(x_k, x_{k+1}) \),

assuming \( \mu(\xi_0 = x_0, \ldots, \xi_k = x_k) > 0 \).

\end{enumerate}

Condition (ii) is the Markov property. Call \( \xi_k \) the position of the Markov chain after \( k \) transitions.

Many questions may be asked about the local and global behaviour of a Markov chain \( \xi \). One could define local behaviour as the behaviour of the Markov chain with respect to the elements of \( X = \{x_1, x_2, \ldots, x_N\} \), while global behaviour as the behavior of the Markov chain with respect to the whole of \( X \) (i.e. little distinction is made between the elements of \( X \)). Alternatively, imagine a black box with a lid containing a Markov-type process. Local questions are questions that would be asked with the lid off, while global questions are questions that would be asked with the lid on.
When we quantise a Markov chain, we can no longer be interested in the local behaviour as the notion of a point is now defunct. Therefore, a central question is for a given chain whether or not the $\xi_k$ display limiting behaviour as $k \to \infty$? If $\xi_{\infty}$ exists, what is its distribution?

Suppose $\xi$ is a Markov chain and the limit $\nu P^n \to \theta$ exists (in particular $\|\nu P^k - \theta\|_1 \to 0$). Loosely speaking, after a long time $N$, $\xi_N$ has distribution $\mu(\xi_N) \sim \theta$:

$$\nu P^N \sim \theta$$
$$\Rightarrow \nu P^N P \sim \theta P$$
$$\Rightarrow \nu P^N+1 \sim \theta P$$

But $\nu P^{N+1} \sim \theta$ also and hence $\theta P \sim \theta$. So if $\xi_{\infty}$ exists then its distribution $\theta$ may have the property $\theta P = \theta$ — it may be stationary. Perron-Frobenious type arguments show that stationary distributions exist.

How many stationary distributions exist? Consider Markov Chains $\xi$ and $\zeta$ on disjoint finite sets $X$ and $Y$, with stochastic operators $P$ and $Q$. The block matrix

$$R = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

is a stochastic operator on $X \cup Y$. If $\pi$ and $\theta$ are stationary distributions for $P$ and $Q$ then

$$\phi_c = (c\pi, (1-c)\theta), \quad c \in [0, 1]$$

is an infinite family of stationary distributions for $R$. The dynamics of this walk are that if the particle is in $X$ it stays in $X$, and vice versa for $Y$ (the graph of $R$ has two disconnected components). This example shows that, in general, the stationary distribution need not be unique.

It can be shown that the Markov chain converges in distribution to a strict probability measure\(^1\) if and only if the Markov chain is ergodic (briefly, connected and aperiodic). In this case, the limiting distribution is the unique stationary distribution.

### 2.2 C*-algebra Quantisation of a Classical Markov Chain

So where is the C*-algebra in a Markov Chain? Well let $\xi$ be a Markov chain on a set $X = \{x_1, \ldots, x_N\}$ with initial distribution $\nu$ and transition probabilities $P[\xi_{k+1} = x_j | \xi_k = x_i] = p(x_i, x_j) = p_{ij}$. The probability distribution of this walk, after $n$ steps, is given by $\nu P^n$. However the probability measures on $X$, $M_p(X)$, lie in the dual of $F(X)$ ($M_p(X) \subset \mathbb{R}^N$ is equipped with the 1-norm while $F(X)$ is equipped with the supremum norm). In fact we can go further, the probability measures comprise the states (defined overleaf) of the C*-algebra $A$ as for any $\theta \in M_p(X) \subset F(X)^*$, $\|\theta\|_1 = 1$ and $\theta \geq 0$.

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\(^1\)a probability measure $\nu \in M_p(X)$ is strict if $\nu(x) > 0$ for all $x \in X$
Actually in this case (X is a finite set) the positivity of the functional θ has two equivalent definitions (the first is the same as saying θ(x) ≥ 0 for all x ∈ X):

1. The dual basis to (1{xi₁}, · · · , 1{xiₙ}), is the basis of delta measures (δₓ¹, · · · , δₓⁿ). In this basis, the coefficients of θ are all positive.

2. θ(f) ∈ C⁺ = R⁺ for all positive functions f ∈ F(X)⁺.

Usually when we talk about functionals on C*-algebras being positive we are talking about definition 2. i.e. a linear map φ : A → B between C*-algebras is said to be positive if φ(A⁺) ⊂ B⁺. The positive elements of a C*-algebra A are given by:

\[ A⁺ = \{ a ∈ A : a^* = a, σ(a) ⊂ R⁺ \}. \]

The states of a general C*-algebra, A, are given by:

\[ S(A) = \{ φ ∈ A^* : \| φ \| = 1, \ φ ≥ 0 \}. \]

In this picture — which looks at the deterministic evolution of \( \{ νP_k : k = 0, 1, \ldots, N \} \) — rather than the random variable picture of the \( ξ_k : (Y, µ) → X \), we thus have an initial distribution \( θ ∈ S(A) \), a stochastic operator:

\[ P : F(X)* → F(X)*, \ θ → θP \]

which is \( S(F(X)) \)-stable, and we look at the set of distributions \( \{ P^k(θ) : k = 1, \ldots, n \} \) to tell us everything we want to know about the Markov chain. For example, if \( P^k(θ) \) is convergent then we know the walk converges and a fixed point of the stochastic operator is a stationary distribution. The only thing left to do is to put some conditions on a stochastic operator being \( S(A) \)-stable — how about the stochastic operator being positive and isometric?

The C*-algebra quantisation is then as follows. Let A be a C*-algebra with dual \( A^* \). Choose an element \( ψ ∈ S(A) \) and a positive linear isometry \( T : A^* → A^* \) — which is automatically \( S(A) \)-stable. The distribution of the quantum Markov chain generated by \( ψ \) and \( T \) after \( k \) steps would then be given by \( T^k(ψ) \).

This construction is leaning towards the fact that the deterministic evolution of the \( T^k(ψ) \) can tell us all about the global behaviour — and this is desirable if we are doing quantisation. Alternatively, we could note that a stochastic operator is a unital, positive operator \( P : F(X) → F(X) \) and work from there.

However, it is equally valid (and indeed far more common), to examine a classical Markov chain, not as a deterministic evolution \( \{ νP_k \}_{k≥0} \), but rather as a random variable. Therefore, instead of a C*-algebra quantisation we could use the quantum analogue of classical probability theory, and quantise ‘down’ in the random variable picture to quantum random variables, and then lift to a deterministic evolution using the associated quantised stochastic operators and distributions. This diagram should explain what I mean, the right arrows are quantisations; the C*-algebra quantisation on top — the quantum probability quantisation on bottom:

\[
\begin{align*}
(M_p(X))P^k & \quad \xrightarrow{q} \quad T^k(S(A)) \\
\{ξ_k\} & \quad \xrightarrow{q} \quad \{j_k\}
\end{align*}
\]
To quantise a random walk on a finite group this second approach seems inevitable — if we want to encode the structure of the group acting on itself.

3 Random Walks on Finite Groups

A particularly nice class of Markov chain is that of a random walk on a group. The particle moves from group element to group element by choosing an element \( h \) of the group ‘at random’ and moving to the product of \( h \) and the present position \( g \), i.e. the particle moves from \( g \) to \( hg \).

To formulate, let \( G \) be a finite group of order \(|G|\) and identity \( e \). Let \( \nu \in M_p(G) \) and \((Y,\mu)\) be a probability space. Let \( \{\zeta_k\}_{k=0}^n : (Y,\mu) \to G \) be a sequence of i.i.d. random variables with distributions \( \mu(\zeta_0 = g_0) = \delta^e(g_0) \) and \( \mu(\zeta_k = g) = \nu(g) \). The sequence of random variables \( \{\xi_k\}_{k=0}^n : (Y,\mu) \to G \)

\[
\xi_k = \zeta_k \zeta_{k-1} \cdots \zeta_1 \zeta_0 \tag{3}
\]

is a right-invariant random walk on \( G \).

3.1 Random Variable Quantisation

In the case of a random walk on a finite group, we can construct the probability space for \( \xi_k \) from \( Y_k = G^k \) (Cartesian product of \( k \) copies of \( G \)):

\[
\xi_k : G^k \to G.
\]

Consider the \( C^* \)-algebras \( A_k = L^\infty(Y_k) \), \( B = L^\infty(X) \cong F(X) \) (for \( X \) finite). Given \( f \in F(G) \), we have \( f \circ \xi_k \in L^\infty(Y_k) \). The maps

\[
j_k : B \to A_k , \ f \mapsto g \circ \xi_k,
\]

are a family of unital *-homomorphisms.

\[
j_0 : F(G) \to F(G), \ f \mapsto f \circ \zeta_0,
\]

\[
j_1 : F(G) \to F(G \times G), \ f \mapsto f \circ \xi_1 = f \circ (\zeta_1 \zeta_0)
\]

\[
j_2 : F(G) \to F(G \times G \times G), \ f \mapsto f \circ \xi_2 = f \circ (\zeta_2 \zeta_1 \zeta_0)
\]

Now note that if \( G \) is a finite group (set) so that \( F(G^k) \cong \bigotimes_{i=1}^k F(G) \) as vector spaces, where at this point \( \otimes \) is an algebraic tensor product (via the isomorphism \( \varphi(\delta^s \otimes \delta^t) = \delta^{(s,t)} \)). Hence we have a family of *-homomorphisms:

\[
j_k : F(G) \to F(G) \otimes F(G) \otimes \cdots \otimes F(G) , \ (k \text{ copies of } F(G)).
\]

Before we quantise via allowing the \( F(G) \) to become non-commutative we need to examine the structure of \( F(G) \) to distinguish it from, say, the algebra of functions on a set \( X \) with no structure at all.
3.2 The Coalgebra Picture

If $G$ is a group, it seems impossible to encode the group laws using the first, C*-algebra quantisation. If we’re going to encode group laws in a not-necessarily commutative algebra $A$ (quantum group), we will need the second, random variable, quantisation.

What kind of relations do we expect to hold ‘up’ in $F(G)$ — associativity, identity and inverses need to be accounted for. In particular, for all $x, y, z \in G$ and $f \in F(G)$, where $e$ is the identity:

$$f(x \ast (y \ast z)) = f((x \ast y) \ast z),$$  \hspace{1cm} (4)
$$f(x \ast e) = f(x) = f(e \ast x),$$  \hspace{1cm} (5)
$$f(x \ast x^{-1}) = f(e) = f(x^{-1} \ast x).$$  \hspace{1cm} (6)

A non-precise way of saying this is that we want $F(G \times (G \times G)) = F((G \times G) \times G)$, $F(G \times \{e\}) = F(G) = F(\{e\} \times G)$, $F(G \times G^{-1}) = F(\{e\}) = F(G^{-1} \times G)$.

Introduce the following maps:

$$\Delta : F(G) \rightarrow F(G \times G), \Delta f(x, y) = f(xy),$$
$$\varepsilon : F(G) \rightarrow \mathbb{C}, \varepsilon(f) = f(e),$$
$$S : F(G) \rightarrow F(G), Sf(x) = f(x^{-1}).$$

These maps are called the **comultiplication**, the **counit** and the **antipode**. Note that $\Delta$ and $\varepsilon$ are *-homomorphisms and $S$ is linear but anti-multiplicative.

Recall the isomorphism $F(G \times G) \cong F(G) \otimes F(G)$. With this identification, we can now encode the group laws via the relations (where $I = I_{F(G)}$), which can be shown to hold for all $f \in F(G)$:

$$(I \otimes \Delta) \circ \Delta = (\Delta \otimes I) \circ \Delta,$$  \hspace{1cm} (7)
$$(\varepsilon \otimes I) \circ \Delta = I = (I \otimes \varepsilon) \circ \Delta,$$  \hspace{1cm} (8)
$$\nabla \circ (S \otimes I) \circ \Delta(f) = \varepsilon(f)1_{F(G)} = \nabla \circ (I \otimes S) \circ \Delta(f).$$  \hspace{1cm} (9)

The first two conditions here are the coassociativity and counitary relations. Consider now a not-necessarily-commutative finite dimensional algebra $(A, \nabla)$. As there is a bilinear multiplication $\nabla : A \times A \rightarrow A$, there exists a unique linear map (which we also call $\nabla$ by abuse of notation) $\nabla : A \otimes A \rightarrow A$. When we have a vector space $A$, together with linear maps $\nabla$, $\Delta$ and $\varepsilon$ satisfying the coassociativity and the counitary property, we call $A$ a **bialgebra**. If $A$ also has a *-algebra structure we call $A$ a *-bialgebra. If furthermore, there exists a linear map $S : A \rightarrow A$ satisfying $\nabla(S \otimes I_A) \circ \Delta(a) = \varepsilon(a)1_A$ then we call $A$ a *-Hopf algebra. If the algebra of $A$ is a C*-algebra then we call $A$ a C*-Hopf Algebra.
Now we make our quantisation:

3.2.1 Definition

A finite quantum group is a finite dimensional C*-Hopf algebra.

Again we would hope that if we restrict to a commutative subalgebra $B$ of a finite quantum group that we would find a group $G$ such that $B = F(G)$. Cartier$^2$ shows that in fact $B \cong F(\Phi(B))$.

3.3 Random Walk on a Quantum Group

Again we look at the classical situation. Note, that $\Delta(f) = f \circ m$, where $m : G \times G \to G$ is the group multiplication:

$$j_1(f) = f \circ \xi_1 = f \circ m(\xi_1 \otimes \xi_1) = f \circ \xi_1 \xi_0,$$

$$\Rightarrow j_1(f)(x) = \Delta f.$$

Now

$$j_2(f) = f \circ \xi_2 = f \circ \xi_2 \xi_1 \xi_0,$$

$$= f \circ m \circ (m \otimes I)(\xi_2 \otimes \xi_1 \otimes \xi_0),$$

$$= (\Delta \otimes I) \circ \Delta f.$$

Iteratively,

$$j_n(f) = (\Delta \otimes I) \circ j_{n-1}(f).$$

Now we need to quantise probability theory — probabilities, conditional expectations, independence, etc. Here we just give a flavour. Let $A$ be a finite dimensional C*-algebra. As mentioned above, states on $A$ are the quantum generalisation of probability measures on a finite set $X$. Form the infinite tensor product $\hat{A} = \bigotimes_{i=0}^{\infty} A$ (with the natural inclusions $x \mapsto x \otimes I_A$). Now consider states on $A$, $\psi$, $\phi$ and form product states:

$$\Psi_k = \psi \otimes \bigotimes_{i=1}^{k} \phi,$$

$$\Psi = \psi \otimes \bigotimes_{i=1}^{\infty} \phi.$$

Now we can think of the $j_n$ as quantum random variables with distributions $\Psi \circ j_k$, and $\{j_k\}_{k \geq 0}$ as a quantum stochastic process. We call $\psi$ the initial state and $\phi$ the transition state. We can now go further and look at quantum conditional expectations and quantum stochastic operators — and crucially, show that it does indeed satisfy a Markov property. This is my plan for the immediate future.

$^2$attributed to Cartier: Theorem 2.22 in http://www.mate.uncor.edu/ ggarcia/encuentros/notas-curso-qg-ha.pdf
4 Research Plan

1. Study quantum probability theory — the non-commutative generalisation of classical probability theory especially conditional expectation, stochastic operators and state convolution.

2. Address topological and analytic concerns; e.g. direct limits of C*-algebras, complete positivity, etc.

3. Examine the limiting behaviour and stationary states of random walks on finite quantum groups.

4. Reconstruct some elementary examples and construct my own.

5. Survey the field for all recent advances. This field is relatively dynamic with papers being written as we speak. I should be on top of these.

6. Extend the definition of a finite quantum group to compact groups and infinite groups. This should use Von Neumann algebras as things ‘go infinite’.

7. Understand the notion of q-deformations of algebras and how they arise as quantum groups.

8. Come to have an appreciation of the quantum mechanical motivations for studying this area (!).