MS 2001: Additional but Harder Exercises for Definitions II

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You can test your understanding of the definitions by answering these questions — i.e. you'll only be able to answer these questions if you really, really know your definitions. I have graded them as follows (all based on the assumption that you know the definitions):

(**A**): Straightforward

(**B**): Hard.

(**C**): Very hard.

(®) Random — in the modern, bastardised meaning of the word.

Questions:

1. Prove that a continuous function is *continuous on any non-empty closed interval* $[a, b] \subset \mathbb{R}$ (**A**).

2. Construct a function which is not continuous (everywhere) but is *continuous on* $[0, 1]$ (**A**).

3. The singleton set[1] $\{x\} \subset \mathbb{R}$ is equal to the closed interval $[x, x]$. Verify that all the conclusions of the *Intermediate Value Theorem* hold when $f : \{x\} \to \mathbb{R}$ is a *continuous function* got by restricting a continuous function $f : \mathbb{R} \to \mathbb{R}$ to $\{x\}$ (**A**). Find *ALL* such continuous functions $\{x\} \to \mathbb{R}$.

4. Prove that the *Mean Value Theorem* holds on any non-empty closed interval for a smooth function $f : \mathbb{R} \to \mathbb{R}$ (**A**).

5. Find the *local maxima and minima* of the constant function $f(x) = 0$ (**A**).

6. Show that if $f : [a, b] \to \mathbb{R}$ is discontinuous at $c \in (a, b)$, then c is a *critical point* (**A**).

7. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4 + 12x^3 + 54x^2 - 12x + 5$ is *concave up* on $\mathbb{R}$ (**A**).

8. If $h(x)$ is an *asymptotic* of $f(x)$, then $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} f(x)$. True or False? (**A**)

9. Prove that $\tan x$ has a *vertical asymptote* at $x = \pi/2$ (**A**).

[1] containing the single real number x

10. Give an example of a function which is *continuous on* $(0, 1]$ but not *on* $[0, 1]$ (**B**).

11. Use the *Intermediate Value Theorem* to prove that $\sqrt{2} \in \mathbb{R}$ exists (**B**).

12. Use the *Intermediate Value Theorem* to prove that if $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous functions such that $f(a) = g(b)$ and $f(b) = g(a)$, then there exists a point $c \in [a, b]$ such that $f(c) = g(c)$ (**B**).

13. It can be shown that a continuous function obtains its *absolute maximum and minimum* on finite unions of closed intervals. Consider the set S defined by:

$$S = [0, 2\pi] \cup \left[2\pi, 2\pi + \frac{1}{2\pi}\right] \cup \left[3\pi, 3\pi + \frac{1}{3\pi}\right] \cup \left[4\pi, 4 + \frac{1}{4\pi}\right] \cup \cdots \cup \left[100\pi, 100\pi + \frac{1}{100\pi}\right],$$
$$= [0, 2\pi] \cup \left(\bigcup_{i=2}^{100} \left[i\pi, i\pi + \frac{1}{i\pi}\right]\right).$$

Find the absolute maxima and minima of $\sin : S \to \mathbb{R}$ (**B**).

14. If $f : (a, b) \to \mathbb{R}$ is a smooth function such that $f' = 0$ for all $x \in (a, b)$ then f is constant on (a, b). Find an example of a differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to\infty} g'(x) = 0 \text{ , but } \lim_{x\to\infty} g(x)$$

is not a constant (**B**).

15. Sketch continuous functions $f_i : [0, 1] \to \mathbb{R}$ for $i = 1, 2, 3, 4, \ldots$ such that f_i has i stationary points; i.e. solutions to $f'(c) = 0$ for $c \in (0, 1)$ (**B**).

16. Is it possible for a function $f : \mathbb{R} \to \mathbb{R}$ to have a *local maximum/ minimum* at a point $a \in \mathbb{R}$ where f is discontinuous? (**B**).

17. In the proof of the *Closed Interval Method*, we make the assumption that $f : [a, b] \to \mathbb{R}$ is continuous. Is this assumption necessary? (**B**).

18. Let $\varepsilon \in (0, 1)$ be a constant. Using algebraic techniques, find the *critical points* of $f : [0, 1] \to \mathbb{R}$

$$f(x) := \begin{cases} 0 & \text{if } -1 \le x < -\varepsilon \\ \frac{1}{\varepsilon}x + 1 & \text{if } -\varepsilon \le x < 0 \\ -\frac{1}{\varepsilon}x + 1 & \text{if } 0 \le x < \varepsilon \\ 0 & \text{if } \varepsilon \le x \le 1 \end{cases} \quad \mathbf{B}$$

19. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is *twice differentiable* but not *three-times differentiable* (**B**).

20. Show that

$$\lim_{h\to 0} \frac{\cos(\pi + h) - \cos(\pi)}{h} = 0 \ \ (\mathbf{B}).$$

21. Show that if $f : [a, b] \to \mathbb{R}$ is *twice differentiable*, there exists a point $c \in (a, b)$ such that
$$f''(c) = \frac{f'(b) - f'(a)}{b - a} \quad (\mathbf{B})$$

22. Suppose that a smooth function $f : [-1, 1] \to \mathbb{R}$ is *concave down* on $[-1, 1]$ and further that $f'(0) = 0$. Draw a sketch suggesting that f attains its absolute maximum at 0 (**B**).

23. Use the geometric definition of *concavity* to show that $f(x) = |x|$ is *concave up* on $\mathbb{R}$ (**B**).

24. Show that $|x|$ is an *asymptotic* of $\sqrt{x^2 + 3}$ (**B**).

25. Explain why there is no polynomial *asymptotic* of $\sin x$ (**B**).

26. Find all *vertical asymptotes* of $f(x) = \frac{\sin x}{x}$ (**B**).

27. Use the *Intermediate Value Theorem* to prove that if $f : [0, 1] \to [0, 1]$ is a continuous function, then f has a *fixed point*; i.e. a point $x \in [0, 1]$ such that $f(x) = x$ (**C**).

28. Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable function such that $f'(x) \leq K$ for all $x \in [a, b]$. Prove that $f(a) + K(b - a)$ is an upper bound for the *absolute maximum of* f *on* $[a, b]$ [HINT: Assume that $f(x) > f(a) + K(b - a)$ for some $x \in [a, b]$ and show that this contradicts the *Mean Value Theorem.*] (**C**).

29. Call a non-empty set of points $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$ an *antipodal set* for $f : \mathbb{R} \to \mathbb{R}$ if the tangents to f at $x_1, x_2, \ldots, x_n$ are all parallel. Prove that an antipodal sets of a cubic function contain at most two elements (**C**).

30. Verify *Rolles's Theorem* for $\sin : [0, \pi] \to \mathbb{R}$ geometrically by considering the unit circle with parametric equation $(x, y) = (\cos\theta, \sin\theta)$ (**C**).

31. Let $[a, b]$ be a non-empty closed interval and consider $\sin : [a, b] \to \mathbb{R}$. Explain why sin satisfies the hypothesis of the *Mean Value Theorem*. Apply the *Mean Value Theorem* to show that there exists an $x \in (a, b)$ such that
$$|\cos(x)| = \left|\frac{\sin(b) - \sin(a)}{b - a}\right| \quad (\mathbf{B}).$$
Now use this result to show that for any $x_1,\ x_2 \in \mathbb{R}$
$$|\sin(x_2) - \sin(x_1)| \leq |x_2 - x_1|.$$
We could then use this result to prove that sin is a continuous function. What would be wrong with the proof? (**C**)

32. In this exercise we see why we might call the quantity

$$\frac{f(b) - f(a)}{b - a}$$

the *average slope* across $[a, b]$ when we talk about the *Mean Value Theorem*. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Partition the interval $[a, b]$ into n equally spaced points:

$$a =: x_0 < x_1 < x_2 < \cdots x_n := b.$$

If $[a, b]$ is divided into n sub-intervals then each will have width $h = (b - a)/n$ — so that $x_i = a + ih$. Produce a sketch of a smooth function $f : [a, b] \to \mathbb{R}$ and a partition of $[a, b]$. Now for large n, and hence intervals of small length, smooth curves look like lines so approximate the slope of f on the interval $[x_{i-1}, x_i]$ by the secant line joining $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$ (a sketch should help):

$$f'(x) \approx \frac{f(x_i) - f(x_{i-1})}{h} \text{ , for } x \in [x_{i-1}, x_i].$$

Now average over all of the n sub-intervals:

$$\text{average}(f') \approx \frac{\sum_{i=1}^{n} \left(\frac{f(x_i) - f(x_{i-1})}{h} \right)}{n}.$$

Now use the relationship between a, b, n and h and telescoping series techniques[2] to show that

$$\text{average}(f') \approx \frac{f(b) - f(a)}{b - a}.$$

To make this interpretation precise — by taking $n \to \infty$ — we actually have to use the *Mean Value Theorem*! More of this in MS2002 (**C**).

33. Argue that if $f : \mathbb{R} \to \mathbb{R}$ is a smooth, even function such that the only solution to $f'(x) = 0$ is $x = 0$, then $x = 0$ cannot be a *saddle point* (a point $a \in \mathbb{R}$ such that $f'(a) = 0$ but a is not a local maxima or minima). Suppose there is a *local maximum* at $x = 0$. Explain why this is an absolute maximum for f on $\mathbb{R}$ (**C**).

34. Construct a function with *vertical asymptotes* at $x = 0, 1, 2, 3, 4, 4, \dots, 10^6$ (**C**).

35. Suppose that a particle is initially at zero on the numberline. Suppose that its subsequent motion is described by its position, s, as a function of time, t: $s : \mathbb{R}^+ \to \mathbb{R}$. Interpret $s'(t)$ physically (®).

36. Given that the derivative of position $s : \mathbb{R}^+ \to \mathbb{R}$ is velocity $v : \mathbb{R}^+ \to \mathbb{R}$, explain why *Rolle's Theorem* suggests that a ball kicked straight into the air and subsequently caught must have a time t_0 such that $v(t_0) = 0$. What does t_0 correspond to physically? (®)

[2]see Wikipedia