We present here the theory of quantum stochastic processes with independent increments with special emphasis on their structure as Markov processes. To avoid all technical difficulties we restrict ourselves to discrete time and finite quantum groups, i.e. finite-dimensional $C^*$-Hopf algebras, see Appendix A. More details can be found in the lectures of Kümmener and Franz in this volume.

Let $G$ be a finite group. A Markov chain $(X_n)_{n \geq 0}$ with values in $G$ is called a (left-invariant) random walk, if the transition probabilities are invariant under left multiplication, i.e.

$$P(X_{n+1} = g' | X_n = g) = P(X_{n+1} = hg' | X_n = hg) = p_{g^{-1}g'}$$
for all \( n \geq 0 \) and \( g, g', h \in G \), with some probability measure \( p = (p_g)_{g \in G} \) on \( G \). Since every group element can be translated to the unit element by left multiplication with its inverse, this implies that the Markov chain looks the same everywhere in \( G \). In many applications this is a reasonable assumption which simplifies the study of \( (X_n)_{n \geq 0} \) considerably. For a survey on random walks on finite groups focusing in particular on their asymptotic behavior, see [SC04].

A quantum version of the theory of Markov processes arose in the seventies and eighties, see e.g. [AFL82, Küm88] and the references therein. The first examples of quantum random walks were constructed on duals of compact groups, see [vW90b, vW90a, Bia90, Bia91b, Bia91a, Bia92a, Bia92c, Bia92b, Bia94]. Subsequently, this work has been generalized to discrete quantum groups in general, see [Izu02, Col04, NT04, INT04]. We hope that the present lectures will also serve as an appetizer for the “quantum probabilistic potential theory” developed in these references.

It has been realized early that bialgebras and Hopf algebras are closely related to combinatorics, cf. [JR82, NS82]. Therefore it became natural to reformulate the theory of random walks in the language of bialgebras. In particular, the left-invariant Markov transition operator of some probability measure on a group \( G \) is nothing else than the left dual (or regular) action of the corresponding state on the algebra of functions on \( G \). This leads to the algebraic approach to random walks on quantum groups in [Maj93, MRP94, Maj95, Len96, Ell04].

This lecture is organized as follows.

In Section 1, we recall the definition of random walks from classical probability. Section 2 provides a brief introduction to quantum Markov chains. For more detailed information on quantum Markov processes see, e.g., [Par03] and of course Kümmerser’s lecture in this volume.

In Sections 3 and 4, we introduce the main objects of these lectures, namely quantum Markov chains that are invariant under the coaction of a finite quantum group. These constructions can also be carried out in infinite dimension, but require more careful treatment of the topological and analytical properties. For example the properties that use the Haar state become much more delicate, because discrete or locally compact quantum groups in general do not have a two-sided Haar state, but only one-sided Haar weights, cf. [Kus05].

The remainder of these lectures is devoted to three relatively independent topics.

In Section 5, we show how the coupling representation of random walks on finite quantum groups can be constructed using the multiplicative unitary. This also gives a method to extend random walks in a natural way which is related to quantization.

In Section 6, we study the classical stochastic processes that can be obtained from random walks on finite quantum groups. There are basically two methods. Either one can restrict the random walk to some commutative sub-algebra that is invariant under the transition operator, or one can look for a
commutative subalgebra such that the whole process obtained by restriction is commutative. We give an explicit characterisation of the classical processes that arise in this way in several examples.

In Section 7, we study the asymptotic behavior of random walks on finite quantum groups. It is well-known that the Cesaro mean of the marginal distributions of a random walk starting at the identity on a classical group converges to an idempotent measure. These measures are Haar measures on some compact subgroup. We show that the Cesaro limit on finite quantum groups is again idempotent, but here this does not imply that it has to be a Haar state of some quantum subgroup.

Finally, we have collected some background material in the Appendix. In Section A, we summarize the basic theory of finite quantum groups, i.e. finite-dimensional C*-Hopf algebras. The most important results are the existence of a unique two-sided Haar state and the multiplicative unitary, see Theorems A.2 and A.4. In order to illustrate the theory of random walks, we shall present explicit examples and calculations on the eight-dimensional quantum group introduced by Kac and Paljutkin in [KP66]. The defining relations of this quantum groups and the formulas for its Haar state, GNS representation, dual, etc., are collected in Section B.

1 Markov chains and random walks in classical probability

Let \((X_n)_{n\geq 0}\) be a stochastic process with values in a finite set, say \(M = \{1, \ldots, d\}\). It is called Markovian, if the conditional probabilities onto the past of time \(n\) depend only on the value of \((X_n)_{n\geq 0}\) at time \(n\), i.e.

\[
P(X_{n+1} = i_{n+1} | X_0 = i_0, \ldots, X_n = i_n) = P(X_{n+1} = i_{n+1} | X_n = i_n)
\]

for all \(n \geq 0\) and all \(i_0, \ldots, i_{n+1} \in \{1, \ldots, d\}\) with

\[
P(X_0 = i_0, \ldots, X_n = i_n) > 0.
\]

It follows that the distribution of \((X_n)_{n\geq 0}\) is uniquely determined by the initial distribution \((\lambda_i)_{1 \leq i \leq d}\) and transition matrices \((p_{ij}^{(n)})_{1 \leq i, j \leq d}, n \geq 1\), defined by

\[
\lambda_i = P(X_0 = i) \quad \text{and} \quad p_{ij}^{(n)} = P(X_{n+1} = j | X_n = i).
\]

In the following we will only consider the case, where the transition probabilities \(p_{ij}^{(n)} = P(X_{n+1} = j | X_n = i)\) do not depend on \(n\).

Definition 1.1. A stochastic process \((X_n)_{n\geq 0}\) with values in \(M = \{1, \ldots, d\}\) is called a Markov chain on \(M\) with initial distribution \((\lambda_i)_{1 \leq i \leq d}\) and transition matrix \((p_{ij})_{1 \leq i, j \leq d}\), if
1. $P(X_0 = i) = \lambda_i$ for $i = 1, \ldots, d$.
2. $P(X_{n+1} = i_{n+1} | X_0 = i_0, \ldots, X_n = i_n) = p_{i_0i_1i_2\cdots i_{n+1}}$ for all $n \geq 0$ and all $i_0, \ldots, i_{n+1} \in M$ s.t. $P(X_0 = i_0, \ldots, X_n = i_n) > 0$.

The transition matrix of a Markov chain is a stochastic matrix, i.e. it has non-negative entries and the sum over a row is equal to one, $\sum_{j=1}^{d} p_{ij} = 1$, for all $1 \leq i \leq d$.

The following gives an equivalent characterisation of Markov chains, cf. [Nor97, Theorem 1.1.1.].

**Proposition 1.2.** A stochastic process $(X_n)_{n \geq 0}$ is a Markov chain with initial distribution $(\lambda_i)_{1 \leq i \leq d}$ and transition matrix $(p_{ij})_{1 \leq i,j \leq d}$ if and only if

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \lambda_{i_0} p_{i_0i_1} \cdots p_{i_{n-1}i_n}$$

for all $n \geq 0$ and all $i_0, i_1, \ldots, i_n \in M$.

If a group $G$ is acting on the state space $M$ of a Markov chain $(X_n)_{n \geq 0}$, then we can get a family of Markov chains $(g.X_n)_{n \geq 0}$ indexed by group elements $g \in G$. If all these Markov chains have the same transition matrices, then we call $(X_n)_{n \geq 0}$ a left-invariant random walk on $M$ (w.r.t. to the action of $G$). This is the case if and only if the transition probabilities satisfy

$$P(X_{n+1} = hg | X_n = h.x) = P(X_{n+1} = g | X_n = x)$$

for all $x, y \in M$, $h \in G$, and $n \geq 0$. If the state space is itself a group, then we consider the action defined by left multiplication. More precisely, we call a Markov chain $(X_n)_{n \geq 0}$ on a finite group $G$ a random walk on $G$, if

$$P(X_{n+1} = hg' | X_n = hg) = P(X_{n+1} = g' | X_n = g)$$

for all $g, g', h \in G$, $n \geq 0$.

**Example 1.3.** We describe a binary message that is transmitted in a network. During each transmission one of the bits may be flipped with a small probability $p > 0$ and all bits have the same probability to be flipped. But we assume here that two or more errors can not occur during a single transmission.

If the message has length $d$, then the state space for the Markov chain $(X_n)_{n \geq 0}$ describing the message after $n$ transmissions is equal to the $d$-dimensional hypercube $M = \{0,1\}^d \cong \mathbb{Z}_2^d$. The transition matrix is given by

$$p_{ij} = \begin{cases} 1 - p & \text{if } i = j, \\ p/d & \text{if } i,j \text{ differ only in one bit,} \\ 0 & \text{if } i,j \text{ differ in more than one bit.} \end{cases}$$

This random walk is invariant for the group structure of $\mathbb{Z}_2^d$ and also for the action of the symmetry group of the hypercube.
2 Quantum Markov chains

To motivate the definition of quantum Markov chains let us start with a reformulation of the classical situation. Let $M, G$ be (finite) sets. Any map $b : M \times G \to M$ may be called an action of $G$ on $M$. (Later we shall be interested in the case that $G$ is a group but for the moment it is enough to have a set.) Let $C^*M$ respectively $C^*G$ be the $*$-algebra of complex functions on $M$ respectively $G$. For all $g \in G$ we have unital $*$-homomorphisms $\alpha_g : C^*M \to C^*M$ given by $\alpha_g(f)(x) := f(b(x, g))$. They can be put together into a single unital $*$-homomorphism

$$\beta : C^*M \to C^*M \otimes C^*G, \quad f \mapsto \sum_{g \in G} \alpha_g(f) \otimes 1_{\{g\}},$$

where $1_{\{g\}}$ denotes the indicator function of $g$. A nice representation of such a structure can be given by a directed labeled multigraph. For example, the graph

![Graph](https://example.com/graph)

with set of vertices $M = \{x, y\}$ and set of labels $G = \{g, h\}$ represents the map $b : M \times G \to M$ with $b(x, g) = x$, $b(x, h) = y$, $b(y, g) = x = b(y, h)$. We get a natural noncommutative generalization just by allowing the algebras to become noncommutative. In [GKL04] the resulting structure is called a transition and is further analyzed. For us it is interesting to check that this is enough to construct a noncommutative or quantum Markov chain.

Let $B$ and $A$ be unital $C^*$-algebras and $\beta : B \to B \otimes A$ a unital $*$-homomorphism. Here $B \otimes A$ is the minimal $C^*$-tensor product [Sak71]. Then we can build up the following iterative scheme ($n \geq 0$).

$$j_0 : B \to B, \quad b \mapsto b$$

$$j_1 : B \to B \otimes A, \quad b \mapsto \beta(b) = b_{(0)} \otimes b_{(1)}$$

(Sweedler’s notation $b_{(0)} \otimes b_{(1)}$ stands for $\sum_i b_{0i} \otimes b_{1i}$ and is very convenient in writing formulas.)

$$j_n : B \to B \otimes \bigotimes_{1}^{n} A, \quad j_n = (j_{n-1} \otimes \text{id}_A) \circ \beta, \quad b \mapsto j_{n-1}(b_{(0)}) \otimes b_{(1)} \in \left( B \otimes \bigotimes_{1}^{n-1} A \right) \otimes A.$$

Clearly all the $j_n$ are unital $*$-homomorphisms. If we want to have an algebra $\tilde{B}$ which includes all their ranges we can form the infinite tensor product $\tilde{A} := \bigotimes_{1}^{\infty} A$ (the closure of the union of all $\bigotimes_{1}^{n} A$ with the natural inclusions $x \mapsto x \otimes 1$) and then $\tilde{B} := B \otimes \tilde{A}$. 
Denote by $\sigma$ the right shift on $\hat{A}$, i.e., $\sigma(a_1 \otimes a_2 \otimes \ldots) = 1 \otimes a_1 \otimes a_2 \otimes \ldots$
Using this we can also write

$$j_n : B \rightarrow \hat{B}, \ b \mapsto \hat{\beta}^n(b \otimes 1),$$

where $\hat{\beta}$ is a unital $^*$-homomorphism given by

$$\hat{\beta} : \hat{B} \rightarrow \hat{B}, \ b \otimes a \mapsto \beta \circ (id \otimes \sigma)(b \otimes a) = \beta(b) \otimes a,$$

i.e., by applying the shift we first obtain $b \otimes 1 \otimes a \in \hat{B}$ and then interpret “$\beta$” as the operation which replaces $b \otimes 1$ by $\beta(b)$. We may interpret $\hat{\beta}$ as a kind of time evolution producing $j_1, j_2, \ldots$

To do probability theory, consider states $\psi, \phi$ on $B, A$ and form product states

$$\psi \otimes \otimes_{i=1}^{n-1} \phi$$

for $B \otimes \otimes_{i=1}^{n-1} A$ (in particular for $n = \infty$ the infinite product state on $\hat{B}$, which we call $\Psi$). Now we can think of the $j_n$ as noncommutative random variables with distributions $\Psi \circ j_n$, and $(j_n)_{n \geq 0}$ is a noncommutative stochastic process [AFL82]. We call $\psi$ the initial state and $\phi$ the transition state.

In order to analyze this process, we define for $n \geq 1$ linear maps

$$Q_{[0,n-1]} : B \otimes \otimes_{i=1}^{n-1} A \rightarrow B \otimes \otimes_{i=1}^{n-1} A,$$

$$b \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n \mapsto b \otimes a_1 \otimes \ldots \otimes a_{n-1} \phi(a_n)$$

In particular $Q := Q_{[0,0]} = id \otimes \phi : B \otimes A \rightarrow B, \ b \otimes a \mapsto b \phi(a)$.

Such maps are often called slice maps. From a probabilistic point of view, it is common to refer to idempotent norm-one (completely) positive maps onto a $C^*$-subalgebra as (noncommutative) conditional expectations [Sak71].

Clearly the slice map $Q_{[0,n-1]}$ is a conditional expectation (with its range embedded by $x \mapsto x \otimes 1$) and it has the additional property of preserving the state, i.e., $\Psi \circ Q_{[0,n-1]} = \Psi$.

**Proposition 2.1. (Markov property)**

$$Q_{[0,n-1]} \circ j_n = j_{n-1} \circ T_{\phi}$$

where $T_{\phi} : B \rightarrow B, \ b \mapsto Q \beta(b) = (id \otimes \phi) \circ \beta(b) = b_{(0)} \phi(b_{(1)})$.

**Proof.**

$$Q_{[0,n-1]}j_n(b) = Q_{[0,n-1]}(j_{n-1}(b_{(0)}) \otimes b_{(1)}) = j_{n-1}(b_{(0)})\phi(b_{(1)}) = j_{n-1}T_{\phi}(b).$$

$\Box$
We interpret this as a Markov property of the process \((j_n)_{n \geq 0}\). Note that if there are state-preserving conditional expectations \(P_{n-1} \) onto \(j_{n-1}(B)\) and \(P_{[0,n-1]} \) onto the algebraic span of \(j_0(B), \ldots, j_{n-1}(B)\), then because \(P_{n-1}\) is dominated by \(P_{[0,n-1]}\) and \(P_{[0,n-1]}\) is dominated by \(Q_{[0,n-1]}\), we get

\[
P_{[0,n-1]} \circ j_n = j_{n-1} \circ T_\phi \quad (Markov\ property)
\]

The reader should check that for commutative algebras this is the usual Markov property of classical probability. Thus in the general case, we say that \((j_n)_{n \geq 0}\) is a quantum Markov chain on \(B\). The map \(T_\phi\) is called the transition operator of the Markov chain. In the classical case as discussed in Section 1 it can be identified with the transition matrix by choosing indicator functions of single points as a basis, i.e., \(T_\phi(1_{\{j\}}) = \sum_{i=1}^d p_{ij} 1_{\{i\}}\). It is an instructive exercise to start with a given transition matrix \((p_{ij})\) and to realize the classical Markov chain with the construction above.

Analogous to the classical formula in Proposition 1.2 we can also derive the following semigroup property for transition operators from the Markov property. It is one of the main reasons why Markov chains are easier than more general processes.

**Corollary 2.2. (Semigroup property)**

\[
Q j_n = T_\phi^n
\]

Finally we note that if \((\psi \otimes \phi) \circ \beta = \psi\) then \(\Psi \circ \beta = \Psi\). This implies that the Markov chain is stationary, i.e., correlations between the random variables depend only on time differences. In particular, the state \(\psi\) is then preserved by \(T_\phi\), i.e., \(\psi \circ T_\phi = \psi\).

The construction above is called coupling to a shift, and similar structures are typical for quantum Markov processes, see [Küm88, Go04].

### 3 Random walks on comodule algebras

Let us return to the map \(b : M \times G \to M\) considered in the beginning of the previous section. If \(G\) is group, then \(b : M \times G \to M\) is called a (left) action of \(G\) on \(M\), if it satisfies the following axioms expressing associativity and unit,

\[
b(b(x,g), h) = b(x, hg), \quad b(x,e) = x
\]

for all \(x \in M, g, h \in G, e \in G\) the unit of \(G\). In Section 1, we wrote \(g.x\) instead of \(b(x,g)\). As before we have the unital \(*\)-homomorphisms \(\alpha_g : \mathbb{C}^M \to \mathbb{C}^M\). Actually, in order to get a representation of \(G\) on \(\mathbb{C}^M\), i.e., \(\alpha_g \alpha_h = \alpha_{gh}\) for all \(g, h \in G\) we must modify the definition and use \(\alpha_g(f)(x) := f(b(b(x,g^{-1})\). (Otherwise we get an anti-representation. But this is a minor point at the moment.) In the associated coaction \(\beta : \mathbb{C}^M \to \mathbb{C}^M \otimes \mathbb{C}^G\) the axioms above
are turned into the coassociativity and counit properties. These make perfect sense not only for groups but also for quantum groups and we state them at once in this more general setting. We are rewarded with a particular interesting class of quantum Markov chains associated to quantum groups which we call random walks and which are the subject of this lecture.

Let $\mathcal{A}$ be a finite quantum group with comultiplication $\Delta$ and counit $\varepsilon$ (see Appendix A). A $C^*$-algebra $\mathcal{B}$ is called an $\mathcal{A}$-comodule algebra if there exists a unital $*$-algebra homomorphism $\beta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ such that

$$(\beta \otimes \text{id}) \circ \beta = (\text{id} \otimes \Delta) \circ \beta, \quad (\text{id} \otimes \varepsilon) \circ \beta = \text{id}.$$ 

Such a map $\beta$ is called a coaction. In Sweedler’s notation, the first equation applied to $b \in \mathcal{B}$ reads

$$b(0)(0) \otimes b(0)(1) \otimes b(1) = b(0) \otimes b(1)(1) \otimes b(1)(2),$$

which thus can safely be written as $b(0) \otimes b(1) \otimes b(2)$.

If we start with such a coaction $\beta$ then we can look at the quantum Markov chain constructed in the previous section in a different way. Define for $n \geq 1$

$$k_n : \mathcal{A} \to \mathcal{B} \otimes \hat{\mathcal{A}}$$

$$a \mapsto 1_B \otimes 1 \otimes \ldots \otimes 1 \otimes a \otimes 1 \otimes \ldots,$$

where $a$ is inserted at the $n$-th copy of $\mathcal{A}$. We can interpret the $k_n$ as (non-commutative) random variables. Note that the $k_n$ are identically distributed. Further, the sequence $j_0, k_1, k_2, \ldots$ is a sequence of tensor independent random variables, i.e., their ranges commute and the state acts as a product state on them. The convolution $j_0 * k_1$ is defined by

$$j_0 * k_1(b) := j_0(b(0)) k_1(b(1))$$

and it is again a random variable. (Check that tensor independence is needed to get the homomorphism property.) In a similar way we can form the convolution of the $k_n$ among each other. By induction we can prove the following formulas for the random variables $j_n$ of the chain.

**Proposition 3.1.**

$$j_n = (\beta \otimes \text{id} \otimes \ldots \otimes \text{id})(\beta \otimes \text{id} \otimes \ldots)(\beta \otimes \text{id})\beta$$

$$= (\text{id} \otimes \text{id} \otimes \ldots \otimes \Delta)(\text{id} \otimes \Delta)(\text{id} \otimes \Delta)\beta$$

$$= j_0 * k_1 * \ldots * k_n$$

Note that by the properties of coactions and comultiplications the convolution is associative and we do not need to insert brackets. The statement $j_n = j_0 * k_1 * \ldots * k_n$ can be put into words by saying that the Markov chain associated to a coaction is a chain with (tensor-)independent and stationary
Random Walks on Finite Quantum Groups

...increment
tions. Using the convolution of states we can write the distribution of \( j_n = j_0 \star k_1 \star \cdots \star k_n \) as \( \psi \star \phi^n \). For all \( b \in \mathcal{B} \) and \( n \geq 1 \) the transition operator \( T_\phi \) satisfies

\[
\psi(T_\phi^n(b)) = \Psi(j_n(b)) = \psi \star \phi^n(b),
\]

and from this we can verify that

\[
T_\phi^n = (\text{id} \otimes \phi^n) \circ \beta,
\]

i.e., given \( \beta \) the semigroup of transition operators \( T_\phi^n \) and the semigroup \( (\phi^n) \) of convolution powers of the transition state are essentially the same thing.

A quantum Markov chain associated to such a coaction is called a random walk on the \( \mathcal{A} \)-comodule algebra \( \mathcal{B} \). We have seen that in the commutative case this construction describes an action of a group on a set and the random walk derived from it. Because of this background, some authors call an action of a quantum group what we called a coaction. But this should always become clear from the context.

Concerning stationarity we get

**Proposition 3.2.** For a state \( \psi \) on \( \mathcal{B} \) the following assertions are equivalent:

(a) \( (\psi \otimes \text{id}) \circ \beta = \psi(\cdot) \mathbf{1} \).

(b) \( (\psi \otimes \phi) \circ \beta = \psi \) for all states \( \phi \) on \( \mathcal{A} \).

(c) \( (\psi \otimes \eta) \circ \beta = \psi \), where \( \eta \) is the Haar state on \( \mathcal{A} \) (see Appendix A).

**Proof.** (a)\( \Rightarrow \) (b) and (b)\( \Rightarrow \) (c) is clear. Assuming (c) and using the invariance properties of \( \eta \) we get for all states \( \phi \) on \( \mathcal{A} \)

\[
\psi = (\psi \otimes \eta) \beta = (\psi \otimes \eta \otimes \phi)(\text{id} \otimes \Delta) \beta = (\psi \otimes \eta \otimes \phi)(\beta \otimes \text{id}) \beta = (\psi \otimes \phi) \beta,
\]

which is (b).

\( \square \)

Such states are often called invariant for the coaction \( \beta \). Of course for special states \( \phi \) on \( \mathcal{A} \) there may be other states \( \psi \) on \( \mathcal{B} \) which also lead to stationary walks.

**Example 3.3.** For explicit examples we will use the eight-dimensional finite quantum group introduced by Kac and Paljutkin [KP66], see Appendix B.

Consider the commutative algebra \( \mathcal{B} = \mathbb{C}^4 \) with standard basis \( v_1 = (1,0,0,0), \ldots, v_4 = (0,0,0,1) \) (and component-wise multiplication). Defining an \( \mathcal{A} \)-coaction by

\[
\beta(v_1) = v_1 \otimes (e_1 + e_3) + v_2 \otimes (e_2 + e_4)
\]

\[
+v_3 \otimes \frac{1}{2} \left( a_{11} + \frac{1 - i}{\sqrt{2}} a_{12} + \frac{1 + i}{\sqrt{2}} a_{21} + a_{22} \right)
\]

\[
+v_4 \otimes \frac{1}{2} \left( a_{11} - \frac{1 - i}{\sqrt{2}} a_{12} - \frac{1 + i}{\sqrt{2}} a_{21} + a_{22} \right),
\]
\[ \beta(v_2) = v_1 \otimes (e_2 + e_4) + v_2 \otimes (e_1 + e_3) \]
\[ + v_3 \otimes \frac{1}{2} \left( a_{11} - \frac{1-i}{\sqrt{2}} a_{12} - \frac{1+i}{\sqrt{2}} a_{21} + a_{22} \right) \]
\[ + v_4 \otimes \frac{1}{2} \left( a_{11} + \frac{1-i}{\sqrt{2}} a_{12} + \frac{1+i}{\sqrt{2}} a_{21} + a_{22} \right), \]

\[ \beta(v_3) = v_1 \otimes \frac{1}{2} \left( a_{11} + \frac{1+i}{\sqrt{2}} a_{12} + \frac{1-i}{\sqrt{2}} a_{21} + a_{22} \right) \]
\[ + v_2 \otimes \frac{1}{2} \left( a_{11} - \frac{1+i}{\sqrt{2}} a_{12} - \frac{1-i}{\sqrt{2}} a_{21} + a_{22} \right) \]
\[ + v_3 \otimes (e_1 + e_2) + v_4 \otimes (e_3 + e_4), \]

\[ \beta(v_4) = v_1 \otimes \frac{1}{2} \left( a_{11} - \frac{1+i}{\sqrt{2}} a_{12} - \frac{1-i}{\sqrt{2}} a_{21} + a_{22} \right) \]
\[ + v_2 \otimes \frac{1}{2} \left( a_{11} + \frac{1+i}{\sqrt{2}} a_{12} + \frac{1-i}{\sqrt{2}} a_{21} + a_{22} \right) \]
\[ + v_3 \otimes (e_3 + e_4) + v_4 \otimes (e_1 + e_2), \]

\[ \mathbb{C}^4 \] becomes an \( \mathcal{A} \)-comodule algebra.

Let \( \phi \) be an arbitrary state on \( \mathcal{A} \). It can be parametrized by \( \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \geq 0 \) and \( x, y, z \in \mathbb{R} \) with \( \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 = 1 \) and \( x^2 + y^2 + z^2 \leq 1 \), cf. Subsection B.3 in the Appendix. Then the transition operator \( T_\phi = (\text{id} \otimes \phi) \circ \Delta \) on \( \mathbb{C}^4 \) becomes

\[ T_\phi = \begin{pmatrix}
\mu_1 + \mu_3 & \mu_2 + \mu_4 & \frac{\mu_5}{2} \left( 1 + \frac{x+y}{\sqrt{2}} \right) & \frac{\mu_5}{2} \left( 1 - \frac{x+y}{\sqrt{2}} \right) \\
\mu_2 + \mu_4 & \mu_1 + \mu_3 & \frac{\mu_5}{2} \left( 1 - \frac{x+y}{\sqrt{2}} \right) & \frac{\mu_5}{2} \left( 1 + \frac{x+y}{\sqrt{2}} \right) \\
\frac{\mu_5}{2} \left( 1 + \frac{x+y}{\sqrt{2}} \right) & \frac{\mu_5}{2} \left( 1 - \frac{x+y}{\sqrt{2}} \right) & \mu_3 + \mu_4 & \mu_1 + \mu_2 \\
\frac{\mu_5}{2} \left( 1 - \frac{x-y}{\sqrt{2}} \right) & \frac{\mu_5}{2} \left( 1 + \frac{x-y}{\sqrt{2}} \right) & \mu_3 + \mu_4 & \mu_1 + \mu_2
\end{pmatrix} \tag{3.1} \]

w.r.t. to the basis \( v_1, v_2, v_3, v_4 \).

The state \( \psi_0 : \mathcal{B} \rightarrow \mathbb{C} \) defined by \( \psi_0(v_1) = \psi_0(v_2) = \psi_0(v_3) = \psi_0(v_4) = \frac{1}{4} \) is invariant, i.e. we have

\[ \psi_0 \star \phi = (\psi_0 \otimes \phi) \circ \beta = \psi_0 \]

for any state \( \phi \) on \( \mathcal{A} \).
4 Random walks on finite quantum groups

The most important special case of the construction in the previous section is obtained when we choose $B = A$ and $\beta = \Delta$. Then we have a random walk on the finite quantum group $A$. Let us first show that this is indeed a generalization of a left invariant random walk as discussed in the Introduction and in Section 1. Using the coassociativity of $\Delta$ we see that the transition operator $T_\phi = (\text{id} \otimes \phi) \circ \Delta$ satisfies the formula

$$\Delta \circ T_\phi = (\text{id} \otimes T_\phi) \circ \Delta.$$ 

Suppose now that $B = A$ consists of functions on a finite group $G$ and $\beta = \Delta$ is the comultiplication which encodes the group multiplication, i.e.

$$\Delta(1_{\{g'\}}) = \sum_{h \in G} 1_{\{g'h^{-1}\}} \otimes 1_{\{h\}} = \sum_{h \in G} 1_{\{h^{-1}\}} \otimes 1_{\{hg'\}},$$

where $1_{\{g\}}$ denotes the indicator function of $g$. We also have

$$T_\phi(1_{\{g'\}}) = \sum_{g \in G} p_{g,g'} 1_{\{g\}},$$

where $(p_{g,g'})$ is the transition matrix. Compare Sections 1 and 2. Inserting these formulas yields

$$(\Delta \circ T_\phi) 1_{\{g'\}} = \Delta(\sum_{g \in G} p_{g,g'} 1_{\{g\}}) = \sum_{h \in G} 1_{\{h^{-1}\}} \otimes \sum_{g \in G} p_{g,g'} 1_{\{hg\}},$$

and

$$[\text{id} \otimes T_\phi \circ \Delta] 1_{\{g'\}} = (\text{id} \otimes T_\phi) \sum_{h \in G} 1_{\{h^{-1}\}} \otimes 1_{\{hg'\}} = \sum_{h \in G} 1_{\{h^{-1}\}} \otimes \sum_{g \in G} p_{hg,hg'} 1_{\{hg\}}.$$ 

We conclude that $p_{g,g'} = p_{hg,hg'}$ for all $g, g', h \in G$. This is the left invariance of the random walk which was already stated in the introduction in a more probabilistic language.

For random walks on a finite quantum group there are some natural special choices for the initial distribution $\psi$. On the one hand, one may choose $\psi = \varepsilon$ (the counit) which in the commutative case (i.e., for a group) corresponds to starting in the unit element of the group. Then the time evolution of the distributions is given by $\varepsilon \star \phi^n = \phi^n$. In other words, we get a convolution semigroup of states.

On the other hand, stationarity of the random walk can be obtained if $\psi$ is chosen such that

$$\psi \otimes \phi \circ \Delta = \psi.$$
(Note that stationarity of a random walk must be clearly distinguished from stationarity of the increments which for our definition of a random walk is automatic.) In particular we may choose the unique Haar state \( \eta \) of the finite quantum group \( A \) (see Appendix A).

**Proposition 4.1.** The random walks on a finite quantum group are stationary for all choices of \( \phi \) if and only if \( \psi = \eta \).

**Proof.** This follows by Proposition 3.2 together with the fact that the Haar state is characterized by its right invariance (see Appendix A).

\[ \square \]

## 5 Spatial Implementation

In this section we want to represent the algebras on Hilbert spaces and obtain spatial implementations for the random walks. On a finite quantum group \( A \) we can introduce an inner product

\[ \langle a, b \rangle = \eta(a^*b), \]

where \( a, b \in A \) and \( \eta \) is the Haar state. Because the Haar state is faithful (see Appendix A) we can think of \( A \) as a finite dimensional Hilbert space which we denote by \( \mathcal{H} \). Further we denote by \( \| \cdot \| \) the norm associated to this inner product. We consider the linear operator

\[ W : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \quad b \otimes a \mapsto \Delta(b)(1 \otimes a). \]

It turns out that this operator contains all information about the quantum group and thus it is called its fundamental operator. We discuss some of its properties.

(a) \( W \) is unitary.

**Proof.** Using \( (\eta \otimes \text{id}) \circ \Delta = \eta(\cdot)1 \) it follows that

\[ \| W b \otimes a \|^2 = \| \Delta(b)(1 \otimes a) \|^2 = \eta \otimes \eta((1 \otimes a^*) \Delta(b^*b)(1 \otimes a)) \]
\[ = \eta(a^*[(\eta \otimes \text{id})\Delta(b^*b)]a) = \eta(a^* \eta(b^*b)a) = \eta(b^*b) \eta(a^*a) \]
\[ = \eta \otimes \eta(b^*b \otimes a^*a) = \| b \otimes a \|^2. \]

A similar computation works for \( \sum b_i \otimes a_i \) instead of \( b \otimes a \). Thus \( W \) is isometric and, because \( \mathcal{H} \) is finite dimensional, also unitary. It can be easily checked using Sweedler’s notation that with the antipode \( S \) the inverse \( W^{-1} = W^* \) can be written explicitly as

\[ W^{-1}(b \otimes a) = [(1 \otimes S)\Delta(b)](1 \otimes a). \]

\[ \square \]
(b) $W$ satisfies the Pentagon Equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$.

This is an equation on $H \otimes H \otimes H$ and we have used the leg notation $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$, $W_{13} = (1 \otimes \tau) \circ W_{12} \circ (1 \otimes \tau)$, where $\tau$ is the flip, $\tau : H \otimes H \to H \otimes H$, $\tau(a \otimes b) = b \otimes a$.

Proof.

\[ W_{12}W_{13}W_{23}a \otimes b \otimes c = W_{12}W_{13}(a \otimes b_{(1)} \otimes b_{(2)})c = W_{12}a_{(1)} \otimes a_{(2)}b_{(2)}c = W_{23}a_{(1)} \otimes a_{(2)}b \otimes c = W_{23}W_{12}a \otimes b \otimes c. \]

\[ \square \]

Remark 5.1. The pentagon equation expresses the coassociativity of the co-
multiplication $\Delta$. Unitaries satisfying the pentagon equation have been called

\textit{multiplicative unitaries} in [BS93].

The operator $L_a$ of left multiplication by $a \in A$ on $H$

\[ L_a : H \to H, \quad c \mapsto ac \]

will often simply be written as $a$ in the following. It is always clear from the
context whether $a \in A$ or $a : H \to H$ is meant. We can also look at left
multiplication as a faithful representation $L$ of the $C^*$-algebra $A$ on $H$. In
this sense we have

\[ \Delta(a) = W(a \otimes 1)W^* \quad \text{for all } a \in A \]

Proof. Here $\Delta(a)$ and $a \otimes 1$ are left multiplication operators on $H \otimes H$. The
formula can be checked as follows.

\[ W(a \otimes 1)W^* b \otimes c = W(a \otimes 1)b_{(1)} \otimes (Sh_{(2)})c = Wab_{(1)} \otimes (Sh_{(2)})c \]

\[ = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}(Sh_{(3)})c = a_{(1)}b_{(1)} \otimes a_{(2)}c(b_{(2)})c \]

\[ = a_{(1)}b \otimes a_{(2)}c = \Delta(a)(b \otimes c) \]

\[ \square \]

By left multiplication we can also represent a random walk on a finite
quantum group $A$. Then $j_n(a)$ becomes an operator on an $(n+1)$-fold tensor
product of $H$. To get used to it let us show how the pentagon equation is
related to our Proposition 3.1 above.

\textbf{Theorem 5.2.}

\[ j_n(a) = W_{01}W_{02}\ldots W_{0n} (a \otimes 1 \otimes \ldots \otimes 1) W_{n}^* \ldots W_{02}^*W_{01}^*. \]

\[ W_{01}W_{02}\ldots W_{0n}|_H = W_{n-1,n}W_{n-2,n-1} \ldots W_{01}|_H, \]

where $|_H$ means restriction to $H \otimes 1 \otimes \ldots \otimes 1$ and this left position gets the
number zero.
Finally we check that \((id \otimes W)\) is a special case of von Neumann’s bicommutant theorem but of course the consequence of the pentagon equation. It corresponds to now define \(\gamma\) and check that it is a coaction. The property \((id \otimes id) \circ \gamma = \Delta(b)(1 \otimes x'a) = \Delta(b)(1 \otimes x')(1 \otimes a)\)

\[W(1 \otimes x')(b \otimes a) = W(b \otimes x'a) = \Delta(b)(1 \otimes x'a) = \Delta(b)(1 \otimes x')(1 \otimes a)\]

Because \(W\) commutes with all \(1 \otimes x'\) it must be contained in \(B(\mathcal{H}) \otimes A\). (This is a special case of von Neumann’s bicommutant theorem but of course the finite dimensional version used here is older and purely algebraic.) We can now define \(\gamma : B(\mathcal{H}) \rightarrow B(\mathcal{H}) \otimes A, \quad x \mapsto W(x \otimes 1) W^*\), and check that it is a coaction. The property \((\gamma \otimes id) \circ \gamma = (id \otimes \Delta) \circ \gamma\) is a consequence of the pentagon equation. It corresponds to

\[W_{01} W_{02} (x \otimes 1 \otimes \ldots \otimes 1) W_{02}^* W_{01}^* = W_{01}^* W_{02} W_{12} (x \otimes 1 \otimes \ldots \otimes 1) W_{12}^* W_{02}^* W_{01}^* = W_{12} W_{01} (x \otimes 1 \otimes \ldots \otimes 1) W_{01}^* W_{12}^* .\]

Finally we check that \((id \otimes \varepsilon) \circ \gamma = \text{id}\). In fact,

\[\gamma(x)(b \otimes a) = W(x \otimes 1)W^*(b \otimes a) = W(x \otimes 1) b_{(1)}B_{(2)} a\]
\[= [x(b_{1(1)})]_{(1)} \otimes [x(b_{1(1)})]_{(2)} (Sb_{2}) \ a \]

and thus

\[[(\text{id} \otimes \varepsilon) \gamma(x)](b) = [x(b_{1(1)})]_{(1)} \varepsilon([x(b_{1(1)})]_{(2)} \varepsilon(Sb_{2}))\]

\[= x(b_{1(1)}) \varepsilon(b_{2(2)}) = x(b),\]

i.e., \((\text{id} \otimes \varepsilon) \gamma(x) = x.\) Here we used \((\text{id} \otimes \varepsilon) \circ \Delta = \text{id}\) and the fact that \(\varepsilon \circ S = \varepsilon.\)

\[\square\]

**Remark 5.4.** The Haar state \(\eta\) on \(A\) is extended to a vector state on \(B(H)\) given by \(1 \in H.\) Thus we have also an extension of the probabilistic features of the random walk. Note further that arbitrary states on \(A\) can always be extended to vector states on \(B(H)\) (see Appendix A). This means that we also find the random walks with arbitrary initial state \(\psi\) and arbitrary transition state \(\phi\) represented on tensor products of the Hilbert space \(H\) and we have extensions also for them. This is an important remark because for many random walks of interest we would like to start in \(\psi = \varepsilon\) and all the possible steps of the walk are small, i.e., \(\phi\) is not a faithful state.

**Remark 5.5.** It is not possible to give \(B(H)\) the structure of a quantum group. For example, there cannot be a counit because \(B(H)\) as a simple algebra does not have nontrivial multiplicative linear functionals. Thus \(B(H)\) must be treated here as an \(A\)-comodule algebra.

In fact, it is possible to generalize all these results and to work with coactions on \(A\)-comodule algebras from the beginning. Let \(\beta : B \to B \otimes A\) be such a coaction. For convenience we continue to use the Haar state \(\eta\) on \(A\) and assume that there is a faithful stationary state \(\psi\) on \(B.\) As before we can consider \(A\) as a Hilbert space \(H\) and additionally we have on \(B\) an inner product induced by \(\psi\) which yields a Hilbert space \(K.\) By modifying the arguments above the reader should have no problems to verify the following assertions. Their proof is thus left as an exercise.

Define \(V : K \otimes H \to K \otimes H\) by \(b \otimes a \mapsto \beta(b)(1 \otimes a).\) Using Proposition 3.2, one can show that the stationarity of \(\psi\) implies that \(V\) is unitary. The map \(V\) satisfies \(V_{12} V_{13} W_{23} = W_{23} V_{12}\) (with leg notation on \(K \otimes H \otimes H\)) and the inverse can be written explicitly as \(V^{-1}(b \otimes a) = [(\text{id} \otimes S)\beta(b)](1 \otimes a).\) In [Wo96] such a unitary \(V\) is called adapted to \(W.\) We have \(\beta(b) = V (b \otimes 1) V^*\) for all \(b \in B.\) The associated random walk \((j_n)_{n \geq 0}\) on \(B\) can be implemented by

\[j_n(b) = V_0 V_1 \ldots V_n (b \otimes 1 \otimes \ldots \otimes 1) V_n^* \ldots V_2 V_1^*\]

with

\[V_0 V_2 \ldots V_n |K| = W_{n-1,n} W_{n-2,n-1} \ldots W_{12} V_0 |K|\]

These formulas can be used to extend this random walk to a random walk \((J_n)_{n \geq 0}\) on \(B(K).\)
Remark 5.6. There is an extended transition operator $Z : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ corresponding to the extension of the random walk. It can be described explicitly as follows. Define an isometry

$$v : \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}, \quad b \mapsto V^*(b \otimes 1) = b_{(0)} \otimes Sb_{(1)}.$$ 

Then we have

$$Z : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K}), \quad x \mapsto v^* x \otimes 1 v.$$

Because $v$ is isometric, $Z$ is a unital completely positive map which extends $T_\eta$. Such extended transition operators are discussed in the general frame of quantum Markov chains in [Go04]. See also [GKL04] for applications in noncommutative coding.

What is the meaning of these extensions? We think that this is an interesting question which leads to a promising direction of research. Let us indicate an interpretation in terms of quantization.

First we quickly review some facts which are discussed in more detail for example in [Maj95]. On $\mathcal{A}$ we have an action $T$ of its dual $\mathcal{A}^*$ which sends $\phi \in \mathcal{A}^*$ to

$$T_\phi : \mathcal{A} \to \mathcal{A}, \quad a \mapsto a_{(0)} \phi(a_{(1)}).$$

Note that if $\phi$ is a state then $T_\phi$ is nothing but the transition operator considered earlier. It is also possible to consider $T$ as a representation of the (convolution) algebra $\mathcal{A}^*$ on $\mathcal{H}$ which is called the regular representation. We can now form the crossed product $\mathcal{A} \rtimes \mathcal{A}^*$ which as a vector space is $\mathcal{A} \otimes \mathcal{A}^*$ and becomes an algebra with the multiplication

$$(c \otimes \phi)(d \otimes \psi) = c T_{\phi(1)}(d) \otimes \phi_{(2)} \star \psi,$$

where $\Delta \phi = \phi_{(1)} \otimes \phi_{(2)} \in \mathcal{A}^* \otimes \mathcal{A}^* \cong (\mathcal{A} \otimes \mathcal{A})^*$ is defined by $\Delta \phi(a \otimes b) = \phi(ab)$ for $a, b \in \mathcal{A}$.

There is a representation $S$ of $\mathcal{A} \rtimes \mathcal{A}^*$ on $\mathcal{H}$ called the Schrödinger representation and given by

$$S(c \otimes \phi) = L_c T_\phi.$$

Note further that the representations $L$ and $T$ are contained in $S$ by choosing $c \otimes \varepsilon$ and $1 \otimes \phi$.

Theorem 5.7.

$$S(\mathcal{A} \otimes \mathcal{A}^*) = \mathcal{B}(\mathcal{H}).$$

If $(c_i), (\phi_i)$ are dual bases in $\mathcal{A}, \mathcal{A}^*$, then the fundamental operator $W$ can be written as

$$W = \sum_i T_{\phi_i} \otimes L_{c_i}.$$

Proof. See [Maj95], 6.1.6. Note that this once more implies $W \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$ which was used earlier. □
We consider an example. For a finite group $G$ both $\mathcal{A}$ and $\mathcal{A}^*$ can be realized by the vector space of complex functions on $G$, but in the first case we have pointwise multiplication while in the second case we need convolution, i.e., indicator functions $\mathbf{1}_{\{g\}}$ for $g \in G$ are multiplied according to the group rule and for general functions the multiplication is obtained by linear extension. These indicator functions provide dual bases as occurring in the theorem and we obtain

$$W = \sum_{g \in G} T_g \otimes L_g,$$

where

$$L_g := L_{\mathbf{1}_{\{g\}}} : \mathbf{1}_{\{h\}} \mapsto \delta_{g,h} \mathbf{1}_{\{h\}},$$

$$T_g := T_{\mathbf{1}_{\{g\}}} : \mathbf{1}_{\{h\}} \mapsto \mathbf{1}_{\{h_{g^{-1}}\}}.$$

The reader may rediscover here the map $b : M \times G \to M$ (for $M = G$) discussed in the beginning of the Sections 2 and 3. It is also instructive to check the pentagon equation directly.

$$W_{12}W_{13}W_{23} = \sum_{a,b,c} (T_a \otimes L_a \otimes \mathbf{1})(T_b \otimes \mathbf{1} \otimes L_b)(\mathbf{1} \otimes T_c \otimes L_c)$$

$$= \sum_{a,b,c} T_a T_b \otimes L_a T_c \otimes L_b L_c = \sum_{a,c} T_a T_c \otimes L_a \otimes L_c$$

$$= \sum_{a,c} T_a \otimes L_a T_c \otimes L_c = \sum_{a,c} T_a \otimes L_{ac^{-1}} T_c \otimes L_c,$$

where the last equality is obtained by the substitution $a \mapsto ac^{-1}$. This coincides with

$$W_{23}W_{12} = \sum_{a,c} (\mathbf{1} \otimes T_c \otimes L_c)(T_a \otimes L_a \otimes \mathbf{1}) = \sum_{a,c} T_a \otimes T_c L_a \otimes L_c$$

precisely because of the relations

$$T_c L_a = L_{ac^{-1}} T_c \quad \text{for all } a, c \in G.$$

This is a version of the canonical commutation relations. In quantum mechanics, for $G = \mathbb{R}$, they encode Heisenberg’s uncertainty principle. This explains why $\mathcal{S}$ is called a Schrödinger representation. Its irreducibility in the case $G = \mathbb{R}$ is a well-known theorem. For more details see [Maj95, Chapter 6.1].

Thus Theorem 5.7 may be interpreted as a generalization of these facts to quantum groups. Our purpose here has been to give an interpretation of the extension of random walks to $\mathcal{B}(\mathcal{H})$ in terms of quantization. Indeed, we see that $\mathcal{B}(\mathcal{H})$ can be obtained as a crossed product, and similarly as in Heisenberg’s situation where the algebra $\mathcal{B}(\mathcal{H})$ occurs by appending to the observable of position a noncommuting observable of momentum, in our case we get $\mathcal{B}(\mathcal{H})$ by appending to the original algebra of observables all the transition operators of potential random walks.
6 Classical versions

In this section we will show how one can recover a classical Markov chain from a quantum Markov chain. We will apply a folklore theorem that says that one gets a classical Markov process, if a quantum Markov process can be restricted to a commutative algebra, cf. [AFL82, Küm88, BP95, Bia98, BKS97].

For random walks on quantum groups we have the following result.

**Theorem 6.1.** Let $\mathcal{A}$ be a finite quantum group, $(j_n)_{n \geq 0}$ a random walk on a finite dimensional $\mathcal{A}$-comodule algebra $\mathcal{B}$, and $\mathcal{B}_0$ a unital abelian sub-$\ast$-algebra of $\mathcal{B}$. The algebra $\mathcal{B}_0$ is isomorphic to the algebra of functions on a finite set, say $\mathcal{B}_0 \cong \mathbb{C}\{1, \ldots, d\}$.

If the transition operator $T_\phi$ of $(j_n)_{n \geq 0}$ leaves $\mathcal{B}_0$ invariant, then there exists a classical Markov chain $(X_n)_{n \geq 0}$ with values in $\{1, \ldots, d\}$, whose probabilities can be computed as time-ordered moments of $(j_n)_{n \in \mathbb{N}}$, i.e.,

$$P(X_0 = i_0, \ldots, X_\ell = i_\ell) = \Psi(j_0(1_{\{i_0\}}) \cdots j_\ell(1_{\{i_\ell\}}))$$

for all $\ell \geq 0$ and $i_0, \ldots, i_\ell \in \{1, \ldots, d\}$.

**Proof.** We use the indicator functions $1_{\{1\}}, \ldots, 1_{\{d\}}$,

$$1_{\{i\}}(j) = \delta_{ij}, \quad 1 \leq i, j \leq d,$

as a basis for $\mathcal{B}_0 \subseteq \mathcal{B}$. They are positive, therefore $\lambda_1 = \Psi(j_0(1_{\{i_1\}})), \ldots, \lambda_d = \Psi(j_0(1_{\{i_d\}}))$ are non-negative. Since furthermore

$$\lambda_1 + \cdots + \lambda_d = \Psi(j_0(1_{\{1\}})) + \cdots + \Psi(j_0(1_{\{d\}})) = \Psi(j_0(1)) = \Psi(1) = 1,$$

these numbers define a probability measure on $\{1, \ldots, d\}$.

Define now $(p_{ij})_{1 \leq i, j \leq d}$ by

$$T_\phi(1_{\{j\}}) = \sum_{i=1}^d p_{ij} 1_{\{i\}}.$$

Since $T_\phi = (\text{id} \otimes \phi) \circ \beta$ is positive, we have $p_{ij} \geq 0$ for $1 \leq i, j \leq d$. Furthermore, $T_\phi(1) = 1$ implies

$$1 = T_\phi(1) = T_\phi \left( \sum_{j=1}^d 1_{\{j\}} \right) = \sum_{j=1}^d \sum_{i=1}^d p_{ij} 1_{\{i\}}$$

i.e. $\sum_{j=1}^d p_{ij} = 1$ and so $(p_{ij})_{1 \leq i, j \leq d}$ is a stochastic matrix.

Therefore there exists a unique Markov chain $(X_n)_{n \geq 0}$ with initial distribution $(\lambda_i)_{1 \leq i \leq d}$ and transition matrix $(p_{ij})_{1 \leq i, j \leq d}$.

We show by induction that Equation (6.1) holds.
For $\ell = 0$ this is clear by definition of $\lambda_1, \ldots, \lambda_d$. Let now $\ell \geq 1$ and $i_0, \ldots, i_\ell \in \{1, \ldots, d\}$. Then we have
\[
\Psi(j_0(1_{i_0}) \cdots j_\ell(1_{i_\ell})) = \Psi(j_0(1_{i_0}) \cdots j_{\ell-1}(1_{i_{\ell-1}}) j_{\ell-1}(1_{i_{\ell}})) = \Psi(j_0(1_{i_0}) \cdots j_{\ell-1}(1_{i_{\ell-1}}) T_\phi(1_{i_\ell}))
\]
by Proposition 1.2.

Remark 6.2. If the condition that $T_\phi$ leaves $A_0$ invariant is dropped, then one can still compute the “probabilities”
\[
\text{“} P(X_0 = i_0, \ldots, X_\ell = i_\ell) \text{“} = \Psi(j_0(1_{i_0}) \cdots j_\ell(1_{i_\ell}))
\]

but in general they are no longer positive or even real, and so it is impossible to construct a classical stochastic process $(X_n)_{n \geq 0}$ from them. We give an example where no classical process exists in Example 6.4.

Example 6.3. The comodule algebra $B = \mathbb{C}^4$ that we considered in Example 3.3 is abelian, so we can take $B_0 = B$. For any pair of a state $\psi$ on $B$ and a state $\phi$ on $A$, we get a random walk on $B$ and a corresponding Markov chain $(X_n)_{n \geq 0}$ on $\{1, 2, 3, 4\}$. We identify $\mathbb{C}^{\{1,2,3,4\}}$ with $B$ by $v_i \equiv 1_{i\{i\}}$ for $i = 1, 2, 3, 4$.

The initial distribution of $(X_n)_{n \geq 0}$ is given by $\lambda_i = \psi(v_i)$ and the transition matrix is given in Equation (3.1).

Example 6.4. Let us now consider random walks on the Kac-Paljutkin quantum group $A$ itself. For the defining relations, the calculation of the dual of $A$ and a parametrization of all states on $A$, see Appendix B. Let us consider here transition states of the form
\[
\phi = \mu_1 \eta_1 + \mu_2 \eta_2 + \mu_3 \eta_3 + \mu_4 \eta_4,
\]
with $\mu_1, \mu_2, \mu_3, \mu_4 \in [0, 1]$, $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1$. 

The transition operators $T_\phi = (\text{id} \otimes \phi) \circ \Delta$ of these states leave the abelian subalgebra $A_0 = \text{span} \{e_1, e_2, e_3, e_4\} \cong \mathbb{C}^4$ invariant. The transition matrix of the associated classical Markov chain on $\{1, 2, 3, 4\}$ that arises by identifying $e_i \equiv 1_{\{i\}}$ for $i = 1, 2, 3, 4$ has the form

$$
\begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 & \mu_4 \\
\mu_2 & \mu_1 & \mu_4 & \mu_3 \\
\mu_3 & \mu_4 & \mu_1 & \mu_2 \\
\mu_4 & \mu_3 & \mu_2 & \mu_1
\end{pmatrix}.
$$

This is actually the transition matrix of a random walk on the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The subalgebra span $\{a_{11}, a_{12}, a_{21}, a_{22}\} \cong M_2$ is also invariant under these states, $T_\phi$ acts on it by

$$
T_\phi(X) = \mu_1 X + \mu_2 V_2^* XV_2 + \mu_3 V_3^* XV_3 + \mu_4 V_4^* XV_4
$$

for $X = a a_{11} + b a_{12} + c a_{21} + d a_{22} \cong (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, $a, b, c, d \in \mathbb{C}$, with

$$
V_2 = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Let $u = \begin{pmatrix} \cos \vartheta \\ e^{i \vartheta} \sin \vartheta \end{pmatrix}$ be a unit vector in $\mathbb{C}$ and denote by $p_u$ the orthogonal projection onto $p_u$. The maximal abelian subalgebra $A_u = \text{span} \{p_u, 1 - p_u\}$ in $M_2 \subset \mathcal{A}$ is in general not invariant under $T_\phi$.

E.g., for $u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we get the algebra $A_u = \text{span} \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \big| a, b \in \mathbb{C} \right\}$.

It can be identified with $\mathbb{C}^{1,2}$ via $\begin{pmatrix} a & b \\ b & a \end{pmatrix} \equiv (a + b) 1_{\{1\}} + (a - b) 1_{\{2\}}$.

Specializing to the transition state $\phi = \eta_2$ and starting from the Haar measure $\psi = \eta$, we see that the time-ordered joint moment

$$
\Psi(\{\eta_1\} \{1_{\{1\}}\} \{1_{\{2\}}\}) = \eta \left( 1_{\{1\}} T_{\eta_2} (1_{\{1\}} T_{\eta_2} (1_{\{2\}} T_{\eta_2} (1_{\{2\}}))) \right)
$$

is negative and can not be obtained from a classical Markov chain.

**Example 6.5.** For states in span $\{\eta_1, \eta_2, \eta_3, \eta_4, \alpha_{11} + \alpha_{22}\}$, the center $Z(A) = \text{span} \{e_1, e_2, e_3, e_4, a_{11} + a_{22}\}$ of $A$ is invariant under $T_\phi$, see also [NT04, Proposition 2.1]. A state on $A$, parametrized as in Equation (B.1), belongs to this set if and only if $x = y = z = 0$. With respect to the basis $e_1, e_2, e_3, e_4, a_{11} + a_{22}$ of $Z(A)$ we get
$T_\phi|_{Z(A)} = \begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\
\mu_2 & \mu_1 & \mu_4 & \mu_3 & \mu_5 \\
\mu_3 & \mu_4 & \mu_1 & \mu_2 & \mu_5 \\
\mu_4 & \mu_3 & \mu_2 & \mu_1 & \mu_5 \\
1 - \mu_5 & 1 - \mu_5 & 1 - \mu_5 & 1 - \mu_5 & 1 - \mu_5
\end{pmatrix}$

for the transition matrix of the classical Markov process that has the same time-ordered joint moments.

For Lévy processes or random walks on quantum groups there exists another way to prove the existence of a classical version that does not use the Markov property. We will illustrate this on an example.

**Example 6.6.** We consider restrictions to the center $Z(A)$ of $A$. If $a \in Z(A)$, then $a \otimes 1 \in Z(A \otimes A)$ and therefore

$$[a \otimes 1, \Delta(b)] = 0 \quad \text{for all } a, b \in Z(A).$$

This implies that the range of the restriction $(j_n|_{Z(A)})_{n \geq 0}$ of any random walk on $A$ to $Z(A)$ is commutative, i.e.

\[
[j_\ell(a), j_n(b)] = [(j_0 \ast k_1 \ast \cdots \ast k_\ell)(a), (j_0 \ast k_1 \ast \cdots \ast k_n)(b)]
\]

\[
= [(j_0 \ast k_1 \ast \cdots \ast k_\ell)(a), (j_0 \ast k_1 \ast \cdots \ast k_\ell)(b_{(1)}) \ast (k_{\ell+1} \ast \cdots \ast k_n)(b_{(2)})]
\]

\[
= m(j_\ell \otimes (k_{\ell+1} \ast \cdots \ast k_n)([a \otimes 1, \Delta(b)])) = 0
\]

for all $0 \leq \ell \leq n$ and $a, b \in Z(A)$. Here $m$ denotes the multiplication, $m : A \otimes A \to A$, $m(a \otimes b) = ab$ for $a, b \in A$. Therefore the restriction $(j_n|_{Z(A)})_{n \geq 0}$ corresponds to a classical process, see also [Sch93, Proposition 4.2.3] and [Fra99, Theorem 2.1].

Let us now take states for which $T_\phi$ does not leave the center of $A$ invariant, e.g. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = x = y = 0$, $\mu_5 = 1$, $z \in [-1, 1]$, i.e.

$$\phi_z = \frac{1 + z}{2} \alpha_{11} + \frac{1 - z}{2} \alpha_{22}.$$

In this particular case we have the invariant commutative subalgebra $A_0 = \text{span} \{e_1, e_2, e_3, e_4, a_{11}, a_{22}\}$ which contains the center $Z(A)$. If we identify $A_0$ with $\mathbb{C}^{(1, \ldots, 6)}$ via $e_1 \equiv \mathbf{1}_{(1)}, \ldots, e_4 \equiv \mathbf{1}_{(4)}, a_{11} \equiv \mathbf{1}_{(5)}, a_{22} \equiv \mathbf{1}_{(6)}$, then the transition matrix of the associated classical Markov chain is

$$\begin{pmatrix}
0 & 0 & 0 & \frac{1 + z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} \\
0 & 0 & 0 & \frac{1 + z}{2} & \frac{1 + z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} \\
0 & 0 & 0 & \frac{1 - z}{2} & \frac{1 + z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} \\
\frac{1 + z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1 - z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1 - z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1 - z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1 - z}{2} & \frac{1 - z}{2} & \frac{1 - z}{2} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$
The classical process corresponding to the center $Z(A)$ arises from this Markov chain by “gluing” the two states 5 and 6 into one. More precisely, if $(X_n)_{n \geq 0}$ is a Markov chain that has the same time-ordered moments as $(j_n)_{n \geq 0}$ restricted to $A_0$, and if $g : \{1, \ldots, 6\} \to \{1, \ldots, 5\}$ is the mapping defined by $g(i) = i$ for $i = 1, \ldots, 5$ and $g(6) = 5$, then $(Y_n)_{n \geq 0}$ with $Y_n = g(X_n)$, for $n \geq 0$, has the same joint moments as $(j_n)_{n \geq 0}$ restricted to the center $Z(A)$ of $A$. Note that $(Y_n)_{n \geq 0}$ is not a Markov process.

7 Asymptotic behavior

Theorem 7.1. Let $\phi$ be a state on a finite quantum group $A$. Then the Cesaro mean
\[
\phi_n = \frac{1}{n} \sum_{k=1}^{n} \phi^* k, \quad n \in \mathbb{N}
\]
converges to an idempotent state on $A$, i.e. to a state $\phi_\infty$ such that $\phi_\infty \ast \phi_\infty = \phi_\infty$.

Proof. Let $\phi'$ be an accumulation point of $(\phi_n)_{n \geq 0}$, this exists since the states on $A$ form a compact set. We have
\[
||\phi_n - \phi \ast \phi_n|| = \frac{1}{n} ||\phi - \phi^{*n+1}|| \leq \frac{2}{n}
\]
and choosing a sequence $(n_k)_{k \geq 0}$ such that $\phi_{n_k} \to \phi'$, we get $\phi \ast \phi' = \phi'$ and similarly $\phi' \ast \phi = \phi'$. By linearity this implies $\phi_n \ast \phi' = \phi' = \phi' \ast \phi_n$.

If $\phi''$ is another accumulation point of $(\phi_n)$ and $(m_l)_{l \geq 0}$ a sequence such that $\phi_{m_l} \to \phi''$, then we get $\phi'' \ast \phi' = \phi' = \phi' \ast \phi''$ and thus $\phi' = \phi''$ by symmetry. Therefore the sequence $(\phi_n)$ has a unique accumulation point, i.e., it converges. $\square$

Remark 7.2. If $\phi$ is faithful, then the Cesaro limit $\phi_\infty$ is the Haar state on $A$.

Remark 7.3. Due to “cyclicity” the sequence $(\phi^* n)_{n \in \mathbb{N}}$ does not converge in general. Take, e.g., the state $\phi = \eta_2$ on the Kac-Paljutkin quantum group $A$, then we have
\[
\eta_2^n = \begin{cases} 
\eta_2 & \text{if } n \text{ is odd}, \\
\varepsilon & \text{if } n \text{ is even},
\end{cases}
\]
but
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \eta_2^k = \frac{\varepsilon + \eta_2}{2}.
\]

Example 7.4. Pala [Pala96] has shown that there exist exactly the following eight idempotent states on the Kac-Paljutkin quantum group [KP66],...
\[
\begin{align*}
\rho_1 &= \eta_1 = \varepsilon, \\
\rho_2 &= \frac{1}{2}(\eta_1 + \eta_2), \\
\rho_3 &= \frac{1}{2}(\eta_1 + \eta_3), \\
\rho_4 &= \frac{1}{2}(\eta_1 + \eta_4), \\
\rho_5 &= \frac{1}{4}(\eta_1 + \eta_2 + \eta_3 + \eta_4), \\
\rho_6 &= \frac{1}{4}(\eta_1 + \eta_4) + \frac{1}{2} \alpha_{11}, \\
\rho_7 &= \frac{1}{4}(\eta_1 + \eta_4) + \frac{1}{2} \alpha_{22}, \\
\rho_8 &= \frac{1}{8}(\eta_1 + \eta_2 + \eta_3 + \eta_4) + \frac{1}{4}(\alpha_{11} + \alpha_{22}) = \eta.
\end{align*}
\]

On locally compact groups idempotent probability measures are Haar measures on some compact subgroup, cf. [Hey77, 1.5.6]. But Pal has shown that \(\rho_6\) and \(\rho_7\) are not Haar states on some “quantum sub-group” of \(\mathcal{A}\).

To understand this, we compute the null spaces \(N_\rho = \{a | \rho(a^*a) = 0\}\) for the idempotent states. We get

\[
\begin{align*}
N_\varepsilon &= \text{span}\{e_2, e_3, e_4, a_{11}, a_{12}, a_{21}, a_{22}\}, \\
N_{\rho_2} &= \text{span}\{e_3, e_4, a_{11}, a_{12}, a_{21}, a_{22}\}, \\
N_{\rho_3} &= \text{span}\{e_2, e_4, a_{11}, a_{12}, a_{21}, a_{22}\}, \\
N_{\rho_4} &= \text{span}\{e_2, e_3, a_{11}, a_{12}, a_{21}, a_{22}\}, \\
N_{\rho_5} &= \text{span}\{a_{11}, a_{12}, a_{21}, a_{22}\}, \\
N_{\rho_6} &= \text{span}\{e_2, e_3, a_{12}, a_{22}\}, \\
N_{\rho_7} &= \text{span}\{e_2, e_3, a_{11}, a_{21}\}, \\
N_\eta &= \{0\}.
\end{align*}
\]

All null spaces of idempotent states are coideals. \(N_\varepsilon, N_{\rho_2}, N_{\rho_3}, N_{\rho_4}, N_{\rho_5}, N_\eta\) are even Hopf ideals, so that we can obtain new quantum groups by dividing out these null spaces. The idempotent states \(\varepsilon, \rho_2, \rho_3, \rho_4, \rho_5, \eta\) are equal to the composition of the canonical projection onto this quotient and the Haar state of the quotient. In this sense they can be understood as Haar states on quantum subgroups of \(\mathcal{A}\). We obtain the following quantum groups,

\[
\begin{align*}
\mathcal{A}/N_\varepsilon &\cong \mathbb{C} \cong \text{functions on the trivial group}, \\
\mathcal{A}/N_{\rho_2} &\cong \mathcal{A}/N_{\rho_3} \cong \mathcal{A}/N_{\rho_4} \cong \text{functions on the group } \mathbb{Z}_2, \\
\mathcal{A}/N_{\rho_5} &\cong \text{functions on the group } \mathbb{Z}_2 \times \mathbb{Z}_2, \\
\mathcal{A}/N_\eta &\cong \mathcal{A}.
\end{align*}
\]

But the null spaces of \(\rho_6\) and \(\rho_7\) are only coideals and left ideals. Therefore the quotients \(\mathcal{A}/N_{\rho_6}\) and \(\mathcal{A}/N_{\rho_7}\) inherit only a \(\mathcal{A}\)-module coalgebra structure,
but no quantum group structure, and $\rho_6, \rho_7$ can not be interpreted as Haar states on some quantum subgroup of $\mathcal{A}$, cf. [Pal96].

We define an order for states on $\mathcal{A}$ by

$$\phi_1 \preceq \phi_2 \iff N_{\phi_1} \subseteq N_{\phi_2}.$$  

The resulting lattice structure for the idempotent states on $\mathcal{A}$ can be represented by the following Hasse diagram,

![Hasse diagram](image)

Note that the convolution product of two idempotent states is equal to their greatest lower bound in this lattice, $\rho_i * \rho_j = \rho_i \wedge \rho_j$ for $i, j, \in \{1, \ldots, 8\}$.

### A Finite quantum groups

In this section we briefly summarize the facts on finite quantum groups that are used throughout the main text. For proofs and more details, see [KP66, Maj95, VD97].

Recall that a bialgebra is a unital associative algebra $\mathcal{A}$ equipped with two unital algebra homomorphisms $\varepsilon : \mathcal{A} \to \mathbb{C}$ and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$$

$$(\text{id} \otimes \varepsilon) \circ \Delta = \text{id} = (\varepsilon \otimes \text{id}) \circ \Delta.$$  

We call $\varepsilon$ and $\Delta$ the counit and the comultiplication or coproduct of $\mathcal{A}$.

For the coproduct $\Delta(a) = \sum_i a_{(1)i} \otimes a_{(2)i} \in \mathcal{A} \otimes \mathcal{A}$ we will often suppress the summation symbol and use the shorthand notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ introduced by Sweedler [Swe69].

If $\mathcal{A}$ has an involution $* : \mathcal{A} \to \mathcal{A}$ such that $\varepsilon$ and $\Delta$ are $*$-algebra homomorphisms, then we call $\mathcal{A}$ a $*$-bialgebra or an involutive bialgebra.

If there exists furthermore a linear map $S : \mathcal{A} \to \mathcal{A}$ (called antipode) satisfying

$$a_{(1)} S(a_{(2)}) = \varepsilon(a) \mathbf{1} = S(a_{(1)}) a_{(2)}$$

for all $a \in \mathcal{A}$, then we call $\mathcal{A}$ a $*$-Hopf algebra or an involutive Hopf algebra.
Definition A.1. A finite quantum group is a finite dimensional $C^*$-Hopf algebra, i.e. a $*$-Hopf algebra $\mathcal{A}$, whose algebra is a finite dimensional $C^*$-algebra.

Note that finite dimensional $C^*$-algebras are very concrete objects, namely they are multi-matrix algebras $\bigoplus_{n=1}^{N} M_{k_n}$, where $M_k$ denotes the algebra of $k \times k$-matrices. Not every multi-matrix algebra carries a Hopf algebra structure. For example, the direct sum must contain a one-dimensional summand to make possible the existence of a counit.

First examples are of course the group algebras of finite groups. Another example is examined in detail in Appendix B.

Theorem A.2. Let $\mathcal{A}$ be a finite quantum group. Then there exists a unique state $\eta$ on $\mathcal{A}$ such that

$$(\text{id} \otimes \eta) \circ \Delta(a) = \eta(a)1$$

for all $a \in \mathcal{A}$.

The state $\eta$ is called the Haar state of $\mathcal{A}$. The defining property (A.1) is called left invariance. On finite (and more generally on compact) quantum groups left invariance is equivalent to right invariance, i.e. the Haar state satisfies also

$$(\eta \otimes \text{id}) \circ \Delta(a) = \eta(a)1.$$

One can show that it is even a faithful trace, i.e. $\eta(a^*a) = 0$ implies $a = 0$ and

$$\eta(ab) = \eta(ba)$$

for all $a, b \in \mathcal{A}$.

This is a nontrivial result. See [VD97] for a careful discussion of it. Using the unique Haar state we also get a distinguished inner product on $\mathcal{A}$, namely for $a, b \in \mathcal{A}$

$$\langle a, b \rangle = \eta(a^*b).$$

The corresponding Hilbert space is denoted by $\mathcal{H}$.

Proposition A.3. Every state on $\mathcal{A}$ can be realized as a vector state in $\mathcal{H}$.

Proof. Because $\mathcal{A}$ is finite dimensional every linear functional can be written in the form

$$\phi_a : b \mapsto \eta(a^*b) = \langle a, b \rangle.$$

Such a functional is positive iff $a \in \mathcal{A}$ is positive. In fact, since $\eta$ is a trace, it is clear that $a \geq 0$ implies $\phi_a \geq 0$. Conversely, assume $\phi_a \geq 0$. Convince yourself that it is enough to consider $a, b \in M_k$ where $M_k$ is one of the summands of the multi-matrix algebra $\mathcal{A}$. The restriction of $\eta$ is a multiple of the usual trace. Inserting the one-dimensional projections for $b$ shows that $a$ is positive.

Because $a$ is positive there is a unique positive square root. We can now write $\phi_a = \langle a^{\frac{1}{2}}, a^{\frac{1}{2}} \rangle$ and if $\phi_a$ is a state then $a^{\frac{1}{2}}$ is a unit vector in $\mathcal{H}$.

$\square$
Note that an equation \( \phi = \langle d, \cdot \rangle \) does not determine \( d \) uniquely. But the vector constructed in the proof is unique and all these vectors together generate a positive cone associated to \( \eta \).

The following result was already introduced and used in Section 5.

**Theorem A.4.** Let \( A \) be a finite quantum group with Haar state \( \eta \). Then the map \( W : A \otimes A \to A \otimes A \) defined by

\[
W(b \otimes a) = \Delta(b)(1 \otimes a), \quad a, b \in A,
\]

is unitary with respect to the inner product defined by

\[
\langle b \otimes a, d \otimes c \rangle = \eta(b^*d) \eta(a^*c),
\]

for \( a, b, c, d \in A \).

Furthermore, it satisfies the pentagon equation

\[
W_{12}W_{13}W_{23} = W_{23}W_{12}.
\]

We used the leg notation \( W_{12} = W \otimes \text{id} \), \( W_{23} = \text{id} \otimes W \), \( W_{13} = (\text{id} \otimes \tau) \circ W_{12} \circ (\text{id} \otimes \tau) \), where \( \tau \) is the flip, \( \tau : A \otimes A \to A \otimes A \), \( \tau(a \otimes b) = b \otimes a \).

**Remark A.5.** The operator \( W : A \otimes A \to A \otimes A \) is called the fundamental operator or multiplicative unitary of \( A \), cf. [BS93, BBS99].

**B The eight-dimensional Kac-Paljutkin quantum group**

In this section we give the defining relations and the main structure of an eight-dimensional quantum group introduced by Kac and Paljutkin [KP66]. This is actually the smallest finite quantum groups that does not come from a group as the group algebra or the algebra of functions on the group. In other words, it is the \( C^* \)-Hopf algebra with the smallest dimension, which is neither commutative nor cocommutative.

Consider the multi-matrix algebra \( A = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \), with the usual multiplication and involution. We shall use the basis

\[
\begin{align*}
  e_1 &= 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0, & a_{11} &= 0 \oplus 0 \oplus 0 \oplus 0 \oplus 10, \\
  e_2 &= 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0, & a_{12} &= 0 \oplus 0 \oplus 0 \oplus 0 \oplus 01, \\
  e_3 &= 0 \oplus 0 \oplus 1 \oplus 0 \oplus 0, & a_{21} &= 0 \oplus 0 \oplus 0 \oplus 0 \oplus 00, \\
  e_4 &= 0 \oplus 0 \oplus 0 \oplus 1 \oplus 0, & a_{22} &= 0 \oplus 0 \oplus 0 \oplus 0 \oplus 01.
\end{align*}
\]

The algebra \( A \) is an eight-dimensional \( C^* \)-algebra. Its unit is of course \( 1 = e_1 + e_2 + e_3 + e_4 + a_{11} + a_{22} \). We shall need the trace \( \text{Tr} \) on \( A \),
Finite quantum groups have unique Haar elements

B.1 The Haar state

The following defines a coproduct on $A$,

$$
\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4 \\
+ \frac{1}{2} a_{11} \otimes a_{11} + \frac{1}{2} a_{12} \otimes a_{12} + \frac{1}{2} a_{21} \otimes a_{21} + \frac{1}{2} a_{22} \otimes a_{22},
$$

$$
\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3 \\
+ \frac{1}{2} a_{11} \otimes a_{22} + \frac{1}{2} a_{22} \otimes a_{11} + \frac{i}{2} a_{21} \otimes a_{12} + \frac{i}{2} a_{12} \otimes a_{21},
$$

$$
\Delta(e_3) = e_1 \otimes e_3 + e_2 \otimes e_4 + e_3 \otimes e_1 + e_4 \otimes e_2 \\
+ \frac{1}{2} a_{11} \otimes a_{22} + \frac{1}{2} a_{22} \otimes a_{11} + \frac{i}{2} a_{21} \otimes a_{12} + \frac{i}{2} a_{12} \otimes a_{21},
$$

$$
\Delta(e_4) = e_1 \otimes e_4 + e_2 \otimes e_3 + e_3 \otimes e_2 + e_4 \otimes e_1 \\
+ \frac{1}{2} a_{11} \otimes a_{11} + \frac{1}{2} a_{22} \otimes a_{22} + \frac{1}{2} a_{12} \otimes a_{12} + \frac{1}{2} a_{22} \otimes a_{22},
$$

$$
\Delta(a_{11}) = e_1 \otimes a_{11} + a_{11} \otimes e_1 + e_2 \otimes a_{22} + a_{22} \otimes e_2 \\
+ e_3 \otimes a_{22} + a_{22} \otimes e_3 + e_4 \otimes a_{11} + a_{11} \otimes e_4,
$$

$$
\Delta(a_{12}) = e_1 \otimes a_{12} + a_{12} \otimes e_1 + i e_2 \otimes a_{21} - i a_{21} \otimes e_2 \\
- i e_3 \otimes a_{21} + i a_{21} \otimes e_3 - e_4 \otimes a_{12} - a_{12} \otimes e_4,
$$

$$
\Delta(a_{21}) = e_1 \otimes a_{21} + a_{21} \otimes e_1 - i e_2 \otimes a_{12} + i a_{12} \otimes e_2 \\
+ i e_3 \otimes a_{12} - i a_{12} \otimes e_3 - e_4 \otimes a_{21} - a_{21} \otimes e_4,
$$

$$
\Delta(a_{22}) = e_1 \otimes a_{22} + a_{22} \otimes e_1 + e_2 \otimes a_{11} + a_{11} \otimes e_2 \\
+ e_3 \otimes a_{11} + a_{11} \otimes e_3 + e_4 \otimes a_{22} + a_{22} \otimes e_4.
$$

The counit is given by

$$
\varepsilon \left( x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right) = x_1
$$

The antipode is the transpose map, i.e.

$$
S(e_i) = e_i, \quad S(a_{jk}) = a_{kj},
$$

for $i = 1, 2, 3, 4, j, k = 1, 2$.

B.1 The Haar state

Finite quantum groups have unique Haar elements $h$ satisfying $h^* = h = h^2,$

$\varepsilon(h) = 1,$ and
\[ ah = \varepsilon(a)h = ha \quad \text{for all } a \in \mathcal{A}, \]
cf. \cite{VD97}. For the Kac-Paljutkin quantum group it is given by \( h = e_1 \). An invariant functional is given by \( \phi(a) = \text{Tr}(aK^{-1}) \), with \( K = (\text{Tr} \otimes \text{id})\Delta(h) = e_1 + e_2 + e_3 + e_4 + \frac{1}{2}(a_{11} + a_{22}) \) and \( K^{-1} = e_1 + e_2 + e_3 + e_4 + 2(a_{11} + a_{22}) \).

On an arbitrary element of \( \mathcal{A} \) the action of \( \phi \) is given by
\[
\phi \left(x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right) = x_1 + x_2 + x_3 + x_4 + 2c_{11} + 2c_{22}.
\]
Normalizing \( \eta \) so that \( \eta(1) = 1 \), we get the Haar state \( \eta = \frac{1}{8}\phi \).

### B.2 The dual of \( \mathcal{A} \)

The dual \( \mathcal{A}^* \) of a finite quantum group \( \mathcal{A} \) is again a finite quantum group, see \cite{VD97}. Its morphisms are the duals of the morphisms of \( \mathcal{A} \), e.g.
\[
m_{\mathcal{A}^*} = \Delta_{\mathcal{A}^*} : \mathcal{A}^* \otimes \mathcal{A}^* \cong (\mathcal{A} \otimes \mathcal{A})^* \to \mathcal{A}^*, \quad m_{\mathcal{A}^*}(\phi_1 \otimes \phi_2) = (\phi_1 \otimes \phi_2) \circ \Delta
\]
and
\[
\Delta_{\mathcal{A}^*} = m_{\mathcal{A}^*} : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^* \cong (\mathcal{A} \otimes \mathcal{A})^*, \quad \Delta_{\mathcal{A}^*} \phi = \phi \circ m_{\mathcal{A}^*}.
\]
The involution of \( \mathcal{A}^* \) is given by \( \phi^*(a) = \overline{\phi((Sa)^*)} \) for \( \phi \in \mathcal{A}^*, a \in \mathcal{A} \). To show that \( \mathcal{A}^* \) is indeed a \( C^* \)-algebra, one can show that the dual regular action of \( \mathcal{A}^* \) on \( \mathcal{A} \) defined by
\[
T_\phi a = \phi(a_{(2)})a_{(1)} \quad \text{for } \phi \in \mathcal{A}^*, a \in \mathcal{A},
\]
is a faithful \( * \)-representation of \( \mathcal{A}^* \) w.r.t. the inner product on \( \mathcal{A} \) defined by
\[
\langle a, b \rangle = \eta(a^* b)
\]
for \( a, b \in \mathcal{A} \), cf. \cite[Proposition 2.3]{VD97}.

For the Kac-Paljutkin quantum group \( \mathcal{A} \) the dual \( \mathcal{A}^* \) actually turns out to be isomorphic to \( \mathcal{A} \) itself.

Denote by \( \{ \eta_1, \eta_2, \eta_3, \eta_4, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \} \) the basis of \( \mathcal{A}^* \) that is dual to \( \{ e_1, e_2, e_3, e_4, a_{11}, a_{12}, a_{21}, a_{22} \} \), i.e. the functionals on \( \mathcal{A} \) defined by
\[
\eta_i(e_j) = \delta_{ij}, \quad \eta_i(a_{rs}) = 0, \quad \alpha_{kl}(e_j) = 0, \quad \alpha_{kl}(a_{rs}) = \delta_{kr}\delta_{ls},
\]
for \( i, j = 1, 2, 3, 4, k, \ell, r, s = 1, 2 \).

We leave the verification of the following as an exercise.

The functionals
\[
\begin{align*}
f_1 &= \frac{1}{8}(\eta_1 + \eta_2 + \eta_3 + \eta_4 + 2\alpha_{11} + 2\alpha_{22}), \\
f_2 &= \frac{1}{8}(\eta_1 - \eta_2 - \eta_3 + \eta_4 - 2\alpha_{11} + 2\alpha_{22}), \\
f_3 &= \frac{1}{8}(\eta_1 - \eta_2 - \eta_3 + \eta_4 + 2\alpha_{11} - 2\alpha_{22}), \\
f_4 &= \frac{1}{8}(\eta_1 + \eta_2 + \eta_3 + \eta_4 - 2\alpha_{11} - 2\alpha_{22}),
\end{align*}
\]
are minimal projections in $\mathcal{A}^*$. Furthermore
\begin{align*}
b_{11} &= \frac{1}{4} (\eta_1 + \eta_2 - \eta_3 - \eta_4), \\
b_{12} &= \frac{1 - i}{2\sqrt{2}} (\alpha_{12} + i\alpha_{21}), \\
b_{21} &= \frac{1 + i}{2\sqrt{2}} (\alpha_{12} - i\alpha_{21}), \\
b_{22} &= \frac{1}{4} (\eta_1 - \eta_2 + \eta_3 - \eta_4),
\end{align*}
are matrix units, i.e. satisfy the relations
\begin{align*}
b_{ij}b_{k\ell} &= \delta_{jk}b_{i\ell} \quad \text{and} \quad (b_{ij})^* = b_{ji},
\end{align*}
and the “mixed” products vanish,
\begin{align*}
f_i b_{jk} &= 0 = b_{jk} f_i, \quad i = 1, 2, 3, 4, \quad j, k = 1, 2.
\end{align*}
Therefore $\mathcal{A}^* \cong \mathbb{C}^4 \oplus \mathcal{M}_2(\mathbb{C}) \cong \mathcal{A}$ as an algebra. But actually, $e_i \mapsto f_i$ and $a_{ij} \mapsto b_{ij}$ defines even a $C^*$-Hopf algebra isomorphism from $\mathcal{A}$ to $\mathcal{A}^*$.

### B.3 The states on $\mathcal{A}$

On $\mathbb{C}$ there exists only one state, the identity map. States on $\mathcal{M}_2(\mathbb{C})$ are given by density matrices, i.e., positive semi-definite matrices with trace one. More precisely, for any state $\phi$ on $\mathcal{M}_2(\mathbb{C})$ there exists a unique density matrix $\rho \in \mathcal{M}_2(\mathbb{C})$ such that
\begin{align*}
\phi(A) &= \text{Tr}(\rho A),
\end{align*}
for all $A \in \mathcal{M}_2(\mathbb{C})$. The $2 \times 2$ density matrices can be parametrized by the unit ball $B_1 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$,
\begin{align*}
\rho(x, y, z) &= \frac{1}{2} \left( \begin{array}{cc}
1 + z & x + iy \\
x - iy & 1 - z
\end{array} \right).
\end{align*}
A state on $\mathcal{A}$ is a convex combination of states on the four copies of $\mathbb{C}$ and a state on $\mathcal{M}_2(\mathbb{C})$. All states on $\mathcal{A}$ can therefore be parametrized by the set $\{(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, x, y, z) \in \mathbb{R}^9 | x^2 + y^2 + z^2 = 1; \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 = 1; \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \geq 0\}$. They are given by
\begin{align*}
\phi &= \text{Tr}(\mu \cdot) = 8\eta(K \mu \cdot),
\end{align*}
where
\begin{align*}
\mu &= \mu_1 \oplus \mu_2 \oplus \mu_3 \oplus \mu_4 \oplus \frac{\mu_5}{2} \left( \begin{array}{cc}
1 + z & x + iy \\
x - iy & 1 - z
\end{array} \right).
\end{align*}
With respect to the dual basis, the state $\phi$ can be written as
\[
\phi = \mu_1 \eta_1 + \mu_2 \eta_2 + \mu_3 \eta_3 + \mu_4 \eta_4 + \frac{\mu_5}{2} (1 + z) \alpha_{11} + (x - iy) \alpha_{12} + (x + iy) \alpha_{21} + (1 - z) \alpha_{22}.
\] (B.1)

The regular representation \( T_\phi = (\text{id} \otimes \phi) \circ \Delta \) of \( \phi \) on \( \mathcal{A} \) has the matrix
\[
\begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} \\
\mu_2 & \mu_1 & \mu_4 & \mu_3 & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} \\
\mu_3 & \mu_4 & \mu_1 & \mu_2 & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} \\
\mu_4 & \mu_3 & \mu_2 & \mu_1 & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} \\
\frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} \\
\frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} \\
\frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} \\
\frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_1 + 2}{2\sqrt{2}} \\
\frac{\mu_5}{2} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} \\
\frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}} & \frac{\mu_5}{2} & \frac{\mu_1 + 2}{2\sqrt{2}} & \frac{\mu_2 - 2}{2\sqrt{2}} & \frac{\mu_3 - 2}{2\sqrt{2}} & \frac{\mu_4 - 2}{2\sqrt{2}}
\end{pmatrix}
\]

with respect to the basis \((2\sqrt{2}e_1, 2\sqrt{2}e_2, 2\sqrt{2}e_3, 2\sqrt{2}e_4, 2a_{11}, 2a_{12}, 2a_{21}, 2a_{22})\).

In terms of the basis of matrix units of \( \mathcal{A}^* \), \( \phi \) takes the form
\[
\phi = (\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5) f_1 + (\mu_1 - \mu_2 - \mu_3 + \mu_4 - z\mu_5) f_2 \\
+ (\mu_1 - \mu_2 - \mu_3 + \mu_4 + z\mu_5) f_3 + (\mu_1 + \mu_2 + \mu_3 + \mu_4 - \mu_5) f_4 \\
+ (\mu_1 + \mu_2 - \mu_3 + \mu_4) b_{11} + (\mu_1 - \mu_2 + \mu_3 - \mu_4) b_{22} \\
+ \frac{x + y}{\sqrt{2}} \mu_5 b_{12} + \frac{x - y}{\sqrt{2}} \mu_5 b_{21}
\]
or
\[
\phi = (\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5) \otimes (\mu_1 - \mu_2 - \mu_3 + \mu_4 - z\mu_5) \otimes \\
\otimes (\mu_1 - \mu_2 - \mu_3 + \mu_4 + z\mu_5) \otimes (\mu_1 + \mu_2 + \mu_3 + \mu_4 - \mu_5) \otimes \\
\otimes \left( \begin{pmatrix}
\mu_1 + \mu_2 - \mu_3 - \mu_4 & \frac{x + y}{\sqrt{2}} \\
\frac{x - y}{\sqrt{2}} & \mu_5 \\
\mu_1 - \mu_2 + \mu_3 - \mu_4
\end{pmatrix}
\right)
\]
in matrix form.

**Remark:** Note that the states on \( \mathcal{A} \) are in general not positive for the \( * \)-algebra structure of \( \mathcal{A}^* \).

If \( \phi \in \mathcal{A}^* \) is positive for the \( * \)-algebra structure of \( \mathcal{A}^* \), then \( T_\phi \) is positive definite on the GNS Hilbert space \( \mathcal{H} \cong \mathcal{A} \) of the Haar state \( \eta \), since the regular representation is a \( * \)-representation, cf. [VD97].

On the other hand, if \( \phi \in \mathcal{A}^* \) is positive as a functional on \( \mathcal{A} \), then \( T_\phi = \text{(id} \otimes \phi) \circ \Delta \) is completely positive as a map from the \( C^* \)-algebra \( \mathcal{A} \) to itself.

**References**


