1. (a) Using *integration by parts* please evaluate:

$$\int_0^1 xe^x \, dx$$

(7 marks)

(b) Determine the *partial derivatives*:

(i) $\frac{\partial}{\partial x}\left(e^y \log(x+y)\right)$.

(ii) $\frac{\partial}{\partial y}\left(\frac{xy}{\cos y}\right)$.

(iii) $\frac{\partial}{\partial z}\log_e(\tan(xyz))$.

(9 marks)

(c) Use *Simpson's Rule* with $n = 6$ to approximate, to one decimal place, the definite integral

$$\int_0^4 \sqrt{1+x}\sin x \, dx$$

(7 marks)

(d) Find the *Laplace Transforms* of the following functions:

(i) $f(t) = \frac{(t^2+4t)^2}{t}$.

(ii) $g(t) = e^{-3t}t^4$

(iii) $h(t) = 4t^2 - 5\cos 3t$.

(12 marks)

(e) The differential equation governing the displacement $x(t)$ of a *door closer* is given by:

$$x''(t) + 8x'(t) + 41x(t) = 0,$$

with the initial conditions $x(0) = 2$ and $x'(0) = 0$.

(i) Solve the differential equation using *Laplace transforms*.

(ii) Is the door closer well designed? Justify your answer by making a suitable analysis of the original differential equation, or otherwise.

(15 marks)

2. (a) Using the method of *partial fractions*, or otherwise, please integrate

$$\int \frac{x^3}{x^2 - 1}\, dx$$

(8 marks)

(b) Use the *Bisection method* to find a solution to the equation

$$e^x = 2.$$

Please find this solution correct to one decimal places.

(6 marks)

(c) Suppose that $f(t)$ has Laplace transform $F(s)$. By integrating by parts, please find, from first principles, the *Laplace transform*:

$$\mathcal{L}\{f'(t)\}.$$

(11 marks)

3. (a) Using *integration by parts* please evaluate:

$$\int_1^2 \frac{\log x}{x^2}\, dt.$$

(7 marks)

(b) If we are using the *Trapezoidal Rule* to approximate a definite integral

$$T_n \approx \int_a^b f(x)\, dx,$$

the maximum error is given by:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2},$$

where $K = \max_{x\in[a,b]} |f''(x)|$.

How large do we have to make n to ensure that the Trapezoidal Rule approximation to

$$\int_1^3 x^2(x^2+1)\, dx,$$

is correct to within 0.001?

(8 marks)

(c) Find the *inverse Laplace Transform*:

$$\mathcal{L}^{-1}\left\{\frac{s+11}{2s^2+4s+52}\right\}$$

(10 marks)

4. (a) The resonant frequency of a series-connected electrical circuit is given by

$$f = \frac{1}{2\pi\sqrt{LC}}, \tag{1}$$

where L is the inductance in Henrys and C is the capacitance in Farads. Determine the *approximate percentage change* in the resonant frequency when L is increased by 1.7% and C is reduced by 3.4%.

(8 marks)

(b) Use *Euler's method* with step size 0.2 to estimate $y(1.8)$, where $y(x)$ is the solution of the initial-value problem

$$\frac{dy}{dx} - x + xy = 0$$
$$y(1) = 0.$$

(5 marks)

(c) It can be shown that a particle of mass m falling under gravity but experiencing air drag satisfies the equation of motion:

$$m\frac{d^2x}{dt^2} = mg - \lambda\frac{dx}{dt},$$

where g is the acceleration due to gravity and λ is the drag coefficient.

Using the initial conditions, $dx/dt(0) = 0$ and $x(0) = 0$; and taking $m = 1$, $\lambda = 5$ and $g = 10$, use Laplace transforms to solve the differential equation for $x(t)$.

This model takes the downwards position as positive. Show that as time tends to infinity, show that the speed of the particle approaches the terminal velocity $v = 2$.

[HINT: The velocity is the derivative of position: $v = dx/dt$.]

(12 marks)