MS 2001: Exercises

November 25, 2010

1 Inequalities

1. *Sketch a rough graph of:*
   - $(x-2)^2$
   - $x^3 - 2x - 3$
   - $-2x^2 + x - 5$
2. *Find the solution set of the inequality*
$$\frac{x}{x+2} \leq \frac{3}{x-2}$$
3. *Find the solution set of the inequality*
$$|x+4| > |3x-8|$$
   and mark this set on a diagram.
4. *Find a positive number* $N > 0$ *such that*
$$\left| x^3 - 3x\cos x + \frac{4}{x} \right| \leq N$$
   for all $1 \leq x \leq 3$.

2 Limits & Continuity

Investigate the limit as $x \to \infty$ *of the following functions:*
$$g(x) = \frac{x^5 - 4x^2 + 2}{5 + 2x^4 - 7x^5}$$
$$h(x) = \frac{x^2 - x + 1}{x - 2}$$

Investigate the limits
$$\lim_{x \to 1} g(x)$$
$$\lim_{x \to 2} h(x)$$

3 Differentiation

Define a function G by

$$G(x) = \frac{1}{(4x^3 + 7x^2)^{10}} \tag{1}$$

Deduce that G is differentiable whenever $x \neq 0$, $-7/4$ and find $G'(x)$.

4 Curve Sketching and MinMax Problems

1. *Find the critical points of the following functions on the intervals $[0, 1]$, $[0, 2]$ and $[-1, 2]$ respectively:*

$$f(x) = 3x^2 - 2x - 1 \tag{2}$$
$$g(x) = -4x^3 + 3x^2 + 18x \tag{3}$$
$$h(x) = x^4 + \frac{4}{3}x^3 - 4x^2 \tag{4}$$

2. *Using the closed interval method, find the locations of the absolute maxima/ minima of the following functions on the intervals $[-3, -1]$ and $[-4, 0]$ respectively:*

$$f(x) = x^3 + 5x - 4 \tag{5}$$
$$g(x) = x^4 - 8x^2 + 16 \tag{6}$$

3. *Examine the critical points of the function $f : [-3, 3] \to \mathbb{R}$ defined by $f(x) = x^3 - 3x$, and sketch its graph.*

4. (a) *Sketch the graph of a function whose first and second derivatives are always negative.*

   (b) *Find the intervals on which f is increasing or decreasing. Find the local maxima and minima. Find the intervals where the function is concave up or concave down. Find the inflection points.*

   (i) $f(x) = x^3 - 12x + 1$

   (ii) $f(x) = 5 - 3x^2 + x^3$

   (iii) $f(x) = x^4 - 2x^2 + 3$

   (iv)
$$f(x) = \frac{x^2}{x^2 + 3}$$

   (c) *Find the local maxima and minima of f using both the First and Second Derivative Tests.*

   (i) $f(x) = x^5 - 5x + 3$

   (ii)
$$f(x) = \frac{x}{x^2 + 4}$$

   (iii) $f(x) = x + \sqrt{1 - x}$

5. (a) *Evaluate the limit.*

(i)

$$\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8}$$

(ii)

$$\lim_{x\to\infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$$

(b) *Find the limit.*

$$\lim_{x\to\infty} \frac{1}{2x+3} \tag{7}$$

$$\lim_{x\to\infty} \frac{3x+5}{x-4} \tag{8}$$

$$\lim_{x\to\infty} \frac{1 - x - x^2}{2x^2 - 7} \tag{9}$$

$$\lim_{y\to\infty} \frac{2 - 3y^2}{5y^2 + 4y} \tag{10}$$

$$\lim_{x\to\infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4} \tag{11}$$

$$\lim_{t\to-\infty} \frac{t^2 + 2}{t^3 + t^2 - 1} \tag{12}$$

$$\lim_{u\to\infty} \frac{4u^2 + 5}{(u^2 - 2)(2u^2 - 1)} \tag{13}$$

$$\lim_{x\to\infty} \frac{x + 2}{\sqrt{9x^2 + 1}} \tag{14}$$

$$\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x + 1} \tag{15}$$

$$\lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} \tag{16}$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2 + 4}} \tag{17}$$

$$\lim_{x\to\infty} (\sqrt{x^4 + 6x^2} - x^2) \tag{18}$$

$$\lim_{x\to\infty} (\sqrt{9x^2 + x} - 3x) \tag{19}$$

$$\lim_{x\to-\infty} (x + \sqrt{x^2 + 2x}) \tag{20}$$

$$\lim_{x\to\infty} (\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x}) \tag{21}$$

$$\lim_{x\to\infty} (x - \sqrt{x}) \tag{22}$$

$$\lim_{x\to\infty} \frac{x^3 - 2x + 3}{5 - 2x^2} \tag{23}$$

$$\lim_{x\to-\infty} (x^4 + x^5) \tag{24}$$

$$\lim_{x\to\infty} x \sin\frac{1}{x} \tag{25}$$

(c) *Find the horizontal asymptotes of each of the following functions.*

$$y = \frac{x}{x+4} \tag{26}$$

$$y = \frac{x^2+4}{x^2-1} \tag{27}$$

$$y = \frac{x^3}{x^2+3x-10} \tag{28}$$

$$y = \frac{x^3+x}{x^3+1} \tag{29}$$

$$h(x) = \frac{x}{\sqrt[4]{x^4+1}} \tag{30}$$

$$F(x) = \frac{x-9}{\sqrt{4x^2+3x-2}} \tag{31}$$

(d) *Find the horizontal asymptotes of the function and use them, together with the concavity and intervals of increase and decrease, to sketch the curve.*

$$y = \frac{1-x}{x+1} \tag{32}$$

$$y = \frac{1+2x^2}{x^2+1} \tag{33}$$

$$y = \frac{x}{x^2+1} \tag{34}$$

$$y = \frac{x}{\sqrt{x^2+1}} \tag{35}$$

6. *A woman arrives at a point A on the shore of a circular lake with radius 2 km wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at a rate of 4 km/hr, and row a boat at 2 km/hr. How should she proceed?*

7. *Find the area of the largest rectangle that can be inscribed in a semicircle of radius r, with one side of the rectangle on the straight side of the rectangle.*