MATH7019 — Technological Maths 311

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Contents

0.1 Introduction

Lecturer

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Office

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This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for the tests. This page shall also house such resources as links (such as to exam papers), as well supplementary material. Please note that not all items here are relevant to MATH7019; only those in the category 'MATH7019'. Feel free to use the comment function therein as a point of contact.

Module Objective

This module covers: Taylor series in one and two variables; first and second order differential equations with constant coefficients; probability distributions, statistical inference and control charts.

Module Content

Further Calculus

Maclaurin and Taylor series of a function of a single variable. Review of partial differentiation. Taylor series expansions of functions of two variables. Differentials.

Differential Equations

Review of the solution of first order differential equations using the method of Separable Variables. Euler's method and the Three Term Taylor method for obtaining numerical solutions to first order differential equations. Solution of second order differential equations using the method of Separable Variables and the method of Undetermined Coefficients. Step functions. Solution of differential equations to include those that occur in the theory of beams and beam struts.

Probability

Laws of probability. Probability distributions such as the Binomial, Poisson Distribution and Normal distributions. Applications of these distributions to engineering problems.

Sampling Theory

Sampling from a Normal population. Confidence intervals for the population mean. Hypothesis tests for population means using the z-test and the t-test.

Quality Control

Control charts for sample means and sample ranges. Process capability.

Assessment

Total Marks 100: End of Year Written Examination 70 marks; Continuous Assessment 30 marks.

Continuous Assessment

The Continuous Assessment will be divided between two in-class tests, each worth 15%, in weeks 6 & 9.
Absence from a test will not be considered accept in truly extraordinary cases. Plenty of notice will be given of the test date. For example, routine medical and dental appointments will not be considered an adequate excuse for missing the test.

Lectures

It will be vital to attend all lectures as many of the examples, proofs, etc. will be completed by us in class.

Tutorials

The aim of the tutorials will be to help you achieve your best performance in the tests and exam.

Exercises

There are many ways to learn maths. Two methods which arent going to work are

1. reading your notes and hoping it will all sink in
2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.
There are exercises in the notes for your consumption. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise*. How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6038, they will not detail all there is to know. Further references are to be found in the library. Good references include:

- Douglas C. Montgomery, George C., Runger 2007, *Applied statistics and probability for engineers*, Fourth Ed., John Wiley & Sons Hoboken, NJ.
- J. Bird, 2006, *Higher Engineering Mathematics*, Fifth Ed., Newnes.

The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isn't too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for further maths modules.

Exam

The exam format will roughly follow last year's. Acceding to the maxim that learning off a few key examples, solutions, etc. is bad and doing exercises is good, solutions to past papers shall not be made available (by me at least). Only by trying to do the exam papers yourself can you guarantee proficiency. If you are still stuck at this stage feel free to ask the question come tutorial time.

0.2 Motivation: When is an Approximation Good Enough?

> *Although this may seem a paradox, all exact science is dominated by the idea of approximation.*
>
> Bertrand Russell

Figure 1: A good door closer should close automatically, close in a gentle manner and close as fast as possible.

One possible design would be to put a mass on the door and attach a spring to it (just for ease of explanation we'll only worry about one dimension).

Assuming that the door is swinging freely the only force closing the door is the force of the spring. Now *Hooke's Law* states that the force of a spring is directly proportion to it's distance from the equilibrium position. If the door is designed so that the equilibrium position of the spring corresponds to when the door is closed flush, then if $x(t)$ is the position of the door t seconds after release, then the force of the spring at time t is given by:

where $k \in \mathbb{R}$ is known as the spring constant.

It can be shown that this system *does* close the door automatically but the balance between closing the door gently and closing the door quickly is lost. Indeed if the door is released from rest at $t = 0$, then the speed of the door will have the following behaviour:

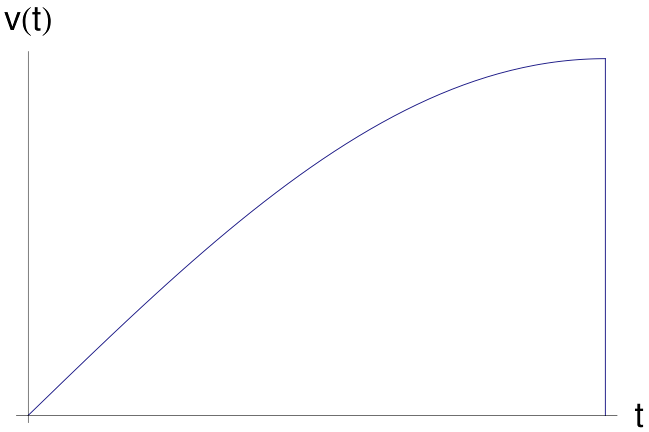


Figure 2: With a spring system alone, the door will quickly pick up speed and slam into the door-frame at maximum speed.

Clearly we need to slow down the door as it approaches the door-frame. A simple model uses a *hydraulic damper*:

Figure 3: A hydraulic damper increases its resistance to motion in direct proportion to speed.

With the force due to the hydraulic damper proportional to speed, the force of the hydraulic damper at time t will be:

for some $\lambda \in \mathbb{R}$. Now by Newton's Second Law:

and the fact that speed is the first derivative of distance, and in turn acceleration is the first derivative of speed, means that the *equation of motion* is given by:

It can be shown that a suitably chosen k and λ will provide us with a system that closes automatically, closes in a gentle manner and closes as fast as possible.
Equations of this form turn up in many branches of physics and engineering. For example, the oscillations of an electric circuit containing an inductance L, resistance R and capacitance C in series are described by

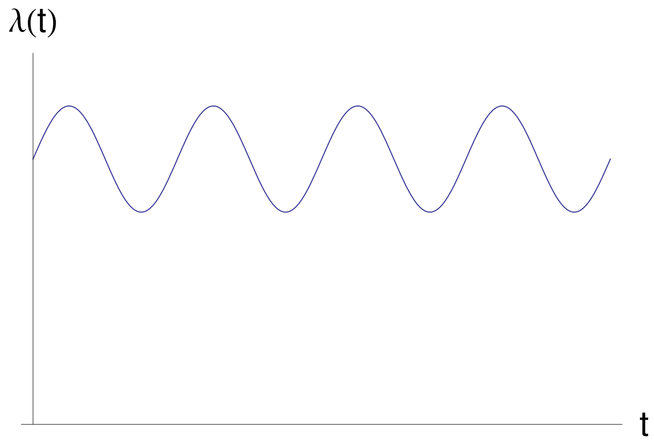
$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0, \qquad (1)$$

in which the variable $q(t)$ represents the charge on one plate of the capacitor. These class of equations, *linear differential equations*,

may be solved in various different ways. In this module we will explore one such method — that of *the method of undetermined coefficients*.

Figure 4: Top Gear dropped a VW Beetle from a height of 1 mile and it spun in the air as it fell.

If we are trying to formulate a model for the fall of this car we would have to try and account for the way the roll of the car means that the coefficient of the drag term ($\lambda v(t)$) varies between its maximum and minimum in a wave-like way:



A function with this behaviour is:

$$\lambda(t) = \frac{1}{2}(M+m) + \frac{1}{2}(M-m)\sin\omega t \tag{2}$$

where M and m are the maximum and minimum of $\lambda(t)$ and ω is a constant related to the angular frequency. Then the equation of motion is of the form:

Neither the method of undetermined coefficients nor any other straightforward method I know of solves this differential equation.

Unfortunately this is typical, and for many systems for which a differential equation may be drawn, it may be impossible to solve the equations. There are a number of numerical techniques which can give approximate answers. However if we are participating in some industrial project with millions spent on it we don't want to be chancing our arms on any old estimate or guess. *Approximation Theory* aims to control these errors as follows. Suppose we have a Differential Equation with solution $y(x)$. An approximate solution $A_y(x)$ to the equation can be found using some numerical method. If the approximation method is sufficiently 'nice' we may be able to come up with a measure of the error:

Here $|\cdot|$ is some measure of the *distance* between $y(x)$ and $A_y(x)$. The most common measure here would be maximum error:

We would call the parameter ε here the *control* or the *acceptable error*. Some classes of problem are even nicer in that with increasing computational power we can develop a sequence of approximate solutions $\{A_y^1(x), A_y^2(x), A_y^3(x), \ldots\}$ with decreasing errors $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$:

Even nicer still from a mathematical point of view if we can find a sequence of approximations with errors decreasing to zero:

In this case we say that the sequence of approximations *converges*.

In this module we will take a first foray into the approximation theory of numerical methods by estimating the solutions of differential equations.

We will not however be measuring *how* accurate our approximate solutions are. However *statistics*, particularly sampling theory, can tell us when our approximations are good enough. For example, suppose we have a business which constructs a machine component. Suppose the company ordering the component wishes to know what 'stress-level' the component can take. Due to natural variations some samples will have a larger tolerance than others — so how can we approach the business and say that our components can take a stress level of S? In practise we can't, but we can make statements along the line of:

> *On average, our components can withstand a stress-level of* S*.*

However we can't go around testing every single one of the components produced. So what we do is we take a sample of 100 or 1,000 of these components away and have them tested. In this module we will see that we can be 'quite' confident that the average ability to withstand stress of all the components we produce is very well estimated by the sample average. *Sampling Theory* makes precise this idea.

Chapter 1

Differential Equations

What is the origin of the urge, the fascination that drives physicists, mathematicians, and presumably other scientists as well? Psychoanalysis suggests that it is sexual curiosity. You start by asking where little babies come from, one thing leads to another, and you find yourself preparing nitroglycerine or solving differential equations. This explanation is somewhat irritating, and therefore probably basically correct.

David Ruelle

1.1 Review of Separable First Order Differential Equations

A differential equation is an equation containing one or more derivatives, e.g.,

$$y' = x^2,$$
$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \sin x.$$

Most laws in physics and engineering are differential equations.

The *order* of a differential equation is the order of the highest derivative that appears;

$$y' = x^2 \quad \text{is first order}$$
$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 1 \quad \text{is second order}$$
$$(\cos x)\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = y \quad \text{is second order.}$$

A function $y = f(x)$ is a *solution* of a differential equation if when you substitute y and its derivatives into the differential equation, the differential equation is satisfied. For example, $y = \tan x$ is a solution of the differential equation $y' = 1 + y^2$ since if $y = \tan x$, then

A differential equation can have many solutions: $y' = 2$ has a solution $y = 2x + C$ for every constant C. The *general solution* of a differential equation is the set of all possible solutions. The differential equation $y' = 2$ has general solution $y = 2x + C$, where the constant C is arbitrary. It can be very difficult to find the general solution of a differential equation. We shall consider only certain first-order differential equations that can be solved fairly readily.

1.2 First-order Separable Differential Equations

A *separable* first-order differential equation is one that can be written in the form

In this situation we can *separate the variables*:

Each side can now be integrated:

The point of separating the variables is that we cannot usually integrate expressions like $\int y\,dx$ where both variables appear.

Example

Solve the separable first order differential equation:

$$y' = xy.$$

Solution: First separate the variables and integrate:

Usually we want to solve for y:

Here there are two infinite families of solutions. The solution of a first-order differential equation will always contain an unknown constant — and might have different families of solutions also (e.g. the solution $y^2 = x + C$ has the families $y = +\sqrt{x+C}$ and $-\sqrt{x+C}$). However an extra piece of numerical data such as "$y = 2$ when $x = 1$" sometimes reduces this to a unique solution. Note that this will usually be written as $y(1) = 2$ — for the input $x = 1$, the output is $y = 2$. This extra data is called an *initial condition* or *boundary condition* and the entire problem (differential equation and boundary condition) is often called an *initial-value problem* or *boundary-value problem*.

Example

Solve the initial-value problem

$$\frac{dy}{dx} = \frac{1+x}{xy} \quad \text{for } x > 0, \quad \text{where } y(1) = -4.$$

Solution: First separate the variables and integrate:

Now apply the boundary condition:

Now substitute in the constant and hopefully solve for $y(x)$:

Now the fact that $y = -4$ at $x = 1$ and that $\sqrt{x} > 0$ where defined implies that the solution is $y(x) = -\sqrt{2(\log_e x + x + 7)}$. [Ex:] Show that this solves the differential equation and satisfies the boundary condition.

Further Remarks: Picard's Existence Theorem

There is a theorem in the analysis of differential equations which states that if a differential equation is suitably *nice* in an interval about the boundary condition then not only does a solution exist but it is unique. This allows us to define functions as solutions to differential equations. For example, an alternate definition of the exponential function, e^x, is the unique solution to the differential equation:

$$\frac{dy}{dx} = y\,,\ y(0) = 1.$$

Exercises

1. Solve the following differential equations:

   (a) $y' = 3x^2 + 2x - 7$ Ans: $y = x^3 + x^2 - 7x + C$

   (b) $y' = 3xy^2$ Ans: $3x^2y + Cy + 2 = 0$

   (c) $\dfrac{dy}{dx} = \dfrac{3x\sqrt{1+y^2}}{y}$ Ans: $2\sqrt{1+y^2} = 3x^2 + C$

   (d) $\dfrac{dy}{dx} = \dfrac{x}{4y},\ y(4) = -2$ Ans: $x^2 = 4y^2$

2. The point $(3, 2)$ is on a curve, and at any point (x, y) on the curve the tangent line has slope $2x - 3$. Find the equation of the curve. Ans: $y = x^2 - 3x + 2$

3. The slope of the tangent line to a curve at any point (x, y) on the curve is equal to $3x^2y^2$. Find the equation of the curve, given that the point $(2, 1)$ lies on the curve. Ans: $-\dfrac{1}{y} = x^3 - 9$

1.3 Numerical Methods

1.3.1 Direction Fields

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. In this section, we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's Method and the Three-Term-Taylor Method).

Suppose we are asked to sketch the graph of the solution of the initial value problem:

$$\frac{dy}{dx} = x + y\ ,\ \ y(0) = 1.$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph of $y(x)$ is equal to the sum of the x- and y-coordinates at that point. In particular, because the curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1:

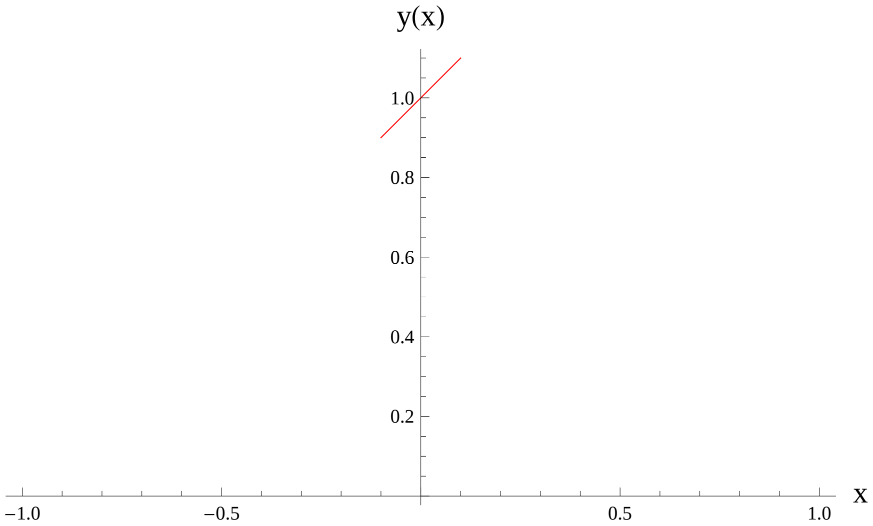


Figure 1.1: Near the point $(0, 1)$, the slope of the solution curve is 1.

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown below:

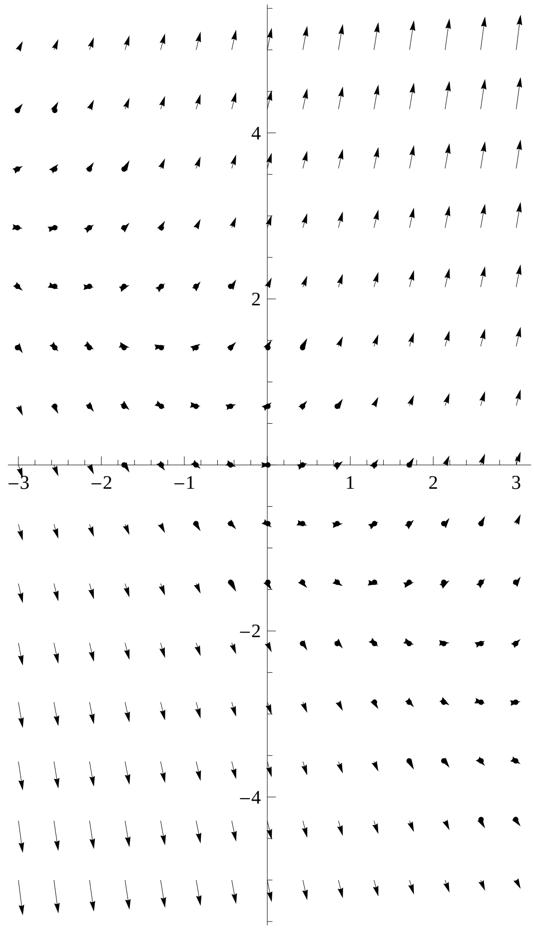


Figure 1.2: For example, the line segment at the point $(1, 2)$ has slope $1+2 = 3$. The direction field allows us to visualise the general shape of the solution by indicating the direction in which the curve proceeds at each point.

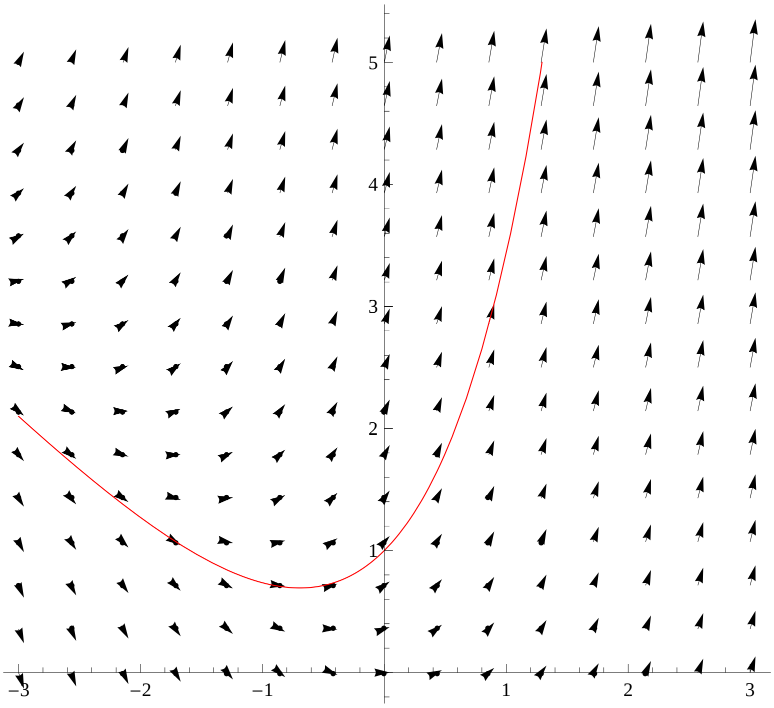


Figure 1.3: We can sketch the solution curve through the point $(0, 1)$ by following the direction field. Notice that we have drawn the curve so that it is parallel to nearby line segments.

1.3.2 Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations. We illustrate the methods on the initial-value problem that we used to introduce direction fields:

$$\frac{dy}{dx} = x + y \ , \quad y(0) = 1.$$

The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point $(0, 1)$. As a first approximation to the solution we could use the linear approximation $L(x) = 1x + 1$. In other words we could use the tangent line at $(0, 1)$ as a rough approximation to the solution curve.

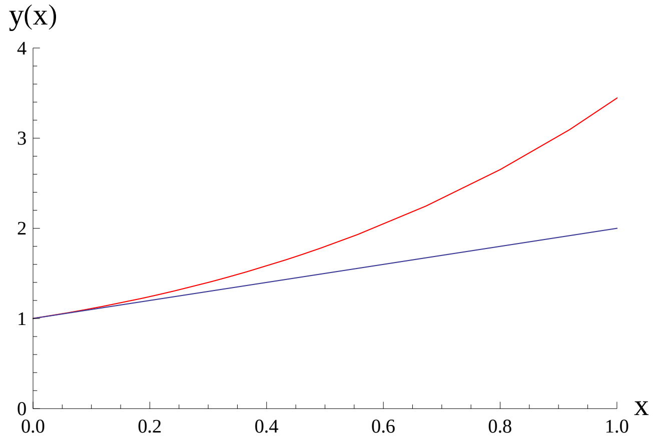


Figure 1.4: The tangent at $(0, 1)$ approximates the solution curve for values near $x = 0$.

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a correction by changing direction according to the direction field:

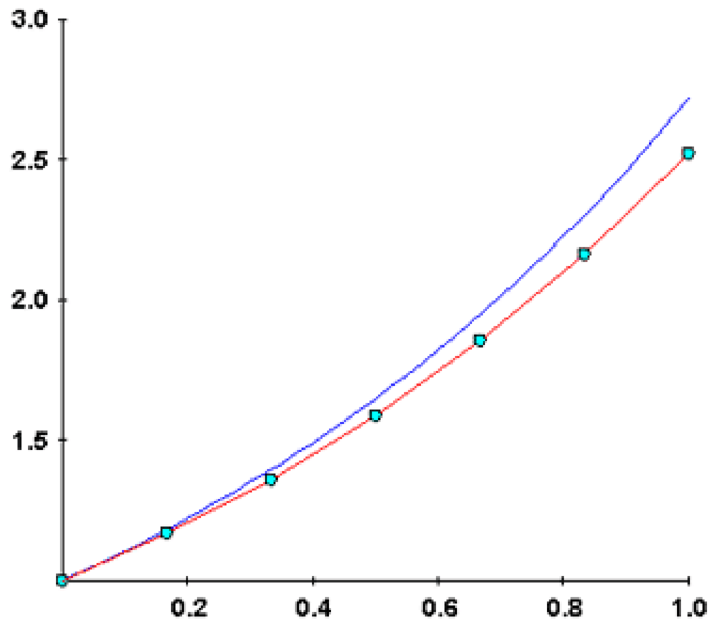


Figure 1.5: Euler's Method starts at some initial point (here $(x_0, y_0) = (0, 1)$), and proceeds for a distance h (in this plot $h = 1/6.$) at a slope that is equal to the slope at that point $y' = x_0 + y_0$. At the point $(x_1, y_1) = (x_0 + h, y_1)$, the slope is changed to what it is at (x_1, y_1), namely $x_1 + y + 1$, and proceeds for another distance h until it changes direction again.

Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short time, look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field. Euler's method does not produce an exact solution to the initial-value problem — it gives approximations. But by decreasing the step size (and therefore increasing the amount of corrections), we obtain successively better approximations to the correct solution.

For the general first-order initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, our aim is to find approximate values for the solution at equally spaced numbers x_0, $x_1 = x_0 + h$,$x_2 = x_0 + 2h = x_1 + h ...$, where h is the step size. The differential equation tells us that the slope at (x_0, y_0) is $y' = F(x_0, y_0)$:

This shows us that the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly,

$$y_1 = y_0 + hF(x_0, y_0)$$

Euler's Method

If

$$\frac{dy}{dx} = F(x, y)\ ,\ \ y(x_0) = y_0$$

is an initial value problem. If we are using Euler's method with step size h then

$$y(x_{n+1}) \approx y_{n+1} = y_n + hF(x_n, y_n) \tag{1.1}$$

for $n \geq 0$.

Example

Use Euler's method with step size $h = 0.1$ to approximate $y(1)$, where $y(1)$ is the solution of the initial value problem:

$$\frac{dy}{dx} = x + y\ ,\ \ y(0) = 1$$

Solution: We are given that $h = 0.1$, $x_0 = 0$ and $y_0 = 1$, and $F(x, y) = x + y$. So we have

$$\begin{aligned} y_1 &= y_0 + F(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1 \\ y_2 &= y_1 + F(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22 \\ y_3 &= y_2 + F(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362 \end{aligned}$$

Continue this process [Exercise] to get $y_{10} = 3.187485$, which approximates $y(x_{10}) = y(x_0 + 10(0.1)) = y(1)$, as required.

Exercises

1. *Use Euler's method with step size 0.5 to compute the approximate* y*-values* y_1, y_2, y_3 *and* y_4 *of the initial value problem* $y' = y - 2x$, $y(1) = 0$.

2. *Use Euler's method with step size to estimate* $y(1)$, *where* $y(x)$ *is the solution of the initial value problem* $y' = 1 - xy$, $y(0) = 0$.

3. *Use Euler's method with step size* 0.1 *to estimate* $y(0.5)$, *where* $y(x)$ *is the solution of the initial value problem* $y' - y = xy$, $y(0) = 1$.

4. *Use Euler's method with step size* 0.2 *to estimate* $y(1.4)$, *where* $y(x)$ *is the solution of the initial-value problem* $y' - x + xy = 0$, $y(1) = 0$.

1.3.3 The Three Term Taylor Method

1.4 Second Order Differential Equations

1.4.1 Separable Second Order Differential Equations

1.4.2 Second Order Linear Differential Equations

1.5 Impulse & Step Functions

1.5.1 Introduction

In this section we examine *impulse* and *step functions*. They arise naturally in the theory of beams. They model various *discontinuous* phenomena.

1.5.2 Step Functions

For example, consider a simple switch that has two states: on (1) and off (0). Suppose the switch is turned on at time $t = 0$. We use the *Heaviside Function* to model this:

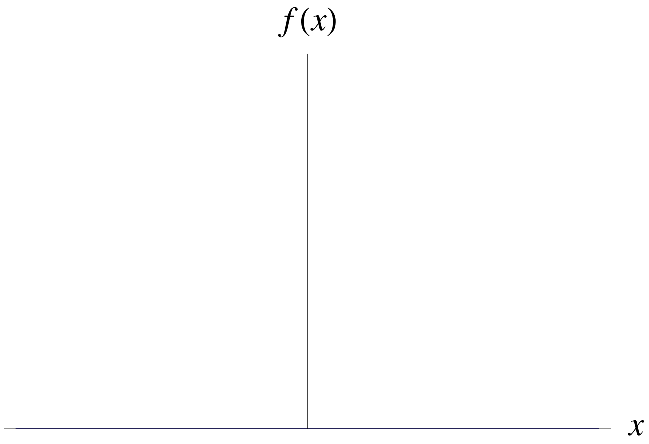


Figure 1.6: The graph of the Heaviside Function.

It is no problem writing down a switch which starts at a time $x = a$:

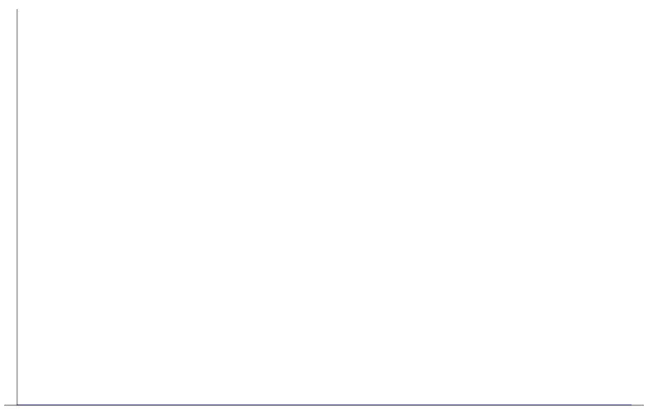


Figure 1.7: The graph of the Heaviside Function $H(x - a)$.

We can also use a combination of Heaviside functions to write an expression for a switch that is on between times $x = a$ and $x = b$:

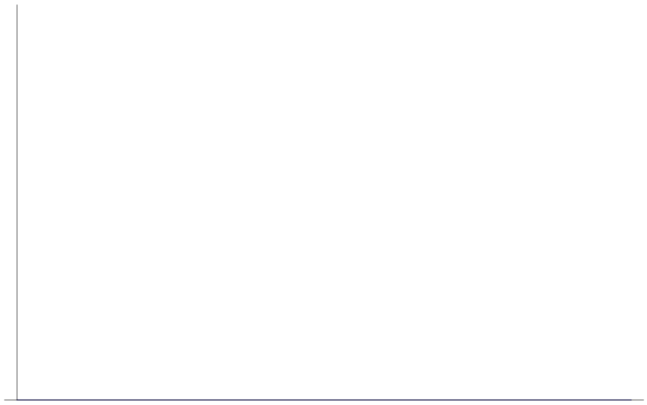


Figure 1.8: This function is equal to $H(x - a) - H(x - b)$. It is an exercise to show that this function is equal to one for $a < x < b$ and zero elsewhere.

Of course the output value doesn't have to be one, it could be 3 say; or we could have a switch equal to 4 between $x = 2$ and $x = 5$:

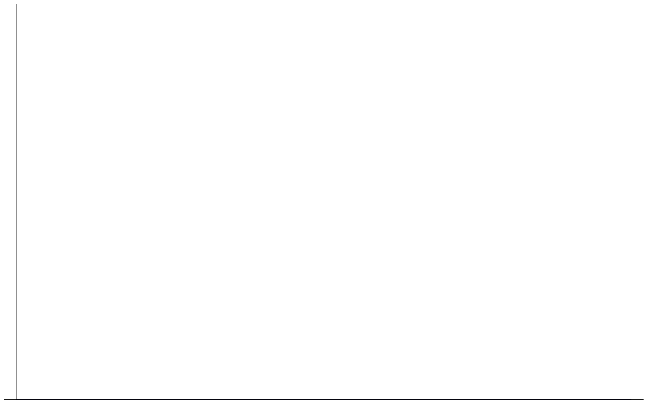


Figure 1.9: The graphs of $3H(x)$ and $4H(x - 2) - 4H(x - 5)$.

Examples

Write down an formula for the following step functions:

1. equal to 7 for $x > -2$ and zero otherwise.
2. equal to 1/2 for $-1 < x < 1$ and zero otherwise.

Solution: 1. $7H(x + 2)$ 2. $\frac{1}{2}H(x + 1) - \frac{1}{2}H(x - 1)$.

What is the derivative of $H(x)$? The slope is 0 except at $x = 0$ where it is infinite. This is the *Dirac Delta Function*:

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

There is a more precise definition that can be done such that the integral of $\delta(x)$ is $H(x)$. Using this function we can model a 'jolt' acting at the time $t = a$ or a point force acting at position $x = a$ of magnitude F is given by $F\delta(x - a)$.

We can also integrate the Heaviside function (this is what we will need to do). First some notation:

1.5.3 Notation

$$[x] := xH(x). \tag{1.3}$$

Note now that $[f(x)]$ is a function whose value if $f(x)$ when $f(x) \geq 0$ and zero otherwise. In essence, $[x]$ is the function which 'chops off' the negative part of the graph of $f(x)$:

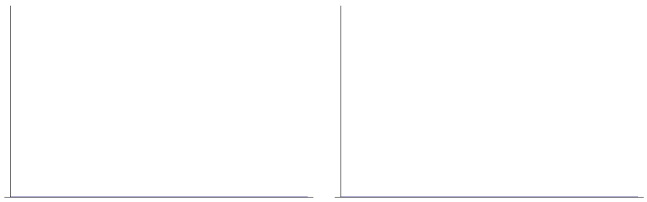


Figure 1.10: The graph of a function $f(x)$ and $[f(x)]$.

Examples

Write down formulas for the following functions:

1. Equal to x^2 for $x > 0$ and zero otherwise.
2. Equal to $x - 4$ for $x > 4$ and zero otherwise.
3. Equal to x for $2 < x < 5$ and zero otherwise.

Solution: 1. $[x^2] = x^2H(x^2)$ 2. $[x - 4] = (x - 4)H(x - 4)$ 3. $[x - 2] - [x - 4]$.

1.5.4 Integration of $H(x)$

The integral of $H(x)$ is equal to $[x] = xH(x)$.

Proof. We have that

Integrating both sides gives the answer.

1.5.5 Integral of $[x-a]$

The integral of $[x-a]$ *is equal to* $\frac{[x-a]^2}{2}$.

Proof. Now

$$[x-a] = \begin{cases} x-a & \text{if } x > a \\ 0 & \text{otherwise} \end{cases}$$

Hence

But note that

$$\frac{[x-a]^2}{2} = \begin{cases} \frac{x^2-2ax+a^2}{2} & \text{if } x > a \\ 0 & \text{otherwise} \end{cases}$$

which is equal to $\int [x-a]\, dx$.

Similarly we can show that

$$\int [x-a]^n\, dx = \frac{[x-a]^{n+1}}{n+1} + C. \tag{1.4}$$

Examples

Solve the following differential equations:

$$\frac{dy}{dx} = 2H(x) - 2H(x-1). \tag{1.5}$$

$$\frac{d^2y}{dx^2} = 3H(x-2) + \delta(x-1). \tag{1.6}$$

$$\frac{d^2y}{dx^2} = [x-2]^2 - [x-4]^2 \tag{1.7}$$

$$\frac{d^3y}{dx^3} = H[x-2]. \tag{1.8}$$

$$\frac{d^4y}{dx^4} = H[x]. \tag{1.9}$$

Solution On board.

Winter 2010

To find the deflection y *at any point on a beam the differential equation below must be solved where* $[x-4]$ *is a step function and where* R *is a constant*

$$EI\frac{d^2y}{dx^2} = -15[x-4]^2 + Rx.$$

Solve this differential equation where the deflection y *and the slope of* y *are zero at the point* $x = 1$. *Also at the point* $x = 6$ *the deflection is zero.*
Solution: We start with two integrations:

Now we apply the boundary conditions. First $y(1) = 0$:

Now $y'(1) = 0$:

And finally $y(6) = 0$:

Hence to find R, C_1 and C_2 we must solve the linear system:

$$
\begin{aligned}
36R + 6C_1 + C_2 &= 0 \\
\frac{R}{2} + C_1 &= 0 \\
\frac{R}{6} + C_1 + C_2 &= 0
\end{aligned}
$$

We can do this either using matrix methods or a substitution method.

Exercises: Find the general solution of the following second order separable differential equations for $M(x)$:

$$\frac{d^2M}{dx^2} = -\delta(x-1) - \delta(x-5)$$
$$\frac{d^2M}{dx^2} = -18$$
$$\frac{d^2M}{dx^2} = 72H(x-4) - 72H(x-1)$$
$$\frac{d^2M}{dx^2} = 36H(x-5) - 36H(x-1) - 5\delta(x-4)$$
$$\frac{d^2M}{dx^2} = -\delta(x-1) - \delta(x-3) - 144H(x-5) + 144H(x-8) - 3\delta(x-7)$$
$$\frac{d^2M}{dx^2} = -x - 2 - 10\delta(x-2)$$
$$\frac{d^2M}{dx^2} = -3x - 3 - 72H(x-2)$$
$$\frac{d^2M}{dx^2} = -6x - 4 - 72H(x-4) + 72H(x-1)$$
$$EI\frac{d^2y}{dx^2} = -144[x-2]^2$$

Now find the particular solutions to the first three differential equations under the following initial conditions.

1. $M'(0) = 1$ and $M(0) = 0$.
2. $M'(0) = 54$ and $M(0) = 0$.
3. $M'(0) = 108$ and $M(0) = 0$.

Note after we do the next section we can look back and see that these are the beam equations for:

1. A simply supported beam of length 6 m with point loads of magnitude 1 kN at $x = 1$ and $x = 5$.
2. A simply supported beam of length 6 m with a uniform load of 18 kN m^{-1}.
3. A simply supported beam of length 5 m with a uniform load of 72 kN m^{-1} between $x = 1$ and $x = 4$.

1.6 Applications to Beams & Beam Struts

In this section we learn how to formulate and solve beam equations so that we may calculate/estimate the deflection of a beam due to the loads on it. This theory known variously as the *Euler-Bernoulli Beam Theory*, *Engineer's Beam Theory* or *Classical Beam Theory*. It is a simplification of the theory of elasticity which, after it was used in the design of the Eiffel Tower and Ferris wheels, became a cornerstone of engineering. The theory makes a number of underlying assumptions. First we must show a beam in a deformed state and an undeformed state:

The beams for which we apply the model are assumed to be:

1. slender: their length is much greater than their width
2. isotropic and homogeneous: the material behaves the same at all points in the beam and in all directions
3. constant cross-section

The assumptions of the model (Kirchhoff's Assumptions) are:

1. Normals remain straight: they do not bend
2. Normals remain unstretched: they do not change length
3. Normals remain normal: they stay perpendicular to the neutral plane

Now using these assumptions we can derive an equation relating the *deflection* $y(x)$ at a point x and the *load* $w(x)$. The deflection is given as follows:

This equation (whose derivation is outside the remit of our course) is given by:

where E is Young's Modulus and I is the moment of inertia of the beam. Note that in all cases we use kilo-Newtons (kN) rather than Newtons (N). Solving this equation requires *four* integrations.

This analysis can be broken down somewhat if we look at the *bending moment* of a load. The bending moment is a *moment* due to a vertical loading. The moment of a force is, about an axis O, at a point P, the product of the magnitude of the force in the direction parallel to the axis, F, times the distance from the axis L:

The bending moment and the shearing force at a distance x from the left-hand side of a beam are related to the load per unit length $w(x)$ and the shearing force $V(X)$ by the differential equations

$$\frac{dM}{dx} = V(x), \quad \frac{dV}{dx} = -w(x).$$

Definition

The *shearing force* at the point x is the sum of the forces at or to the left of the point x.

Ye probably know more about shearing than me and we just need this working definition. These can be combined to form the second order differential equation

$$\frac{d^2M}{dx^2} = -w. \tag{1.10}$$

Hence if we know the load we can calculate the bending moment, $M(x)$. Then we can appeal to the master equation to solve for the deflection.

We have three (five) main types of loads:

1. *Uniformly distributed loads, U.D.L. over the entire beam*

Mathematically we have

$$w_{\text{UDL}}(x) = w \tag{1.11}$$

where w is the the value of the UDL.

Example

Find the bending moment due to a U.D.L. of 36 kN m^{-1} across a beam of length 7 m where $M(0) = 0$ and $M'(0) = 126$.
Solution: First we integrate twice:

Now we apply the boundary conditions. First $M(0) = 0$:

$$M(0) = 0 = C_2.$$

Now $M'(0) = 126$:

$$M'(0) = C_1 = 126.$$

So the answer is

$$M(x) = -18x^2 + 126x.$$

2. *U.D.L. over a segment of the beam.*

Mathematically we have, for a UDL from $x = a$ to $x = b$:

$$w_{\text{UDL }:x=a\to b}(x) = wH(x-a) - wH(x-b) \tag{1.12}$$

where w is the UDL across $a \leq x \leq b$.

Example

Write down the load due to a UDL of 18 kN m^{-1} from $x = 2$ to $x = 3$ on a beam of length 5 m. Now find the bending moment due to this load. Write your answer in terms of M_A, the bending moment at $x = 0$, and R_A the reaction/shearing force at $x = 0$.
Solution: First draw at a picture:

Now write down the bending moment equation and integrate twice:

Now apply the boundary conditions taking into account that $M'(x)$ is the shearing force:

Therefore the answer is

$$M(x) = -9[x-2]^2 + 9[x-3]^2 + R_A x + M_A.$$

3. *Point loads*

Mathematically we have, for a point load of weight w at a point $x = a$:

$$w(x) = w\delta(x - a) \tag{1.13}$$

Example

Write down the load due to a point mass of 13 kN at $x = 3$ on a beam of length 6 m. Now find the bending moment.
Solution: We simply have $w(x) = 13\,\delta(x-3)$ so we have the bending moment equation:

Although this example doesn't ask for it, we can nearly always give our answer in terms of R_A and M_A:

So our best answer is

$$M(x) = -13[x - 3] + R_A x + M_A$$

4. *Linear loads over the entire beam*

Here we have $w(0) = w_A$ and $w(L) = W_B$. It is not hard to show that a line has equation

$$\text{OUTPUT} = \text{SLOPE} \times \text{INPUT} + Y\text{-INTERCEPT}. \tag{1.14}$$

The slope is the ratio of how much you go up as you go across (rise/run):

$$\text{SLOPE} = \frac{W_B - W_A}{L} \tag{1.15}$$

Therefore we have

$$w(x) = \frac{W_B - W_A}{L} x + W_A \tag{1.16}$$

Example

Write down the load on a 8 m beam due to a linear load that varies from 10 kN m^{-1} at $x = 0$ to 26 kN m^{-1} at $x = 8$. Hence find the bending moment.
Solution: On board.

5. *Linear load over a segment of the beam**

Pretty much the same as the last section except we utilise step functions:

$$w_{\text{linear } x=a\to b}(x) = \left(\frac{W_b - W_a}{b-a}(x-a) + W_a\right) H(x-a) - \left(\frac{W_b - W_a}{b-a}(x-a) + W_a\right) H(x-b) \tag{1.17}$$

where W_a is the load at a and W_b is the load at b with $a < b$.

Example

Write down an expression for a linear load on a 6 m beam which varies linearly from 4 kN m^{-1} at $x = 2$ to 12 kN m^{-1} at $x = 4$. Hence find the bending moment.
Solution: First we draw a picture:

Now we find the equation of the line. This is preferable to using the formula in my opinion. It will not be so easy to use the $y = mx + c$ formula as the line does not cut at $x = 0$ (i.e. it does not cut at A). Hence we use the following:

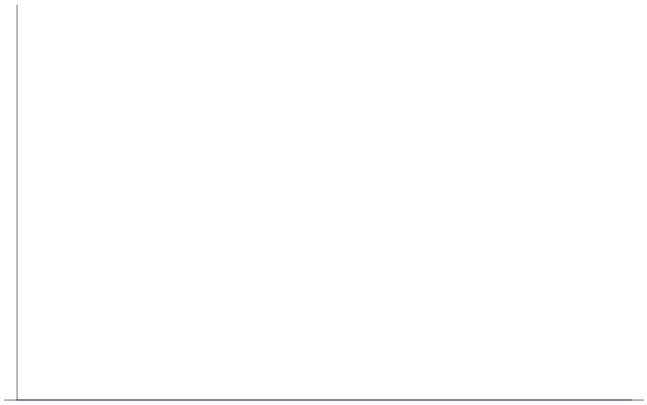


Figure 1.11: The equation of a line can also be written as $y - y_1 = m(x - x_1)$.

Now the only thing is we want this load to be 'switched on' at $x = 2$ and switched off again at $x = 4$ so we use the Heaviside function:

Now to integrate this is not so straightforward but we can take out the Heaviside function rewrite the loading as follows:

Now we want to integrate $-w(x)$ twice to find the bending moment:

That is we have

$$M(x) = \begin{cases} C_1x + C_2 & \text{if } x < 2 \\ -\frac{2}{3}x^3 + 2x^2 + C_1x + C_2 & \text{if } 2 < x < 4 \\ C_1x + C_2 & \text{if } x > 4 \end{cases} \tag{1.18}$$

Now we apply $M(0) = M_A$ and $M'(0) = R_A$:

$$M(x) = \begin{cases} R_Ax + M_A & \text{if } x < 2 \\ -\frac{2}{3}x^3 + 2x^2 + R_Ax + M_A & \text{if } 2 < x < 4 \\ R_Ax + M_A & \text{if } x > 4 \end{cases} \tag{1.19}$$

Exercises What form does the bending moment take for a beam of span 5 m where there is

1. a U.D.L. of 12 kN m^{-1} between $x = 2$ and $x = 5$.
2. a U.D.L. of 10 kN m^{-1} between $x = 0$ and $x = 4$.
3. a U.D.L. of 8 kN m^{-1} between $x = 2$ and $x = 3$.
4. there is a point load of 10 kN at $x = 2$.
5. a linear load varying from 8 kN m^{-1} at $x = 0$ to 18 kN m^{-1} at $x = 5$.
6. a linear load varying from 9 kN m^{-1} at $x = 2$ to 27 kN m^{-1} at $x = 5$.
7. there are point loads of 10 kN and 12 kN at $x = 2$ and $x = 3$ respectively.
8. there is a point load of 12 kN at $x = 3$ and a U.D.L. of 10 kN m^{-1} between $x = 2$ and $x = 5$.
9. there is a point load of 20 kN at $x = 4$ and a U.D.L. of 12 kN m^{-1} between $x = 0$ and $x = 3$.
10. there is a point load of 32 kN at $x = 1$ and a U.D.L. of 16 kN m^{-1} between $x = 2$ and $x = 3$.
11. there is a point load of 10 kN at $x = 2$ and a linear load varying from 5 kN m^{-1} at $x = 3$ to 15 kN m^{-1} at $x = 5$.

1.6.1 A More Pragmatic Method of finding Bending Moments

Note that if you are sure what you are doing you can write down the bending moment straight away. Personally I would prefer to solve the $M'' = -w$ equation but this is an option for you. If you were going to do this I think we would have to show the following to be sure about what you are doing:

Macauley's Method

When solving the differential equation

$$\frac{d^2M}{dx^2} = -w(x), \tag{1.20}$$

the following hold:

1. *If the bending moment due to a load $w_1(x)$ is $M_1(x)$, and the bending moment due to a load $w_2(x)$ is $M_2(x)$, the the bending moment due to the load $w_1(x) + w_2(x)$ is given by $M_1(x) + M_2(x)$.*
2. *The bending moment due to a U.D.L. of w kN m^{-1} is given by*

$$M(x) = -\frac{w^2}{2} + R_A x + M_A, \tag{1.21}$$

where R_A is the reaction or shearing force at $x = 0$ and M_A is the bending moment at $x = 0$.

3. *The bending moment due to a U.D.L. of w kN m^{-1} applied between points $x = a$ and $x = b$ (with $a < b$) is given by*

$$-\frac{w}{2}[x-a]^2 + \frac{w}{2}[x-b]^2 + R_A x + M_A \tag{1.22}$$

4. *The bending moment due to a point load of magnitude w kN at $x = a$ is given by*

$$-w[x-a] + R_A x + M_A \tag{1.23}$$

5. *The bending moment due to a linear load varying from W_A at $x = 0$ to W_B at $x = L$ is given by*

$$-\frac{(W_B - W_A)x^3}{6L} - \frac{W_A x^2}{2} + R_A x + M_A \tag{1.24}$$

Justification

1. This follows from the fact that integration is linear; e.g. $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.
2. The situation looks as follows:

We find by integrating both sides of $\frac{d^2M}{dx^2} = -w$ that that the general solution is given by:

$$M(x) = -\frac{w}{2}x^2 + C_1x + C_2. \tag{1.25}$$

Now we apply the following boundary conditions. The bending moment at $x = 0$ is M_A and the shearing force at $x = 0$ is R_A:

3. The situation is as follows:

Now the loading is of the form $W(x) = wH(x - a)$. Therefore we integrate twice to find the bending moment:

Now apply the boundary conditions that shear at $x = 0$ is R_A and the bending moment at $x = 0$ is M_A:

This yields

$$M(x) = -\frac{w}{2}[x - a]^2 + R_Ax + M_A. \tag{1.26}$$

4. The situation is as follows:

   Now the load function is equal to $w(x) = -W\delta(x-a)$. Integrate twice to find the bending moment:

   Apply the boundary conditions to yield:

$$M(x) = -W[x-a] + R_A x + M_A. \tag{1.27}$$

5. Left as an exercise.

Examples

1. Write down the bending moment due to a U.D.L. of 12 kN m^{-1} between $x = 2$ and $x = 5$ on a 5 m beam:

2. Write down the bending moment due to a U.D.L. of 10 kN m^{-1} between $x = 0$ and $x = 4$ on a 5 m beam:

3. Write down the bending moment due to a U.D.L. of 8 kN m^{-1} between $x = 2$ and $x = 3$ and a point load of 12 kN at $x = 1$ on a 5 m beam:

4. a linear load varying from 2 kN m^{-1} at $x = 0$ to to 27 kN m^{-1} at $x = 5$ on a 5 m beam:

Exercises Write down the bending moment due to the following loads on a 5 m beam:

1. Point loads of 10 and 12 kN at $x = 2$ and $x = 3$ respectively.
2. A U.D.L. of 10 kN m^{-1} from $x = 2$ to $x = 5$ and a point load of 12 kN at $x = 3$.
3. A U.D.L. of 12 kN m^{-1} from $x = 0$ to $x = 3$ and a point load of 20 kN at $x = 4$.
4. A U.D.L. of 16 kN m^{-1} from $x = 2$ to $x = 3$ and a point load of 32 kN at $x = 1$.
5. a linear load varying from 8 kN m^{-1} at $x = 0$ to 18 kN m^{-1} at $x = 5$.

The three different types of beams that we look at are simply supported beams, fixed end beams and linear loads. The first two differ mathematically only in terms of the initial conditions. The master equation involves four integrations so we must have four initial/boundary conditions.

1.6.2 Simply Supported Beams

A simply supported beam looks as follows:

We have the following boundary conditions:

1. the bending moment at each of the ends is zero: i.e. $M(0) = 0 = M(L)$.
2. the deflection at both ends are zero: i.e. $y(0) = 0 = y(L)$.
3. in addition, if the load is symmetric about the the centre $x = L/2$ then we also have $R_A = R_B = W_T/2$, where w_T is the total load.

Example

A light beam of span 5 m is simply supported *at its end points and carries a uniformly distributed load (U.D.L.) of 18 kN* m^{-1} *along the beam. By solving the differential equation*

$$\frac{d^2M}{dx^2} = -w.$$

find the Bending Moment at any point along the beam.
Solution: First we integrate twice to find the general solution:

Now we have to apply some boundary conditions. Here we draw a sketch of the beam.

The (a) uniformly distributed load may be replaced by a point load on the beam at the centre. The total load is the intensity times the length: 18(5) kN = 90 kN. By symmetry the reactions at A and B are given by $R_A = 45$ kN $= R_B$. Also the bending moment about $x = 0$:

$$M(0) = -(90)(2.5) + (45)(5) = 0 \tag{1.28}$$

This is our first boundary condition. From symmetry

$$M(5) = 0. \tag{1.29}$$

Hence we apply these boundary conditions:

This yields the nice solution:

$$M(x) = -9x^2 + 45x \tag{1.30}$$

Here I plot the load, the shearing, and the bending moment on the one graph:

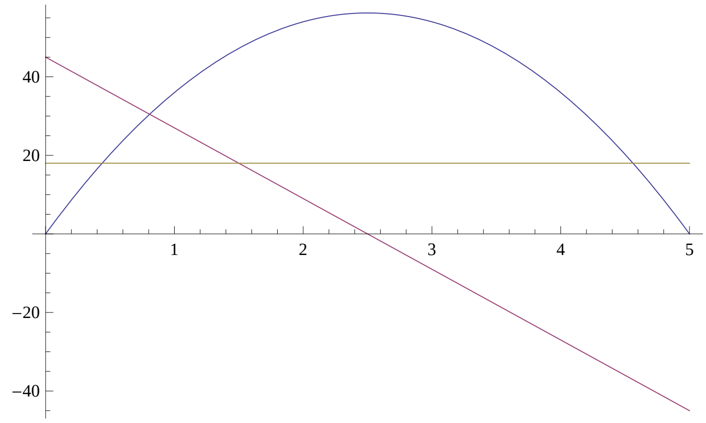


Figure 1.12: A plot of the load, shearing and the bending moment.

Winter 2011: Q. 1(c)

A light beam of span 5 m is simply supported at its end points and carries a load that varies uniformly with x the distance from one end of the beam. The load varies from 18 kN m^{-1} at $x = 0$ to 12 kN m^{-1} at $x = 5$. Find a formula for the load per unit length. By solving the differential equation

$$\frac{d^2M}{dx^2} = -w(x), \tag{1.31}$$

find the bending moment M at any point along the beam. Also find the maximum value of the bending moment.
Solution: First a picture:

Now $-w(x) = \frac{6}{5}x - 18$. We integrate twice to find the bending moment:

Now we apply the boundary conditions that $M(0) = 0 = M(5)$. We may *use* R_A equals half the total load as the load is not symmetric:

This yields a bending moment function

$$M(x) = \frac{1}{5}x^3 - 9x^2 + 40x \tag{1.32}$$

The bending moment looks like

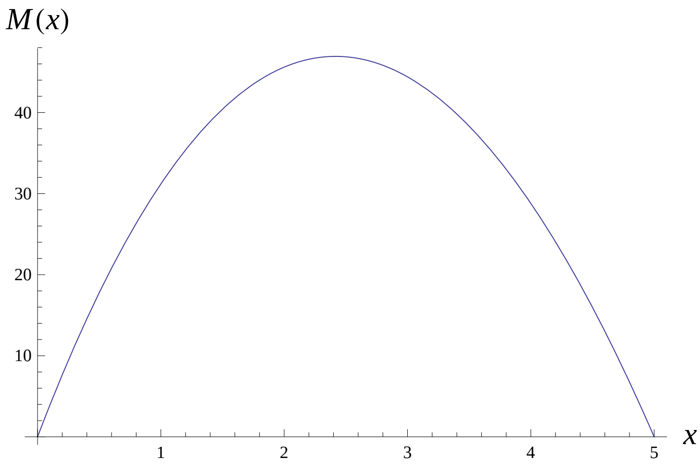


Figure 1.13: A plot of the bending moment.

The maximum occurs when the slope is zero...

Obviously the + here is too big and the maximum is found at $15 - 5\sqrt{57}/3 \approx 2.42$. Throw this into $M(x)$ to find the maximum bending moment:

$$M(2.42) \approx 42.9269 \text{ kN m}.$$

Remark: Normally I would throw the surd into the bending moment function. The use of decimals introduces *rounding errors* into our calculations and pure mathematicians consider unrestricted decimal approximation very gravely indeed. However, at the maximum the bending moment function is quite flat so moving a little bit away from the true value doesn't change $M(x)$ that much:

$$M(x_{\text{max}}) \approx M(x_{\text{max}} + \varepsilon) \tag{1.33}$$

In fact in this example $M(x_{\text{max}})$ and $M(2.42)$ agree to four places of decimals. However doing the same thing around $x = 1$ m is *significant*. In particular $M(1) \approx 31.2$ kN m but $M(1 + 1")\approx 31.789$ kN m — a not insignificant difference — over 500 N m in fact about the torque needed to hold 50 kg stable at an arm's length.

Winter 2011 Q. 1(a)

A light beam of span 6 m is simply supported of its endpoints. Between the points $x = 2$ m and $x = 5$ m there is a U.D.L. of 72 kN m^{-1}. Express the bending moment M in terms of step functions. The deflection y at any point on the beam is found by solving the differential equation

$$EI\frac{d^2y}{dx^2} = -M \tag{1.34}$$

Solve this differential equation where y is zero at both ends of the beam.
Solution: First as always, a picture:

Now I write down the loading and solve $M''(x) = -w$:

Here the loading is not symmetric so the reaction isn't shared equally among the two points. But, as the load is simply supported the bending moment at both of the ends is zero:

$$M(0) = 0 = M(6) \tag{1.35}$$

We can use these two equations to find C_1 and C_2 (R_A and M_A):

Which yields

$$M(x) = 36[x-5]^2 - 36[x-2]^2 + 90x \tag{1.36}$$

Remark: Again, as the picture suggests, most of the reaction force is concentrated at B due to the asymmetry. In fact $M_B = 3(72) - 90 = 126$ kN.

Now integrating twice:

That is we have

$$EIy(x) = 3[x-2]^4 - 3[x-5]^4 - 15x^3 + K_1x + K_2 \tag{1.37}$$

Now we apply the boundary condition that $y(0) = 0 = y(6)$:

Now we can write our final answer:

$$y(x) = \frac{1}{EI}\left(3[x-2]^4 - 3[x-5]^4 - 15x^3 + \frac{825}{2}x\right) \tag{1.38}$$

Winter 2010 Q. 1(a)

A light beam of span 6 m is simply supported of its endpoints. Between the points $x = 4$ m and $x = 6$ m there is a U.D.L. of 72 kN m^{-1}. At the point $x = 2$ there is a load of 72 kN. Express the bending moment M in terms of step functions. Solve the differential equation

$$EI\frac{d^2y}{dx^2} = -M \tag{1.39}$$

to find the deflection at any point on the beam. At both ends the deflection is zero.
Solution: First draw a picture:
We can use Macauley's Method but here I will write down the loading function:

$$w(x) = 72\delta(x-2) + 72H(x-4), \tag{1.40}$$

and integrate $-w(x)$ twice to find $M(x)$:

Now apply the boundary condition $M(0) = 0 = M(6)$:

This yields

$$M(x) = -72[x-2] - 36[x-4]^2 + 72x. \tag{1.41}$$

Now write down the differential equation:

$$EI\frac{d^2y}{dx^2} = -M(x)$$

Now apply the boundary conditions $y(0) = 0 = y(6)$ (as it is simply supported):

Hence we have an answer:

$$y(x) = \frac{1}{EI}\left(36[x-2]^3 + 3[x-4]^4 - 12x^3 + 40x\right) \tag{1.42}$$

1.6.3 Fixed Ends

A fixed end beam of length L looks as follows:

We have the following boundary conditions:

1. the deflection at both ends are zero: i.e. $y(0) = 0 = y(L)$.
2. the slope at both ends is zero: i.e. $y'(0) = 0 = y'(L) = 0$.

Note that $M_A \neq 0 \neq M_B$ necessarily as the wall exerts a bending moment.

Winter 2011: Question 1 (b)

A light beam of span 6 m has both ends embedded in walls. At the point $x = 2$ m there is a load of 36 kN. Between the points $x = 4$ m and $x = 6$ m there is a U.D.L. of 72 kN m^{-1}. Express the bending moment M in terms of step functions. Solve the differential equation

$$EI\frac{d^2y}{dx^2} = -M, \tag{1.43}$$

to find the deflection at any point on the beam.
Solution: First draw a picture:
We can use Macauley's Method but here I will write down the loading function:

$$w(x) = 36\delta(x-2) + 72H(x-4), \tag{1.44}$$

and integrate $-w(x)$ twice to find $M(x)$:

In the case of a simply supported beam we the boundary condition $M(0) = 0 = M(6)$ **but this is not the case when the ends are fixed**. You can carry around an M_A and R_A if you want but I'm just going to use C_1 and C_2.

Now we need to solve

$$\begin{aligned} EI\frac{d^2y}{dx^2} &= -M(x) \\ &= 36[x-2] + 36[x-4]^2 - C_1x - C_2 \end{aligned}$$

Thus

$$EI\frac{d^2y}{dx^2} = 6[x-3]^3 + 3[x-4]^4 - \frac{C_1}{6}x^3 - \frac{C_2}{2}x^2 + C_3x + C_4$$

Four unknowns is tough but we have four boundary conditions which will generate four equations in C_1, C_2, C_3, C_4 which we should then be able to solve. First up let's look at $y(0) = 0 = y'(0)$:

Chapter 2

Probability

2.1 Introduction

Solution:

2.1.1 Conditional Probability

2.1.2 Independence

Solution:

Solution:

Solution:

Solution:

Solution:

2.1.3 Random Variables

2.2 Binomial Distribution

Solution:

Solution:

2.3 Poisson Distribution

Solution:

Solution:

2.4 Normal Distribution

Solution:

Solution:

Solution:

2.5 Applications to Engineering Problems

Chapter 3

Sampling Theory

3.1 The Central Limit Theorem

Solution:

Solution:

3.2 Confidence Intervals

3.3 Hypothesis Testing

3.3.1 Z-Test

3.3.2 T-Test

Solution:

Chapter 4

Quality Control

Chapter 5

Further Calculus

Awesome Calculus quote.

In this chapter we revise and expand on calculus.

5.1 Maclaurin and Taylor Series of a Function of a Single Variable

5.2 Review of Partial Differentiation

5.3 Taylor Series Expansion of Functions of Two Variables

Solution:

Solution:

5.4 Differentials

Further Remarks

Examples

1. ...

5.4.1 Definition

... ...

5.4.2 Theorem

Let... *Then ...*

Remark

...

Proof. ...●

Exercises
Evaluate each of the following.

1. ...

Chapter Checklist

1. ...