MS 2001: Test 2 B

Name:
Student Number:

Answer all questions. Marks may be lost if necessary work is not clearly shown.
Remarks by me in italics and would not be required in a test - J.P.

Question 1

(a) Where are the following functions differentiable on $\mathbb{R}$? Please quote theorems/ rules used:

(i)
$$f_1(x) = \frac{x^2 + x + 2}{x^2 - 1}$$

(ii)
$$f_2(x) = \cos^2 x$$

(iii)
$$f_3(x) = \sqrt{2x + 3} + \log x$$

(b) Consider the curve
$$x^3 + x^2y + xy^2 + y^3 = 1 \qquad (1)$$
Show that the point $(1, 0)$ is on the curve. Find the slope of the tangent to the curve at the point $(1, 0)$.

Solution

(a)

(i) x^2+x+2 and x^2-1 are both differentiable everywhere as polynomials. By the Quotient Rule f_1 is differentiable where $x^2 - 1 \neq 0 \Leftrightarrow x \neq \pm 1$. Ans: $\mathbb{R} \backslash \{1, -1\}$.

(ii) $\cos x$ is differentiable everywhere and by the Product Rule $f_2 = (\cos x)(\cos x)$ is differentiable everywhere. Ans: $\mathbb{R}$.
Alternative Solution: x^2 is differentiable everywhere as a polynomial and $\cos x$ is differentiable everywhere. f_2 is the composition of differentiable functions and hence by the Chain Rule is differentiable. Ans: $\mathbb{R}$.

(iii) $\sqrt{2x + 3}$ is the composition of the function $g(x) = \sqrt{x}$ and the differentiable everywhere function $h(x) = 2x + 3$. Now $g(x)$ is differentiable for all $x > 0$ hence by the Chain Rule $g \circ h(x) = \sqrt{2x + 3}$ is differentiable for all $2x + 3 > 0 \Rightarrow x > -3/2$. $\log x$ is differentiable for $x > 0$. Hence by the Sum Rule $f_3(x)$ is differentiable for $x > 0$. Ans: $(0, \infty)$.

(b) Firstly,

$$(1)^3 + (1)^2(0) + (1)(0)^2 + (0)^3 = 1$$

Hence $(1,0)$ is on the curve.

Now, differentiating with respect to x:

$$\begin{aligned}
3x^2 + x^2.\frac{dy}{dx} + 2x[y(x)] + 2x[y(x)].\frac{dy}{dx} + [y(x)]^2 + 3y^2.\frac{dy}{dx} &= 0 \\
\frac{dy}{dx}\left[x^2 + 2xy + 3y^2\right] &= -3x^2 - 2xy - y^2 \\
\frac{dy}{dx} &= \left.\frac{-3x^2 - 2xy - y^2}{x^2 + 2xy + 3y^2}\right|_{(x,y)=(1,0)} \\
&= \frac{-3(1)^2 - 2(1)(0) - (0)^2}{(1)^2 + 2(1)(0) + 3(0)^2} = -3
\end{aligned}$$

Question 2

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^3 + 1 & \text{if } x \leq 0 \\ x^2 + 1 & \text{if } x > 0 \end{cases} \tag{2}$$

Show that g is differentiable on $\mathbb{R}$. Is g twice differentiable? Justify your answer.

Solution

The answer we give here is simpler than has been done in the past but requires two additional hypothesis. Please see the webpage for a proof that our method still works.
Away from 0 (*on the intervals* $(-\infty, 0), (0, \infty)$) g is a polynomial and hence differentiable:

$$g'(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 2x & \text{if } x > 0 \end{cases} \tag{3}$$

Now

$$\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} 2x = 0$$
$$\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} 3x^2 = 0$$

Hence $\lim_{x \to 0} g'(x)$ exists (*and because it is bounded and* g *continuous at* $x = 0$), and thus g is differentiable at $x = 0$ and so differentiable for all $x \in \mathbb{R}$.

$$g''(x) = \begin{cases} 6x & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases} \tag{4}$$

Now

$$\lim_{x \to 0^+} g''(x) = \lim_{x \to 0^+} 2 = 2$$
$$\lim_{x \to 0^-} g''(x) = \lim_{x \to 0^-} 6x = 0$$

Hence $\lim_{x \to 0} g''(x)$ does not exist and hence g is not twice differentiable.

Question 3

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is *not* continuous at a point $a \in \mathbb{R}$. Which of the following statements are true? (Circle the correct statement)

   (a) $f(x) = 0$ for some $x \in \mathbb{R}$.
   Let
   $$f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 2 & \text{otherwise} \end{cases} \quad (5)$$
   Now f is not continuous at 0 and $f(x) \neq 0$, $\forall\, x \in \mathbb{R}$.

   (b) $f(a) = 0$.
   Let f as in (a).

   (c) f is not differentiable at $a \in \mathbb{R}$. ✓

   (d) f' is continuous at $a \in \mathbb{R}$.
   f(x)=1/x is not continuous at $x = 0$. $f'(x) = -1/x^2$ is not continuous at $x = 0$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and has a local minimum at $c \in \mathbb{R}$. Which of the following are true? (Circle the correct statement)

   (a) There exists an interval $I \subset \mathbb{R}$ such that $c \in I$ and $f(x) \geq f(c)$, $\forall\, x \in I$. ✓

   (b) $f'(c) = 0$
   $f(x) = |x|$ has a local minimum at 0 but f' is undefined at 0 let alone equal to 0.

   (c) $f'(c) = 0$ *and* $f''(c) > 0$.

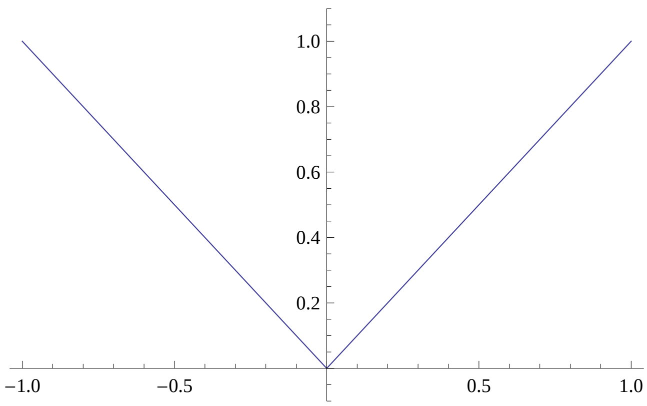

Figure 1: $f(x) = |x|$ has a local minimum at $x = 0$, but $f'(0) \neq 0$.

   $f(x) = |x|$ has a local minimum at 0 but f' and f'' are undefined at 0 let alone equal to 0 and positive.

   (d) None of the above.
   (a) is correct.

3. Which of the following functions satisfy the hypothesis of *Rolle's Theorem* on the closed interval $[a, b]$? (Circle the correct statement)

(a) f is differentiable on (a, b) and $f(a) = f(b)$.
Let

$$f(x) = \begin{cases} x + 1 & \text{if } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

Then for all $x \in (0, 1)$, f *is differentiable, with* $f' = 1$. *Hence there is no point in* $c \in (0, 1)$ *such that*

$$f'(c) = 0. \tag{6}$$

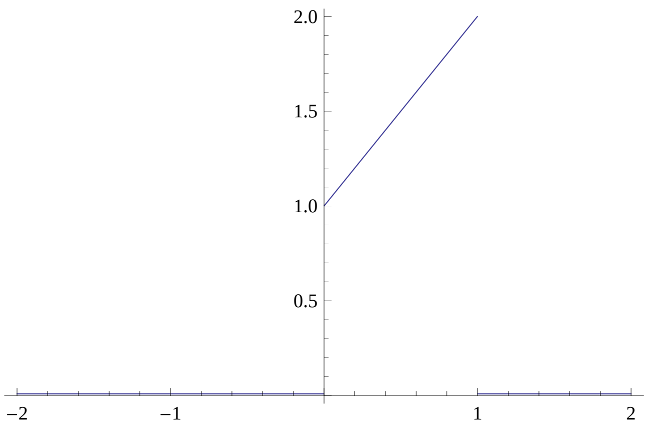

Figure 2: $f(x)$ is differentiable on $(0, 1)$ but does not have a point c with $f'(c) = 0$.) here.

(b) f is continuous on $[a, b]$ and $f(a) = f(b)$.
$f(x) = 1 - |2x - 1|$ *is continuous on* $[0, 1]$ *and* $f(0) = 0 = f(1)$ *but there does not exist a point* $c \in (0, 1)$ *such that*

$$f'(c) = 0. \tag{7}$$

In fact on $(0, 1/2)$, $f' = 2$ *and on* $(1/2, 1)$, $f' = -2$.

(c) f is continuous at on $[a, b]$ and differentiable on (a, b).
$f(x) = x$ *is continuous on* $[0, 1]$ *and differentiable on* $(0, 1)$ *but there does not exist a point* $c \in (0, 1)$ *such that* $f'(c) = 0$. $f'(x) = 1, \forall x \in \mathbb{R}$.

(d) f is differentiable on $[a, b]$ and $f(a) = f(b)$. ✓

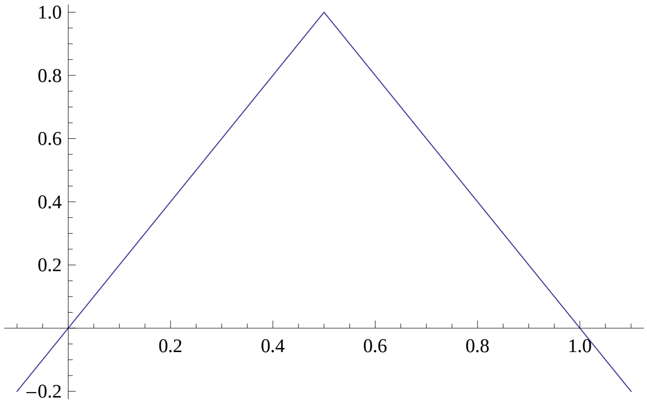

Figure 3: $f(x) = 1 - |2x - 1|$ is continuous on $[0, 1]$ and $f(0) = f(1)$ but does not have a point c with $f'(c) = 0$.