Integration

November 5, 2010

1 Differentiation, Integration and the Fundamental Theorem of Calculus

1.1 Introduction

In this section the derivative and definite integral are introduced in their correct setting. At Leaving Cert level, integration is introduced merely as the inverse of differentiation. This approach simplifies things but greater understanding comes out of a proper treatment. Historically integration was developed separately of differentiation and the link between them was later discovered. The link between them; namely that integration is indeed the inverse of differentiation, is known as the *Fundamental Theorem of Calculus.* The topics of coordinate geometry, limits and functions should be studied in more depth prior to a thorough study calculus, but this brief section is merely intended as an exposition to aid understanding.

1.2 Differentiation

In the figure below, the line from a to b is called a *secant* line.

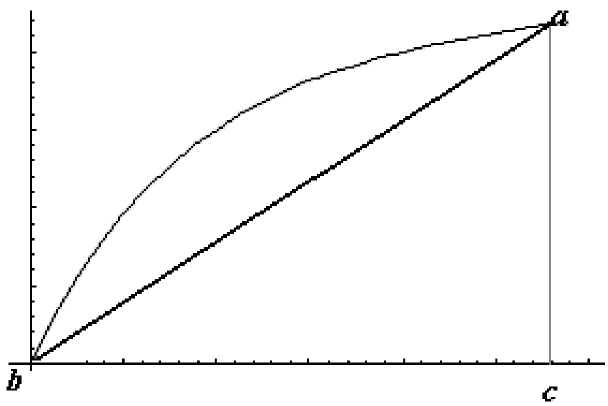


Figure 1: Secant Line.

Introduce the idea of slope. The slope of a line is something intuitive. A steep hill has a greater slope than a gentle rolling hill. The slope of the secant line is simply the ratio of how much the line travels vertically as the line travels horizontally. Denote slope by m:

$$m = \frac{|ac|}{|bc|}. \tag{1}$$

What about the slope of the curve? From a to b it is continuously changing. Maybe at one point its slope is equal to that of the secant but that doesn't tell much. It could be estimated, however, using a ruler the slope at any point. It would be the tangent, as shown:

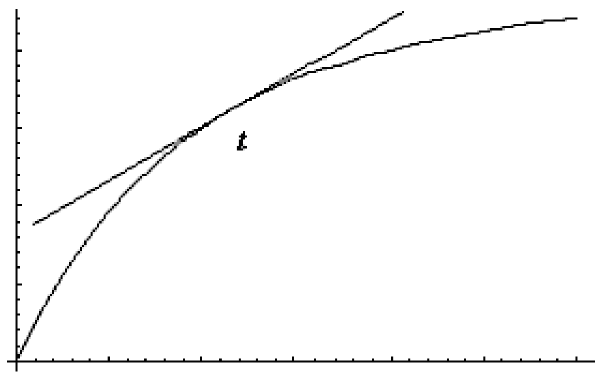


Figure 2: Tangent Line

The above line *is* the slope of the curve at t. Construct a secant line:

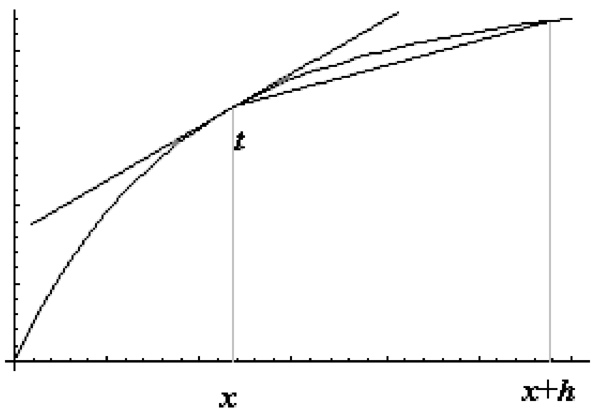


Figure 3: Secant line and Tangent line

Now with respect to analytic geometry, with a function $f(x)$, the slope of this secant is given by:

$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$
$$\Rightarrow m = \frac{f(x+h) - f(x)}{h}$$

It is apparent that the secant line has a slope that is close, in value, to that of the tangent line. Let h become smaller and smaller:

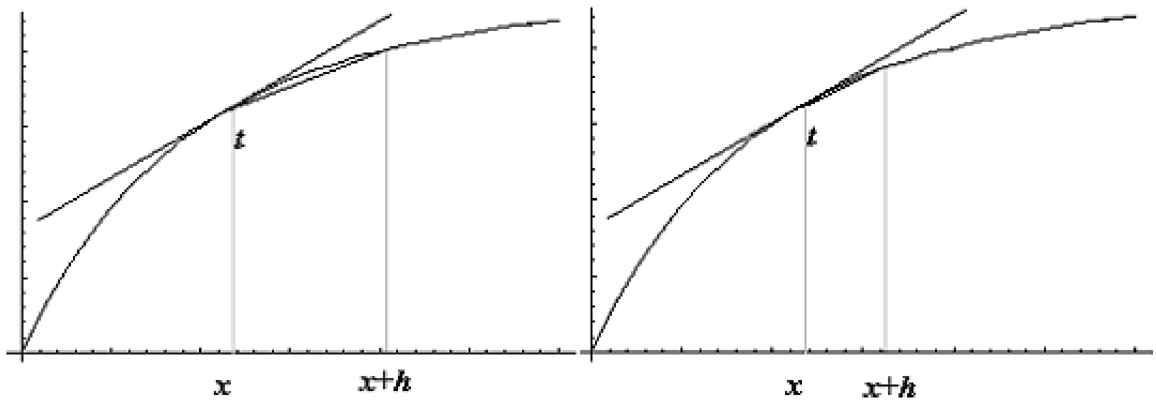


Figure 4: Secant line approaching slope of Tangent line

The slope of the secant line is almost identical to that of our tangent. Let $h \to 0$. Of course, if $h = 0$ there is no secant. But if h got *so close to* 0 *as doesn't matter* then there would be a secant and hence a slope. With respect to analytic geometry, with a function $f(x)$, the slope of this secant, which is indistinguishable from that of the tangent to the point, is given by:

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

This m is the *derivative of f(x)*. This gives the slope of the curve at *every* point on the curve.

1.2.1 Notation

There are two significant branches of notation used to denote the derivative of f. The difference is just notation. They are different ways of writing down the same thing. This subsection is included to fight against simple misunderstandings.

Newtonian Notation

The function is denoted $f(x)$ and the graph is the set of points $(x, f(x))$:

The derivative of $f(x)$ is denoted $f'(x)$. Other names for the derivative of $f(x)$ include:

- the differentiation of $f(x)$
- the derived function for $f(x)$
- the slope of the tangent at $(x, f(x))$
- the gradient
- $\frac{df}{dx}$

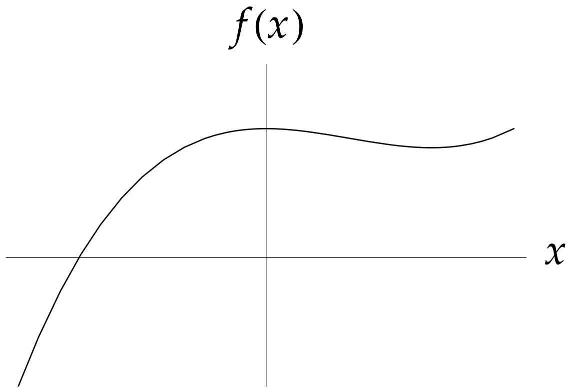


Figure 5: Newtonian notation for functions

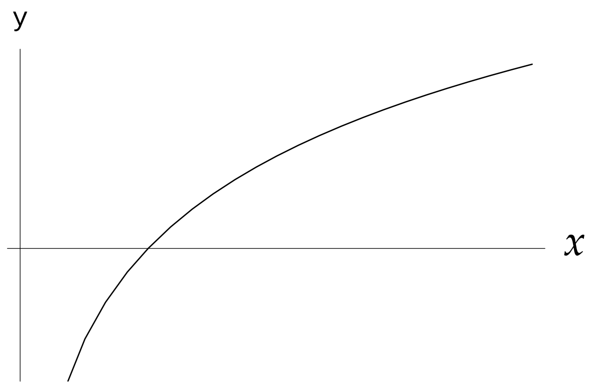


Figure 6: Leibniz notation for functions

Leibniz Notation

The function is denoted $y = f(x)$ (e.g. $y = x^2$); and the graph is the set of points (x, y):

In this notation, y is equivalent to $f(x)$. However, the notation for the derivative of y is:

$$\frac{dy}{dx}. \tag{3}$$

It must be understood that if $y = f(x)$; then

$$f'(x) \equiv \frac{dy}{dx}, \tag{4}$$

and there is no notion of canceling the ds; it is just a notation. It is an illuminating one because if the second graph of figure 4 is magnified about the secant:

If dy is associated with a small variation in $y \sim f(x+h) - f(x)$; and dx associated with a small variation in $x \sim h$; then dy/dx makes sense.

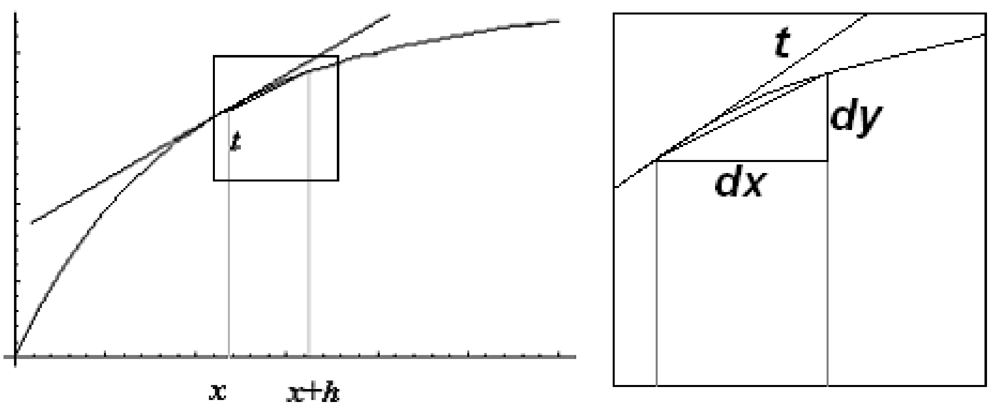


Figure 7: Leibniz notation for the derivative

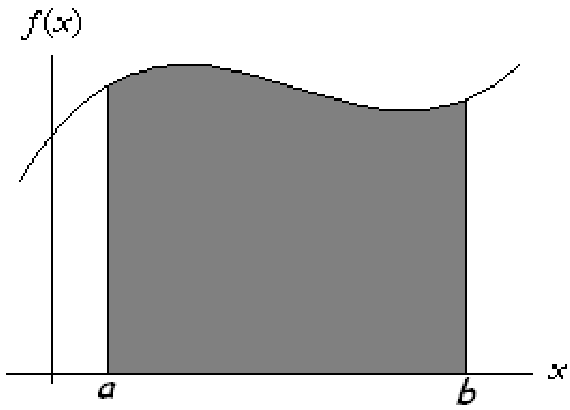
To reiterate if $y = f(x)$; then dy/dx is the same thing as:

- the derivative of $f(x)$
- the differentiation of $f(x)$
- the derived function for $f(x)$
- the slope of the tangent at $(x, f(x))$
- the gradient
- $f'(x)$

So to emphasise $f(x)$ and y are freely interchangeable; as are $f'(x)$ and dy/dx.

1.3 Integration

What is the area of the shaded region under the curve $f(x)$?



Start by subdividing the region into n strips S_1, S_2, ..., S_n of equal width as Figure 8.

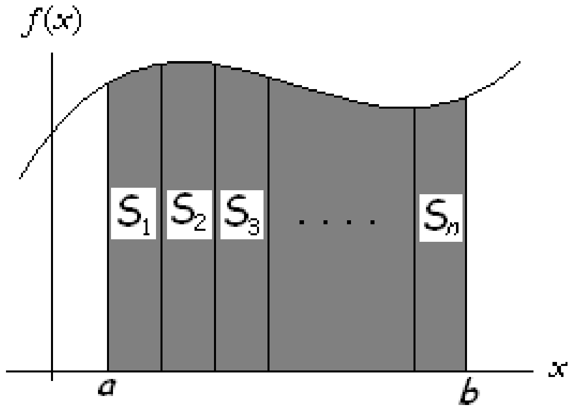
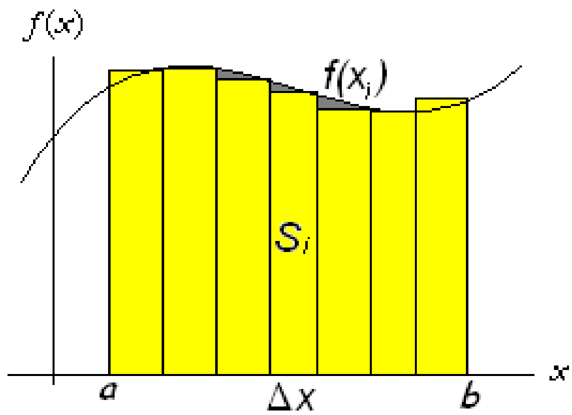


Figure 8:

The width of the interval $[a, b]$ is $b - a$ so the width of each of the n strips is

$$\Delta x = \frac{b-a}{n}.$$

Approximate the ith strip S_i by a rectangle with width Δx_i and height $f(x_i)$, which is the value of f at the right endpoint. Then the area of the ith rectangle is $f(x_i)\,\Delta x_i$:



The area of the original shaded region is approximated by the sum of these rectangles:

$$A \approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x. \tag{5}$$

This approximation becomes better and better as the number of strips increases, that is, as $n \to \infty$. Therefore the area of the shaded region is given by the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots f(x_n)\,\Delta x]. \tag{6}$$

1.3.1 Definition

If $f(x)$ is a function defined in $[a, b]$ and x_i, Δx are as defined above, then the *definite integral of f from a to b* is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x. \tag{7}$$

So an integral is an infinite sum. Associate $\int \sim \lim_{n\to\infty} \sum_n$. Again dx is associated with a small variation in $x \sim \Delta x$.

1.4 Fundamental Theorem of Calculus

1.4.1 Rough Version

If f is a function with derivative f' then

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

Proof

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \\ \Rightarrow \int_a^b f'(x)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f'(x_i)\, \Delta x, \\ \Rightarrow \int_a^b f'(x)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\lim_{h\to 0} \frac{f(x_i+h)-f(x_i)}{h} \right] \Delta x. \end{aligned}$$

Note that $n \to \infty$ and $h \to 0$ is equivalent to $h \to \Delta x$. This is because $\Delta x \underset{n\to\infty}{\to} 0$. Hence let $h \sim \Delta x$;

$$\begin{aligned} \int_a^b f'(x)\, dx &= \lim_{h\to 0} \sum_{i=1}^{n} \left[\lim_{h\to 0} f(x_i+h) - f(x_i) \right] \frac{h}{h}, \\ \Rightarrow \int_a^b f'(x)\, dx &= \lim_{h\to 0} \sum_{i=1}^{n} [f(x_i+h) - f(x_i)]. \end{aligned}$$

Note $h \to \Delta x$ so $x_i + h \to x_{i+1}$:

$$\int_a^b f'(x)\, dx = \sum_{i=1}^{n} [f(x_{i+1}) - f(x_i)].$$

Note $\sum_i [f(x_{i+1}) - f(x_i)]$ forms a telescoping sum:

$$\begin{aligned} \sum_{i=1}^{n} [f(x_{i+1}) - f(x_i)] = f(x_2) - f(x_1) &+ \\ f(x_3) - f(x_2) &+ \\ f(x_4) - f(x_3) &+ \\ &\vdots \\ f(x_{n+1}) - f(x_n)&; \end{aligned}$$

where all but $f(x_{n+1}) - f(x_1)$ cancels. But $x_{n+1} = b$ and $x_1 = a$. Hence $f(x_{n+1}) = f(b)$ and $f(x_1) = f(a)$. Hence

$$\int_a^b f'(x)\,dx = \sum_{i=1}^{n}[f(x_{i+1}) - f(x_i)],$$
$$\Rightarrow \int_a^b f'(x)\,dx = f(b) - f(a).$$

□

1.4.2 The Indefinite Integral

In Leaving Cert where integrals are defined as the inverse of derivatives, an indefinite integral defines integration.

Definition

If $f(x)$ is a function and it differential with respect to x is $f'(x)$, then

$$\int f'(x)\,dx = f(x) + c \tag{8}$$

where c is called the *constant of integration*.

Note the constant of integration. It's inclusion is vital because if $f(x)$ is a function with derivative $f'(x)$ then $f(x) + c$ also has derivative $f'(x)$ as:

$$\frac{d}{dx}(f(x) + c) = \underbrace{\frac{df}{dx}}_{=f'(x)} + \underbrace{\frac{d}{dx}c}_{=0},$$
$$\Rightarrow \frac{d}{dx}(f(x) + c) = f'(x).$$

Geometrically a curve $f(x)$ with slope $f'(x)$ has the same slope as a curve that is shifted upwards; $f(x)+c$. Note that the constant of integration can be disregarded for the indefinite integral. Suppose the integrand is $f'(x)$ and the anti-derivative is $f(x) + c$. Then:

$$\int_a^b f'(x)\,dx = (f(b) + c) - (f(a) + c),$$
$$\Rightarrow \int_a^b f'(x)\,dx = f(b) - f(a).$$

The cs cancel!

1.5 Conclusion

Finding the derivative of a function f at x is finding the slope of the tangent to the curve at x. Integration meanwhile measures the area between two points $x = a$ and $x = b$. The Fundamental Theorem of Calculus states however that differentiation and integration are intimately related; that is given a function f:

$$\frac{d}{dx}\int f(x)\,dx = f(x),$$
$$\int \frac{d}{dx} f(x)\,dx = f(x) + c.$$

i.e. differentiation and integration are essentially inverse processes.

2 Integration

2.1 What We Know

From the Fundamental Theorem of Calculus

$$\int f'(x)\,dx = f(x) + c \tag{9}$$

Thus:

$f(x)$	$\int f(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
e^x	$e^x + c$
$\sec^2 x$	$\tan x + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$

Also because

$$\frac{d}{dx}\sin nx = n\cos nx \text{ , and}$$
$$\frac{d}{dx}\cos nx = -n\sin nx$$
$$\Rightarrow \int \cos nx\,dx = \frac{\sin nx}{n} + c \text{ and}$$
$$\Rightarrow \int \sin nx\,dx = -\frac{\cos nx}{n} + c$$

Also, let $a > 0$;

$$\frac{d}{dx}\frac{1}{a}\arctan\frac{x}{a} = \frac{1}{a}\frac{1}{1+x^2/a^2}.\frac{1}{a}$$
$$\Rightarrow \frac{d}{dx}\frac{1}{a}\arctan\frac{x}{a} = \frac{1}{\cancel{a}}\frac{1}{(a^2+x^2)/\cancel{a^2}}.\frac{1}{\cancel{a}}$$
$$\Rightarrow \frac{d}{dx}\frac{1}{a}\arctan\frac{x}{a} = \frac{1}{a^2+x^2}$$
$$\Rightarrow \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + c$$

Also

$$\frac{d}{dx}\arcsin\frac{x}{a} = \frac{1}{\sqrt{1-x^2/a^2}}\frac{1}{a}$$
$$\Rightarrow \frac{d}{dx}\arcsin\frac{x}{a} = \frac{1}{\sqrt{(a^2-x^2)/a^2}}\frac{1}{a}$$
$$\Rightarrow \frac{d}{dx}\arcsin\frac{x}{a} = \sqrt{\frac{\cancel{a^2}}{a^2-x^2}}\frac{1}{\cancel{\sqrt{a^2}}}$$
$$\Rightarrow \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}$$

2.2 Properties of Integration

2.2.1 Proposition

- $$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx \tag{10}$$
- $$\int k\,f(x)\,dx = k\int f(x)\,dx \ , \ \ where\ k \in \mathbb{R} \tag{11}$$

Proof (Non-Examinable)

- $$\int (f(x) \pm g(x))\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i) \pm g(x_i))\,\Delta x$$
$$\Rightarrow \int (f(x) \pm g(x))\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\,\Delta x \pm \sum_{i=1}^{n} f(x_i)\,\Delta x\right)$$
$$\Rightarrow \int (f(x) \pm g(x))\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \pm \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\,\Delta x$$
$$\Rightarrow \int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

- Let $k \in \mathbb{R}$.

$$\begin{aligned}
\int k\, f(x)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} k f(x_i)\, \Delta x \\
\Rightarrow \int k\, f(x)\, dx &= \lim_{n\to\infty} k \sum_{i=1}^{n} f(x_i)\, \Delta x \\
\Rightarrow \int k\, f(x)\, dx &= k \left(\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\, \Delta x \right) \\
\Rightarrow \int k\, f(x)\, dx &= k \int f(x)\, dx
\end{aligned}$$

Remarks

There is no direct analogue of the product, quotient nor chain rule for integration. Although the substitution method below is like a chain rule for integrals.

2.2.2 The Substitution Method for Evaluating Integrals

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du \tag{12}$$

where $u = g(x)$

Proof (Non-Examinable)

$$\begin{aligned}
u &= g(x) \\
\Rightarrow \frac{du}{dx} &= g'(x) \\
\Rightarrow \frac{dx}{du} &= \frac{1}{g'(x)} \\
\Rightarrow dx &= \frac{du}{g'(x)}
\end{aligned}$$

So

$$\int f(g(x))g'(x)dx = \int f(u)\cancel{g'(x)}\frac{du}{\cancel{g'(x)}} = \int f(u)du$$

2.3 Indefinite Integrals

2.3.1 Proposition

Suppose $f(x)$ has an anti-derivative $F(x)$. Then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{13}$$

Proof (Non-Examinable)

$$\int_a^b f(x)\,dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x)\,dx$$

□

2.4 Techniques of Trigonometric Integration

Consider the integral

$$\int \cos 4x \cos 2x\,dx \tag{14}$$

To differentiate such a function is no problem; there is a product rule. However, there is no such product rule for integration. There are functions, such as

$$\int \frac{\sin x}{x}\,dx \tag{15}$$

which, because there is no product or quotient rule for integration, may not be integrated easily. In fact, (15) has *no* ordinary solution in terms of elementary functions (like polynomials, exponential functions, logarithms, trigonometric functions, inverse trigonometric functions and their combinations). This won't happen at LC level however so (14) may certainly be integrated. The key is to realise that there *is* a sum rule for integration so if $\cos 4x \cos 2x$ can be written as a sum of integrable terms things will work out easily. Luckily such formulae exist and are in the tables:

$$2\cos A\cos B = \cos(A+B) + \cos(A-B) \tag{16}$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B) \tag{17}$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B) \tag{18}$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B) \tag{19}$$

Now (14) may be easily integrated.

2.4.1 Products of Unlike Trigonometric Terms

To integrate functions of the kind:

$$\cos mx \cos nx \tag{20}$$
$$\cos mx \sin nx \tag{21}$$
$$\sin mx \cos nx \tag{22}$$
$$\sin mx \cos nx \tag{23}$$

convert the product to a sum using (16-19)

Consider now the integral

$$I = \int \sin^2\theta \cos^3\theta \, d\theta \tag{24}$$

There is no available formula to express this product as a sum. There is a substitution technique however:

$$\int \sin^2\theta \cos^3\theta \, d\theta = \int \sin^2\theta \cos^2\theta \cos\theta \, d\theta$$

Now because cos and sin are (up to sign) derivatives of each other, spot that $\cos\theta$ is the derivative of $\sin\theta$ so $\cos\theta$ can play the rôle of the $g'(x)$ in (12). Hence let $u = \sin\theta$:

$$\begin{aligned}
\frac{du}{d\theta} &= \cos\theta \\
\Rightarrow d\theta &= \frac{du}{\cos\theta} \\
\Rightarrow I &= \int \sin^2\theta \cos^2\theta \cancel{\cos\theta} \frac{du}{\cancel{\cos\theta}} \\
\Rightarrow I &= \int u^2 \cos^2\theta \, du
\end{aligned}$$

Now it seems all is lost but as

$$\begin{aligned}
\cos^2 x + \sin^2 x &= 1 \\
\Rightarrow \cos^2 x &= 1 - \sin^2 x \\
\Rightarrow I &= \int u^2(1-u^2)\, du
\end{aligned}$$

Typically at LC level the powers of sin and cos in this case will be under 3.

2.4.2 Products of Powers of $\sin$ & $\cos$

To integrate functions of the kind:

$$\sin^n x \cos^m \theta \tag{25}$$

*break the one with an **odd** power into a square plus 1; i.e.*

$$\sin^{2n+1}\theta = \sin^{2n}\theta \sin\theta$$

and then let $u =$ *the other one. If* m *and* n *are both even* ($m = 2 = n$) *then use the formulae:*

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{26}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{27}$$

2.4.3 $\cos^3 x$ and $\sin^3 x$

The integrals of $\cos^3 x$ and $\sin^3 x$ require a mix of the previous two cases. The integrand is manipulated as follows (similar for $\sin^3 x$):

$$\begin{aligned}
\cos^3 x &= \cos^2 x \cos x \\
\Rightarrow \cos^3 x &= \frac{1}{2}(1 + \cos 2x)\cos x \\
\Rightarrow \cos^3 x &= \frac{1}{2}(\cos x + \cos 2x \cos x) \\
\underset{(16)}{\Rightarrow} &\frac{1}{2}\left(\cos x + \frac{\cos 3x + \cos x}{2}\right) \\
\Rightarrow \cos^3 x &= \frac{3\cos x}{4} + \frac{\cos 3x}{4}
\end{aligned}$$

which can be easily integrated.

2.5 Intersecting Curves

Suppose the area between two curves is required:

There are a few facts we must know. By definition, the area of a function between $x = a$ and $x = b$ *is* the definite integral from a to b. Now consider

If $f(x) \geq g(x)$ for $x \in [a, b]$ then, where $A_a^b(f, g)$ is the area between $f(x)$ and $g(x)$ between $x = a$ and $x = b$:

$$A_a^b(f, g) = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx \tag{28}$$

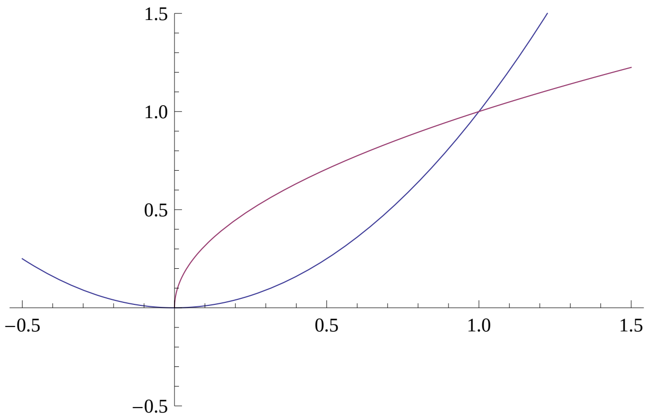


Figure 9: The area enclosed between $f(x) = x^2$ and $g(x) = \sqrt{x}$ is bound inside the points where they intersect: $x = 0, 1$

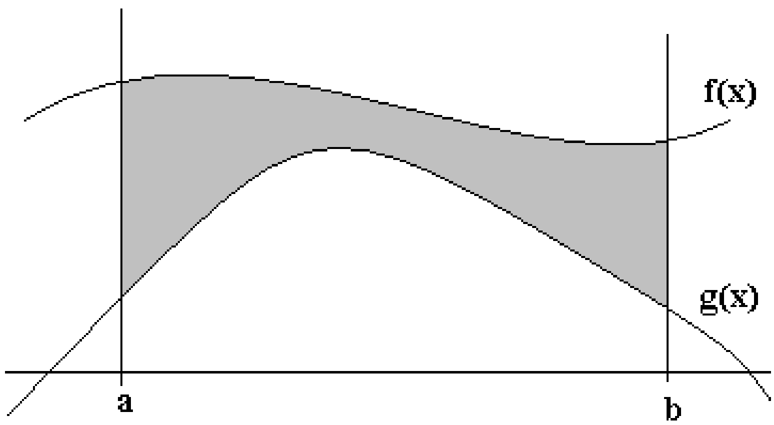


Figure 10: The area between $f(x)$ and $g(x)$ (shaded) is the area under $f(x)$ less the area under $g(x)$.

2.6 Changing Products to Sums and Sums to Products

Frequently it will be helpful to change sums of trigonometrical expressions into products or vice versa. For example to find the roots of

$$\sin 3x - \sin x$$

it would very helpful to change this to a product and using the fact that

$$a.b = 0 \Leftrightarrow a = 0 \text{ or } b = 0 \tag{29}$$

to find the roots. Indeed using (**??**):

$$\sin 3x - \sin x = 2\cos 2x \cos x$$

In contrast, some things are easier as sums. For example there is no *product rule* for integration;

$$\int f(x)g(x)\,dx$$

has no formula in terms of $\int f$ and $\int g$. However

$$\int (f(x)+g(x))dx = \int f(x)\,dx + \int g(x)\,dx \tag{30}$$

so it can be handy to write a product as a sum. For example, using (**??**):

$$\int 2\cos 5x \cos 3x\,dx = \int \cos 8x\,dx + \int \cos 2x\,dx.$$

2.6.1 Substitution Technique for Integration

In general, if we need to make a substitution, we look for the following pattern:

$$\int f(g(x))g'(x)\,dx \tag{31}$$

that is we look for a function and its derivative in the integrand. Then let $u = g(x)$:

$$\begin{aligned} \frac{du}{dx} &= g'(x) \\ \Rightarrow dx &= \frac{du}{g'(x)} \\ \Rightarrow \int f(g(x))g'(x)\,dx = \int f(u)\cancel{g'(x)}\frac{du}{\cancel{g'(x)}} &= \int f(u)\,du \end{aligned}$$

Examples

Spot the patterns:

$$\begin{aligned} &\int 2x^2\sqrt{x^3+1}\,dx \\ &\int t(5+3t^2)^8\,dt \\ &\int x^2 e^{x^3}\,dx \\ &\int s^2\sqrt[5]{7-4s^3}\,ds \\ &\int \sqrt{1+\frac{1}{3x}}\,\frac{dx}{x^2} \\ &\int x^2\sec^2(x^3+1)\,dx \\ &\int \sin^2 x\cos x\,dx \end{aligned}$$

LIATE

If we cannot see a $g(x)$, $g'(x)$ pattern we can use the LIATE rule. Choose u according to the most complicated expression in the following hierarchy:

L ogarithms

I nverses (inverse sine, tan)

A lgebraic (polynomials in x)

T rigonometric

E xponential

In general this works well (also works for Integration by Parts).

2.6.2 Areas under Curves

The Basics

By definition, the area of a function between $x = a$ and $x = b$ *is* the definite integral from a to b. Note however that areas *under* the x-axis will be computed to have negative area so if the function looks like:

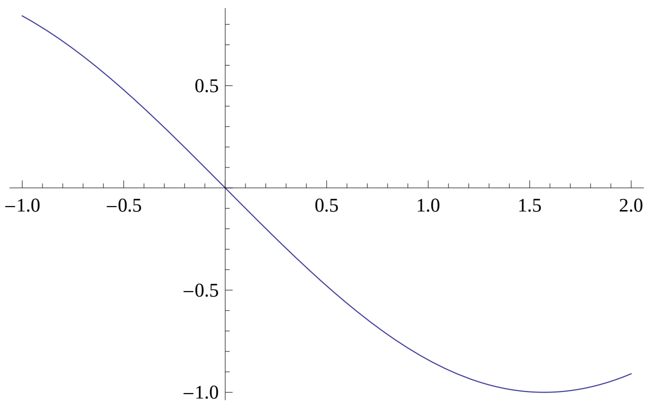


Figure 11: The area under the curve from $x = -1$ to $x = 2$ will be the integral from $x = -1$ to 0, *minus* the integral from $x = 0$ to 2.

Areas between Curves

Now consider

If $f(x) \geq g(x)$ for $x \in [a, b]$ then, where $A_a^b(f, g)$ is the area between $f(x)$ and $g(x)$ between $x = a$ and $x = b$:

$$A_a^b(f, g) = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx \tag{32}$$

Note if $f(x) \geq g(x)$ then $f(x) - g(x)$ is always positive so the problem of negative area doesn't arise.

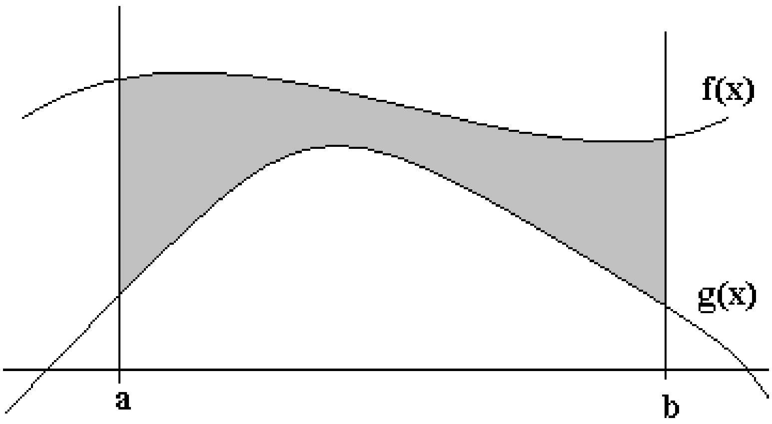


Figure 12: The area between $f(x)$ and $g(x)$ (shaded) is the area under $f(x)$ less the area under $g(x)$.

Intersecting Curves

Suppose we want to find the area bound by two curves, e.g.:

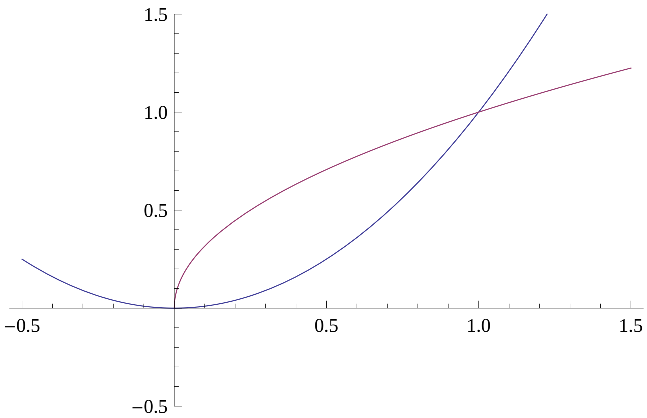


Figure 13: The area enclosed between $f(x) = x^2$ and $g(x) = \sqrt{x}$ is bound inside the points where they intersect: $x = 0, 1$

If the curves are $y = f(x)$ and $y = g(x)$; there intersubsections are the set of points such that:

$$f(x) = g(x) \tag{33}$$

For example for $f(x) = x^2$ and $g(x) = \sqrt{x}$, the intersubsections occur when

$$\begin{aligned} x^2 &= \sqrt{x} \\ \Rightarrow x^2 - \sqrt{x} &= 0 \\ \Rightarrow \sqrt{x}(x^{3/2} - 1) &= 0 \\ \Rightarrow x &= 0, 1 \end{aligned}$$

Hence the area enclosed between the curves is given by:

$$\int_0^1 (\sqrt{x} - x^2)\, dx \tag{34}$$