MS 2001: Test 1 B Solutions

Name:
Student Number:

Answer all questions. Marks may be lost if necessary work is not clearly shown.
Remarks by me in italics and would not be required in a test - J.P.

Question 1

(a) Find the solution set of the following inequality:

$$\left|\frac{3x-2}{x+1}\right| \leq 2\,,\ x \neq -1$$

(b) Evaluate the following using the Calculus of Limits. Hint: Rationalise the numerator:

$$\lim_{x\to 1} \frac{2x-\sqrt{3x+1}}{x-1}$$

Solution

(a)

$$\left|\frac{3x-2}{x+1}\right| \leq 2$$
$$\Rightarrow \frac{|3x-2|}{|x+1|} \leq 2$$

As $|a/b| = |a|/|b|$*. As* $|x+1| > 0$ *we can multiply across. It is not 0 as we have assumed that* $x \neq -1$*:*

$$|3x-2| \leq 2|x+1|$$
$$\Rightarrow |3x-2|^2 \leq 2^2|x+1|^2$$

We can square both sides as they are positive. Now $|x|^2 = x^2$*:*

$$(3x-2)^2 \leq [2(x+1)]^2$$
$$\Rightarrow (3x-2)^2 - [2(x+1)]^2 \leq 0$$
$$\Rightarrow [(3x-2)+2(x+1)][(3x-2)-2(x+1)] \leq 0$$

by the difference of two squares $a^2-b^2 = (a-b)(a+b)$*. Multiplying out and factoring works too*

$$(5x)(x-4) \leq 0 \tag{1}$$

The LHS is zero at $x = 0, 4$. Now test the sign of $(5x)(x-4)$ between the roots, say at $-1, 1, 5$:

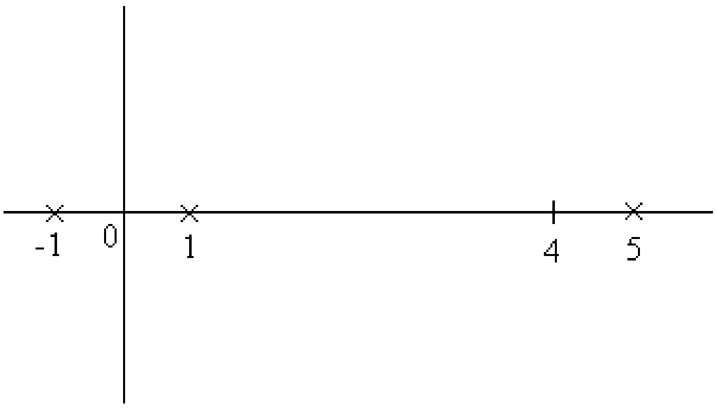

Figure 1: *To find out where a continuous function is positive or negative, find out where it is zero (i.e. the roots). Test the sign of the function between the roots. The function can only change sign at roots so if the function is positive between roots α and β it will be positive for all $x \in (\alpha, \beta)$*

Let $f(x) = (5x)(x-4)$;

$$f(-1) = (-5)(-5) > 0$$
$$f(1) = (5)(-3) < 0$$
$$f(5) = (25)(1) > 0$$

Hence $(5x)(x-4) \leq 0$ for all x between 0 and 4 and we have equality at 0 and 4 (*so we include these points. We show this with square brackets.*)
Solution Set $[0, 4]$.

Alternatively, from (1)
$(5x)(x-4) = 5x^2 - 20$ is a quadratic with $a > 0$ so has geometry:

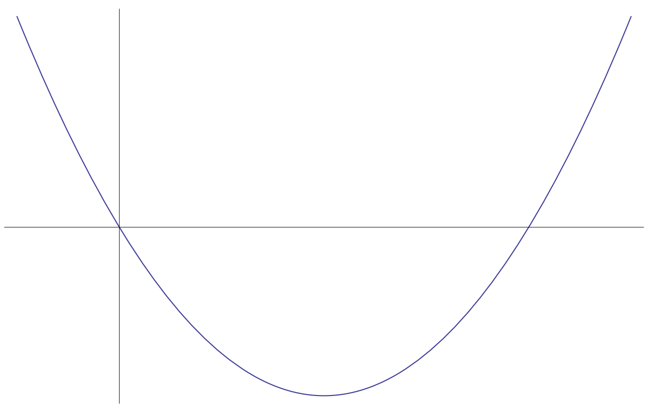

Figure 2: *A quadratic $ax^2 + bx + c$ with $a > 0$ is negative inside the roots*

Hence the solution set is inside the roots, and including the roots: $[0, 4]$.

(b) *Rationalising the numerator means multiplying by its conjugate, and using the difference of two squares:*

$$\begin{aligned}
\lim_{x\to 1} \frac{2x-\sqrt{3x+1}}{x-1} &= \lim_{x\to 1} \frac{2x-\sqrt{3x+1}}{x-1} \times \underbrace{\frac{2x+\sqrt{3x+1}}{2x+\sqrt{3x+1}}}_{=1} \\
&= \lim_{x\to 1} \frac{(2x)^2-(\sqrt{3x+1})^2}{(x-1)(2x+\sqrt{3x+1})} \\
&= \lim_{x\to 1} \frac{4x^2-(3x+1)}{(x-1)(2x+\sqrt{3x+1})} \\
&= \lim_{x\to 1} \frac{4x^2-3x-1}{(x-1)(2x+\sqrt{3x+1})} \\
&= \lim_{x\to 1} \frac{(x-1)(4x+1)}{(x-1)(2x+\sqrt{3x+1})}
\end{aligned}$$

We can cancel the $(x-1)$*s by multiplying above and below by* $1/(x-1)$*. This is only possible if* $(x-1)\neq 0$*. We can guarantee this because a limit as* $x\to 1$ *is concerned with* x *very close to* ***but not equal*** *to* 1.

$$\begin{aligned}
\lim_{x\to 1} \frac{2x-\sqrt{3x+1}}{x-1} &= \lim_{x\to 1} \frac{\cancel{(x-1)}(4x+1)}{\cancel{(x-1)}(2x+\sqrt{3x+1})} \\
&= \lim_{x\to 1} \frac{(4x+1)}{(2x+\sqrt{3x+1})} = \frac{4(1)+1}{2(1)+\sqrt{3(1)+1}} = \frac{5}{4}
\end{aligned}$$

Question 2

For $a \in \mathbb{R}$ consider the function

$$g(x) := \begin{cases} x^2 & \text{for } x > 0 \\ x+1 & \text{for } -1 < x < 0 \\ 3x + a & \text{for } x \leq -1 \end{cases}$$

Is g continuous at 0? Justify your answer. For what value(s) of a is g continuous at $x = -1$? Justify your answer.

Solution

g is not continuous at 0 (*).

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x^2) = 0$$
$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (x+1) = 1$$

As the left- and right-hand limits are not equal at 0, the limit at 0 does not exist (*by a theorem from the notes - see Test A Q. 3. 3 (b)*). Hence the function is not continuous at 0 (*for a function f to be continuous at a, the function must be defined at a, the limit must exist and further they must be equal:*

$$\lim_{x \to a} g(x) = g(a) \tag{2}$$

).
Alternative Solution (from (*)) The function is not continuous at 0 because it is not defined there.

For g to be continuous at -1, we need

$$\lim_{x \to -1} g(x) = g(-1) \tag{3}$$

For the limit to exist we need the left- and right- hand limits equal:

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} (x+1) = 0$$
$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} (3x+a) = -3 + a$$
$$\Rightarrow 0 = -3 + a$$
$$\Rightarrow a = 3$$

Now

$$\lim_{x \to -1} g(x) = 0 = 3(-1) + a = g(-1)$$

Hence g is continuous at -1 if $a = 3$.

Question 3

1. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is a *quadratic* function. Which of the following statements are true? (Circle the correct statement)

   (a) f has two real roots.
   $f(x) = x^2 + 1$ *is a polynomial with two complex roots,* $\pm i$, *and no real roots.*

   (b) $f(x) = (x - m)(x - n)$ for some $m, n \in \mathbb{Z}$.
   By the Factor Theorem $k \in \mathbb{R}$ *is a root of a polynomial if and only if* $(x - k)$ *is a factor.* $x^2 + 1$ *doesn't have integer (i.e.* $\mathbb{Z}$*=whole numbers, positive & negative) so doesn't have any factors of the form* $(x - m)$, $m \in \mathbb{Z}$.

   (c) $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$, with $a \neq 0$. ✓.

   (d) $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ for some $a, b, c, d, e \in \mathbb{R}$, with $a \neq 0$.
   This is a quartic.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function, *decreasing* on the interval $[-1, 1]$. Which of the following are true? (Circle the correct statement)

   (a) For $x, y \in [-1, 1]$, if $y \geq x$, then $|f(y)| \leq |f(x)|$.
   Let
   $$f(x) = \left\{ \begin{array}{cc} 1 & \text{if } x < 0 \\ -2 & \text{if } x \geq 0 \end{array} \right\}$$
   Then f *is decreasing but* $-1/2 \leq 1/2$ *and* $|f(-1/2)| = 1 \not\geq 2 = |f(1/2)|$.

   (b) $f(-1) \neq f(1)$.
   $f(x) = 0$ *is increasing but* $f(-1) = 0 = f(1)$

   (c) If f has a root in $[-1, 1]$, then it is unique.
   $f(x) = 0$ *is increasing but all of* $x \in [-1, 1]$ *are roots.*

   (d) For any $x \in [-1, 1]$, $f(1) \leq f(x)$. ✓

3. Which of the following statements are true for *continuous* functions? (Circle the correct statement)

(a) If $g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions, then so is
$$f(x) = \frac{g(x)}{h(x)}$$
$g(x) = 1$ *and* $h(x) = x$ *are continuous but* $1/x$ *is not continuous at* 0.

(b) If for all $\varepsilon > 0$, there exists a delta $\delta > 0$, such that whenever $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$, then f is continuous at the point a. ✓

(c) If for all $\delta > 0$, there exists an $\varepsilon > 0$, such that whenever $|f(x) - f(a)| < \varepsilon$ then $|x - a| < \delta$, then f is continuous at the point a.
This is an incorrect variation on the correct answer. Let $f : \mathbb{R} \to \mathbb{R}$;
$$f(x) := \left\{ \begin{array}{ll} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{array} \right. \tag{4}$$
Let $\delta > 0$. *If we put* $\varepsilon = 1/2$, *then whenever* $|f(x) - f(0)| < 1/2$ *then* $x = 0$ *and we have* $|0 - 0| < \delta$ *as required. But* f *is not continuous at* 0.

(d) $f(x) = \sqrt{x-1}$ is continuous at 0.
It's not because it's not defined: $f(0) = \sqrt{-1} \notin \mathbb{R}$.