Problem Sets: Solutions

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1 Introduction to Applied Mathematics

1.1 Algebra

1.1.1 Problem

Solve the simultaneous equations

$$\begin{aligned} x - y &= 0, \\ (x+2)^2 + y^2 &= 10. \end{aligned}$$

Solution: Let

$$\begin{aligned} x - y &= 0 \quad (A) \\ (x+2)^2 + y^2 &= 10 \quad (B) \end{aligned}$$

$(A) \Rightarrow x = y$.

$$\begin{aligned} (B) : (x+2)^2 + x^2 &= 10 \\ \Rightarrow x^2 + 4x + 4 + x^2 - 6 &= 0 \\ \Rightarrow 2x^2 + 4x - 6 &= 0 \\ \Rightarrow x^2 + 2x - 3 &= 0 \\ \Rightarrow (x+3)(x-1) &= 0 \\ \Rightarrow x &= -3, \text{ or } 1 \\ \Rightarrow \text{Sol. Set} &= \{(1,1), (-3,-3)\}. \end{aligned}$$

1.1.2 Problem

Show that the following simplifies to a constant when $x \neq 2$

$$\frac{3x-5}{x-2} + \frac{1}{2-x}$$

Solution:

$$\begin{aligned} &\frac{3x-5}{x-2} + \frac{1}{2-x}, \\ &= \frac{3x-5}{x-2} - \frac{1}{x-2}, \\ = \frac{3x-5-1}{x-2} = \frac{3x-6}{x-2} = 3\frac{(x-2)}{x-2} &= 3. \end{aligned}$$

1.1.3 Problem

Show that

$$\frac{-1+\sqrt{3}}{1+\sqrt{3}} = 2 - \sqrt{3}$$

Solution:

$$\begin{aligned} &\frac{-1+\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} \\ &= \frac{-1+\sqrt{3}+\sqrt{3}-3}{1-3} \\ &= \frac{2\sqrt{3}-4}{-2} = 2-\sqrt{3}. \end{aligned}$$

1.1.4 Problem

$x^2 - px + q$ is a factor of $x^3 + 3px^2 + 3qx + r$.

(i) *Show that $q = -2p^2$.*

(ii) *Show that $r = -8p^3$.*

(iii) *Find the three roots of $x^3 + 3px^2 + 3qx + r = 0$ in terms of p.*

Solution: A cubic function $f(x) = x^3 + bx^2 + cx + d$ has factors $(x-\alpha)(x-\beta)(x-\gamma)$ where α, β and γ are the roots of $f(x)$. Similarly a quadratic function $g(x) = x^2 - px + q$ has factors $g(x) = (x-\alpha_1)(x-\alpha_2)$ where α_1 and α_2 are the roots of $g(x)$. If a quadratic $g(x)$ is a factor of a cubic $f(x)$ then the roots of g are roots of f and $f(x) = g(x).(x-\alpha_3)$ where α_3 is the third root of $f(x)$. Hence let $f(x) = x^3 + 3px^2 + 3qx + r$ and $g(x) = x^2 - px + q$.

$$\begin{aligned} f(x) &= (x^2 - px + q)(x - \alpha_3) \\ \Rightarrow f(x) &= x^3 - px^2 + qx - \alpha_3 x^2 + \alpha_3 px - \alpha_3 q \\ \Rightarrow f(x) &= x^3 + (-\alpha_3 - p)x^2 + (q + \alpha_3 p) + (-\alpha_3 q) \end{aligned}$$

(i) Equating the x^2-coefficients:

$$3p = -\alpha_3 - p$$
$$\Rightarrow \alpha_3 = -4p.$$

Equating the x-coefficients:

$$3q = q + (-4p)p$$
$$\Rightarrow 2q = -4p^2$$
$$\Rightarrow q = -2p^2.$$

(ii) Equating the constant coefficient:

$$r = -\alpha_3 q$$
$$\Rightarrow r = -(-4p)(-2p^2)$$
$$\Rightarrow r = -8p^3.$$

(iii) $\alpha_3 = -4p$ is one root. The other roots of $f(x) = g(x)(x - \alpha_3)$ are the roots of $g(x)$;

$$g(x) = x^2 - px - 2p^2 \overset{!}{=} 0$$
$$\Rightarrow x^2 - 2px + px - 2p^2 = 0$$
$$\Rightarrow x(x - 2p) + p(x - 2p) = 0$$
$$\Rightarrow (x - 2p)(x + p)$$

Hence the set of roots is

$$\{x : f(x) = 0\} = \{-4p, 2p, -p\}.$$

1.2 Vectors

1.2.1 Problem

$\mathbf{s} = 4\mathbf{i} + 3\mathbf{j}$ *and* $\mathbf{t} = 2\mathbf{i} - 5\mathbf{j}$.
Find $|\mathbf{st}|$
Solution:

$$\mathbf{st} = \mathbf{t} - \mathbf{s}$$
$$\Rightarrow \mathbf{st} = (2\mathbf{i} - 5\mathbf{j}) - (4\mathbf{i} + 3\mathbf{j}) = -2\mathbf{i} - 8\mathbf{j}.$$

Where $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$;

$$|\mathbf{v}| = \sqrt{x^2 + y^2}. \tag{1}$$

$$\Rightarrow |\mathbf{st}| = \sqrt{(-2)^2 + (-8)^2} = \sqrt{68} = \sqrt{4(17)} = 2\sqrt{17}.$$

1.2.2 Problem

$\mathbf{a} = 2\mathbf{i} + (2k+3)\mathbf{j}$ and $\mathbf{b} = k^2\mathbf{i} + 6\mathbf{j}$, where $k \in \mathbb{Z}$.
$\mathbf{a}$ is perpendicular to $\mathbf{b}$.

(i) Find the value of k.

(ii) Using your value for k, write $\mathbf{a}+\mathbf{b}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(iii) Hence, find the measure of the angle between $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$ correct to the nearest degree.

Solution:

(i)

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0. \tag{2}$$

$$\begin{aligned} \underset{\mathbf{a}\perp\mathbf{b}}{\Rightarrow} \mathbf{a} \cdot \mathbf{b} &= 0 \\ \Rightarrow 2k^2 + 6(2k+3) &\overset{!}{=} 0 \\ \Rightarrow 2k^2 + 12k + 18 &= 0 \\ \Rightarrow k^2 + 6k + 9 &= 0 \\ \Rightarrow (k+3)^2 &= 0 \\ \Rightarrow k &= -3 \end{aligned}$$

(ii)

$$\begin{aligned} \underset{k=-3}{\Rightarrow} \mathbf{a} &= 2\mathbf{i} - 3\mathbf{j} \\ \mathbf{b} &= 9\mathbf{i} + 6\mathbf{j} \\ \Rightarrow \mathbf{a} + \mathbf{b} &= 11\mathbf{i} + 3\mathbf{j} \end{aligned}$$

(iii) By the properties of the Dot Product:

$$\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = \underbrace{\mathbf{a} \cdot \mathbf{a}}_{=|\mathbf{a}|^2} + \underbrace{\mathbf{a} \cdot \mathbf{b}}_{=0}$$

Also, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) &= |\mathbf{a}||\mathbf{a} + \mathbf{b}| \cos\theta \\ \Rightarrow |\mathbf{a}|^2 &= |\mathbf{a}||\mathbf{a} + \mathbf{b}| \cos\theta \\ \Rightarrow \cos\theta &= \frac{|\mathbf{a}|^2}{|\mathbf{a}||\mathbf{a} + \mathbf{b}|} = \frac{|\mathbf{a}|}{|\mathbf{a} + \mathbf{b}|} \\ \Rightarrow \cos\theta &= \frac{\sqrt{4+9}}{\sqrt{121+9}} = \frac{\sqrt{13}}{\sqrt{130}} = \frac{1}{\sqrt{10}} \\ \Rightarrow \theta = \cos^{-1}(1/\sqrt{10}) &= 71.5651^\circ \approx 72^\circ. \end{aligned}$$

1.2.3 Problem

rst is a triangle where $\mathbf{r} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{s} = -4\mathbf{i} - 2\mathbf{j}$ *and* $\mathbf{t} = 3\mathbf{i} - \mathbf{j}$.

(i) *Express* $\mathbf{rs}$, $\mathbf{st}$ *and* $\mathbf{tr}$ *in terms of* $\mathbf{i}$ *and* $\mathbf{j}$.

(ii) *Show that the triangle* rst *is right-angled at* r

(iii) *Find the measure of* $\angle rst$.

Solution:

(i)

$$\begin{aligned} \mathbf{rs} &= \mathbf{s} - \mathbf{r} \\ &= (-4\mathbf{i} - 2\mathbf{j}) + \mathbf{i} - 2\mathbf{j} \\ & -3\mathbf{i} - 4\mathbf{j}. \end{aligned}$$

$$\begin{aligned} \mathbf{st} &= \mathbf{ts} \\ &= 3\mathbf{i} - \mathbf{j} + 4\mathbf{i} + 2\mathbf{j} \\ &= 7\mathbf{i} + \mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{tr} &= \mathbf{r} - \mathbf{t} \\ &= -\mathbf{i} + 2\mathbf{j} - 3\mathbf{i} + \mathbf{j} \\ &= -4\mathbf{i} + 3\mathbf{j} \end{aligned}$$

(ii) For Δrst to be right-angled at r: $\mathbf{rs} \perp \mathbf{tr}$:

$$\begin{aligned} \mathbf{rs} \perp \mathbf{tr} &\Leftrightarrow \mathbf{rs} \cdot \mathbf{tr} = 0, \\ \mathbf{rs} \cdot \mathbf{tr} &= (-3\mathbf{i} - 4\mathbf{j}) \cdot (-4\mathbf{i} + 3\mathbf{j}) = 12 - 12 = 0, \\ &\Rightarrow \Delta rst \text{ right-angled at } r \end{aligned}$$

(iii) Where $\theta := \angle rst$;

$$\begin{aligned} \mathbf{sr} \cdot \mathbf{st} &= |\mathbf{sr}||\mathbf{st}| \cos\theta \\ \Rightarrow \cos\theta &= \frac{\mathbf{sr} \cdot \mathbf{st}}{|\mathbf{sr}||\mathbf{st}|} \\ \underset{\mathbf{sr}=-\mathbf{rs}}{\Rightarrow} \cos\theta &= \frac{21+4}{5\sqrt{50}} \\ \Rightarrow \cos\theta = \frac{25}{5\sqrt{50}} = \frac{5}{\sqrt{50}} = \frac{\sqrt{25}}{\sqrt{50}} &= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \\ \Rightarrow \theta &= 45^\circ. \end{aligned}$$

1.3 Coordinate Geometry

1.3.1 Problem

The line B contains the points $(6, -2)$ and $(-4, 10)$.
The line A with equation $ax + 6y + 21 = 0$ is perpendicular to B.
Find the value of the real number a.
Solution: If (x_1, y_1) and (x_2, y_2) are two points on a line L, then the slope is given by:

$$m_L = \frac{y_2 - y_1}{x_2 - x_1}. \tag{3}$$

$$\Rightarrow m_B = \frac{10+2}{-4-6} = -\frac{6}{5}.$$

For two lines L and K;

$$L \perp K \Leftrightarrow m_L.m_K = -1. \tag{4}$$

When a line is written in the form:

$$L \equiv y = bx + c, \tag{5}$$

then $b = m_L$. Hence

$$\begin{aligned} B &\equiv ax + 6y + 21 = 0 \\ \Rightarrow B &\equiv y = -\frac{a}{6}x - \frac{21}{6}. \end{aligned}$$

$$\begin{aligned} &A \perp B \\ \Rightarrow -\frac{6}{5}\left(-\frac{a}{6}\right) &= -1 \\ \Rightarrow a &= -5. \end{aligned}$$

1.3.2 Problem

The equation of the line L is $14x + 6y + 1 = 0$.
Find the equation of the line perpendicular to L that contains the point $(3, -2)$.
Solution:

$$\begin{aligned} L &\equiv 14x + 6y + 1 = 0 \\ \Rightarrow L &\equiv y = -\frac{14}{6}x - \frac{1}{6} \end{aligned}$$

Hence $m_L = -7/3$. Let $K \perp L$ and $(3, -2) \in K$. $K \perp L \Rightarrow m_K = 3/7$. The equation of a line A containing a point (x_1, y_1) of slope m is given by:

$$y - y_1 = m(x - x_1). \tag{6}$$

$$\begin{aligned} \Rightarrow K \equiv y + 2 &= \frac{3}{7}(x-3) \\ \Rightarrow K \equiv y = \frac{3}{7}x - \frac{9}{7} - 2 &= \frac{3}{7} - \frac{9}{7} - \frac{14}{7} \\ \Rightarrow K \equiv y &= \frac{3}{7}x - \frac{23}{7}. \end{aligned}$$

1.3.3 Problem

Show that the line $6x - 8y - 71 = 0$ *contains the midpoint of* $[ab]$ *where* a *has coordinates* $(8, -6)$ *and* b *has coordinates* $(5, -2)$.
Solution: The mid-point of $[pq]$ where $p = (x_1, y_1)$, $q = (x_2, y_2)$ is given by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \tag{7}$$

Hence the mid-point of $[ab]$:

$$\left(\frac{13}{2}, -4\right).$$

$$\Rightarrow 6\left(\frac{13}{2}\right) - 8(-4) - 71 = 39 + 32 - 71 = 0,$$

$$\Rightarrow \left(\frac{13}{2}, -4\right) \in L \equiv 6x - 8y - 71 = 0.$$

1.3.4 Problem

Find the equation of the line pq *where* p *has coordinates* $(7, -6)$ *and* q *has coordinates* $(-3, 2)$.
Find the point of intersection of pq *and the line* $2x - 3y + 1 = 0$.
Determine the ratio in which the line $2x - 3y + 1 = 0$ *divides* $[pq]$.
Solution:

$$m_{pq} = \frac{2 + 6}{-3 - 7} = -\frac{4}{5}.$$

$$\Rightarrow pq \equiv y - 2 = -\frac{4}{5}(x + 3)$$

$$\Rightarrow pq \equiv 5y - 10 = -4x - 12$$

$$\Rightarrow pq \equiv 4x + 5y = -2.$$

To find the intersection between this line and the line $2x - 3y + 1 = 0$ is the solution of the simultaneous equations:

$$4x + 5y - 2 \quad (A)$$

$$2x - 3y = -1 \quad (B)$$

$$\underset{(B)}{\Rightarrow} 2x = 3y - 1$$

$$\underset{(A)}{\Rightarrow} 6y - 2 + 5y = -2$$

$$\Rightarrow 11y = 0 \Rightarrow y = 0$$

$$\Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$$

$$\text{point of intersection} = \left(-\frac{1}{2}, 0\right).$$

Suppose $2x - 3y + 1 = 0$ divides $[pq]$ in the ratio $m : n$ at $(-1/2, 0)$. Suppose $a = (x_1, y_1)$, $b = (x_2, y_2)$. Then the point that divides $[a, b]$ in the ratio $s : t$ is given by:

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right). \tag{8}$$

Thence

$$\left(-\frac{1}{2}, 0\right) \stackrel{!}{=} \left(\frac{m(-3)+n(y)}{m+n}, \frac{m(2)-6n}{m+n}\right)$$
$$\Rightarrow 2m - 6n \stackrel{!}{=} 0$$
$$\Rightarrow m = 3n$$
$$\Rightarrow \frac{m}{n} = m : n = \frac{3n}{n} = 3 : 1.$$

1.4 Trigonometry

1.4.1 Problem

Find the value of θ for which

$$\cos\theta = -\frac{\sqrt{3}}{2}, \quad 0^\circ \le \theta \le 180^\circ.$$

Solution: In the first instance $\cos 30^\circ = \sqrt{3}/2$. cos is negative in the second quadrant:

$$\cos(180^\circ - \theta) = \underbrace{\cos(180^\circ)}_{=-1}\cos(\theta) + \underbrace{\sin(180^\circ)}_{=0}\cos\theta$$
$$\Rightarrow \cos(180^\circ - \theta) = -\cos\theta$$
$$\Rightarrow \theta = 180^\circ - 30^\circ = 150^\circ.$$

1.4.2 Problem

If $\tan A = 1/2$*, find* $\tan 2A$ *without evaluating A, where A is an acute angle.*
Express $\tan B$ *in the form a/b, where a, $b \in \mathbb{N}$, given that*

$$\tan(2A + B) = \frac{63}{16}.$$

Solution: The double-angle formula for tan:

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \tag{9}$$

$$\tan 2A = \frac{2}{1 - \frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}.$$

The addition formula for tan:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{10}$$

$$\Rightarrow \tan(2A+B) = \frac{\tan 2A + \tan B}{1 - \tan 2A \tan B} \stackrel{!}{=} \frac{63}{16}$$
$$\Rightarrow \frac{63}{16} = \frac{\frac{4}{3} + \tan B}{1 - \frac{4}{3}\tan B}$$
$$\Rightarrow 16\left(\frac{4}{3} + \tan B\right) = 63\left(1 - \frac{4}{3}\tan B\right)$$
$$\Rightarrow 3(16)\left(\frac{4}{3} + \tan B\right) = 3(63)\left(1 - \frac{4}{3}\tan B\right)$$
$$\Rightarrow 64 + 48\tan B = 189 - 252\tan B$$
$$\Rightarrow 300\tan B = 125$$
$$\Rightarrow \tan B = \frac{125}{300} = \frac{5}{12}.$$

1.4.3 Problem

Express $\sin(135° - A)$ *in terms of* $\sin A$ *and* $\cos A$.
Express $\sin(135° - A)\cos(135° + A)$ *in the form* $k(1 + \sin pA)$, *where* $k, p \in \mathbb{R}$.
Find the values of A *for which*

$$\sin(135° - A)\cos(135° + A) = -\frac{3}{4}$$

where $0° \leq A \leq 180°$.
Solution: The subtraction formula for sin:

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \tag{11}$$

$$\sin(135° - A) = \sin 135° \cos A - \cos 135° \sin A$$
$$\Rightarrow \sin(135° - A) = \frac{1}{\sqrt{2}}\cos A + \frac{1}{\sqrt{2}}\sin A = \frac{1}{\sqrt{2}}(\cos A + \sin A)$$

The addition formula for cos:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \tag{12}$$

$$\cos(135° + A) = \cos 135° \cos A - \sin 135° \sin A$$
$$\Rightarrow \cos(135° + A) = -\frac{1}{\sqrt{2}}\cos A - \frac{1}{\sqrt{2}}\sin A = -\frac{1}{\sqrt{2}}(\sin A + \cos A)$$
$$\Rightarrow \sin(135° - A)\cos(135° + A) = -\frac{1}{2}(\cos A + \sin A)^2$$
$$\Rightarrow \sin(135° - A)\cos(135° + A) = -\frac{1}{2}(\underbrace{\cos^2 A + \sin^2 A}_{=1} + \underbrace{2\sin A\cos A}_{=\sin 2A})^2$$
$$\Rightarrow \sin(135° - A)\cos(135° + A) = -\frac{1}{2}(1 + \sin 2A)$$

$$\begin{aligned}
\sin(135^\circ - A)\cos(135^\circ + A) = -\frac{3}{4} &= -\frac{1}{2}(1 + \sin 2A) \\
\Rightarrow \frac{1}{2}(1 + \sin 2A) &= \frac{3}{4} \\
\Rightarrow 1 + \sin 2A &= \frac{3}{2} \\
\Rightarrow \sin 2A = \frac{1}{2} \Rightarrow 2A &= 30^\circ \\
\Rightarrow A &= 15^\circ
\end{aligned}$$

2 Accelerated Linear Motion

2.1 Problem

A lift decelerates from 3 m s^{-1} to rest during the last 6 m of its motion. Find the deceleration and the time taken.

2.1.1 Solution

For this motion

$$\begin{aligned}
s &= 6 \\
t &=? \\
u &= 3 \\
v &= 0 \\
a &=?
\end{aligned}$$

Using $v^2 = u^2 + 2as$;

$$\begin{aligned}
\Rightarrow a &= \frac{v^2 - u^2}{2s} \\
\Rightarrow a &= \frac{0 - 3^2}{12} \\
\Rightarrow a &= -\frac{3}{4} \text{ m/s}^2
\end{aligned}$$

Using

$$\begin{aligned}
t &= \frac{v - u}{a} \\
\Rightarrow t = \frac{0 - 3}{-3/4} = \frac{4}{4} \times \frac{3}{3/4} &= 4 \text{ s}
\end{aligned}$$

Ans: Deceleration $= 3/4$ m/s^2 and time taken $= 4$ s.

2.2 Problem

A train slows down from 70 m s^{-1} to 50 m s^{-1} over an eight-second time interval. Find the deceleration and the distance covered. If the train continues to decelerate at the same uniform rate, how much further will it travel before it comes to rest?

2.2.1 Solution

In the eight-second interval:

$$\begin{aligned} s &= ? \\ t &= 8 \\ u &= 70 \\ v &= 50 \\ a &= ? \end{aligned}$$

Using

$$\begin{aligned} a &= \frac{v-u}{t} \\ \Rightarrow a = \frac{50-70}{8} &= \frac{-20}{8} = -\frac{5}{2} \text{ m/s}^2 \end{aligned}$$

Using

$$\begin{aligned} s &= \left(\frac{u+v}{2}\right)t \\ \Rightarrow s &= \left(\frac{70+50}{2}\right)8 \\ \Rightarrow s &= 60 \times 8 = 480 \text{ m} \end{aligned}$$

Examining now the motion as the train decelerates from 70 m/s to rest:

$$\begin{aligned} s &= ? \\ t &= ? \\ u &= 70 \\ v &= 0 \\ a &= -\frac{5}{2} \end{aligned}$$

Hence using:

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow s &= \frac{v^2-u^2}{2a} \\ \Rightarrow s = \frac{0-4900}{2(-5/2)} &= \frac{4900}{5} = 980 \text{ m} \end{aligned}$$

However

$$\therefore \text{(distance travelled from 50 m/s to rest)} =$$
$$\text{(distance travelled from 70 m/s to rest)} - \text{(distance travelled from 70 m/s to 50 m/s)}$$
$$\Rightarrow \text{(distance travelled from 50 m/s to rest)} = 980 - 480 = 500 \text{ m}.$$

Ans: 500 m.

2.3 Problem

(a) Convert 72 km/hour to metres per second

(b) A train decelerates from 72 km/hour to 48 km/hour over a distance of 1/2 *km. Find in metres/second*^{2} *the deceleration, and the time taken. If the train continues to decelerate at this rate find out how much further it will travel before it comes to rest.*

2.3.1 Solution

(a)

$$72\,\frac{\text{km}}{\text{hr}}$$
$$\Rightarrow 72\,\frac{\text{km}}{\text{hr}} = 72\,\frac{1000\text{ m}}{60\,\text{mins}} = 72\,\frac{1000\text{ m}}{60(60\text{ s})}$$
$$\Rightarrow 72\,\frac{\text{km}}{\text{hr}} = \frac{72(1000)}{3600} = 20\text{ m/s}$$

(b) In the first instance units must be converted to SI units:

$$72\text{ km/hr} = 20\text{ m/s}$$
$$48\text{ km/hr} = \frac{2}{3}72\text{ km/hr} = \frac{40}{3}\text{ m/s}$$
$$\frac{1}{2}\text{ km} = 500\text{ m}$$

Hence over the 500 m:

$$s = 500$$
$$t = ?$$
$$u = 20$$
$$v = \frac{40}{3}$$
$$a = ?$$

Hence using $v^2 = u^2 + 2as$;

$$a = \frac{v^2 - u^2}{2s}$$
$$\Rightarrow a = \frac{(40^2/9) - 400}{1000}$$
$$\Rightarrow a = \frac{9}{9} \times \frac{(40^2/9) - 400}{1000} = \frac{40^2 - 3600}{9000} = \frac{-2000}{9000}$$
$$\Rightarrow a = -\frac{2}{9} \text{ m/s}^2$$

Using

$$t = \frac{v - u}{a}$$
$$\Rightarrow t = \frac{(40/3) - 20}{-(2/9)} = \frac{9}{9} \times \frac{(40/3) - 20}{-(2/9)} = \frac{120 - 180}{-2} = 30 \text{ s}$$

Examining now the motion as the train decelerates from 72 km/hr to rest:

$$s = ?$$
$$t = ?$$
$$u = 20$$
$$v = 0$$
$$a = -\frac{2}{9}$$

Using $v^2 = u^2 + 2as$;

$$s = \frac{v^2 - u^2}{2a}$$
$$\Rightarrow s = \frac{0 - 400}{-(4/9)} = \frac{9}{9} \times \frac{400}{(4/9)} = \frac{3600}{4} = 900 \text{ m}$$

However

$$\therefore \text{(distance travelled from 48 km/hr to rest)}$$
$$= \text{(distance travelled from 72 km/hr to rest)} - \text{(distance travelled from 72 km/hr to 48 km/hr)}$$
$$\Rightarrow \text{(distance travelled from 48 km/hr to rest)} = 900 - 500 = 400 \text{ m.}$$

Ans: 500 m.

2.4 Problem: LC OL 1984

Define velocity and speed.
Show that a speed of 1 km/hour is equivalent to 5/18 m/s. The speed of a car is reduced from 72 km/hour to 54 km/hour over a distance of 35 m. Find the retardation, assuming it is uniform throughout. If the retardation continues, how much farther will the car travel before coming to rest?

2.4.1 Solution

Velocity is speed in a given direction
Speed is the rate of change of distance with respect to time

$$1 \text{ km/hr} = 1\frac{1000 \text{ m}}{60 \text{ mins}} = 1\frac{1000 \text{ m}}{60(60) \text{ s}}$$
$$\Rightarrow 1 \text{ km/hr} = \frac{1000}{3600} \text{ m/s} = \frac{5}{18} \text{ m/s}.$$

First convert to SI units.

$$72 \text{ km/hr} = 72\left(\frac{5}{18} \text{ m/s}\right) = 20 \text{ m/s}$$
$$\Rightarrow 54 \text{ km/hr} = \frac{3}{4}(72 \text{ km/hr}) = 15 \text{ m/s}$$

Examining the motion over the 35 m:

$$s = 35$$
$$t = ?$$
$$u = 20$$
$$v = 15$$
$$a = ?$$

Using $v^2 = u^2 + 2as$;

$$a = \frac{v^2 - u^2}{2s} = \frac{225 - 400}{70} = \frac{-175}{70} = -\frac{5}{2} \text{ m/s}^2.$$

Examining the motion from 15 m/s to rest:

$$s = ?$$
$$t = ?$$
$$u = 15$$
$$v = 0$$
$$a = -5/2$$

Using $v^2 = u^2 + 2as$;

$$s = \frac{v^2 - u^2}{2a}$$
$$\Rightarrow s = \frac{0 - 225}{-5} = \frac{225}{5} = 45 \text{ m}.$$

2.5 Problem: LC OL 1983

A car starts from rest with a uniform acceleration and reaches a velocity of 27 m/s in 9 s. The brakes are then applied and it comes to rest with uniform deceleration after travelling a further 54 m. Calculate:

(i) the uniform acceleration

(ii) the uniform deceleration

(iii) the average speed of the car for the journey

(iv) the two times that the velocity of the car will be 15 m/s

2.5.1 Solution

(i) Using

$$a = \frac{v-u}{t}$$
$$\Rightarrow a = \frac{27-0}{9} = 3 \text{ m/s}^2$$

(ii) In the decelerating part of the motion, $u = 27$, $v = 0$. Using

$$v^2 = u^2 + 2as$$
$$\Rightarrow a = \frac{v^2 - u^2}{2s}$$
$$\underset{27^2=729}{\Rightarrow} a = \frac{0-729}{108} = -6.75 \text{ m/s}^2$$

(iii) The average speed is given by:

$$\bar{v} = \frac{\text{distance travelled}}{\text{time taken}} \tag{13}$$

The motion is acceleration followed immediately by deceleration hence

$$d : a = t_1 : t_2 \tag{14}$$

where t_1 is the time accelerating, t_2 the time decelerating.

$$\Rightarrow \frac{d}{a} = \frac{t_1}{t_2}$$
$$\Rightarrow t_2 = t_1 \frac{a}{d} = 9\frac{3}{6.75} = 4 \text{ s}$$

Therefore the time taken is 9+4=13 s. The distance travelled is the area under the graph. The graph is a triangle of height 27 and base 13:

$$\text{total distance} = \frac{1}{2}(13)(27) = 175.5 \text{ m}$$
$$\Rightarrow \bar{v} = \frac{175.5}{13} = 13.5 \text{ m/s}$$

(iv) Clearly the velocity will be 15 m/s once when $t < 9$ and once when $9 < t < 13$. To find the first time, using

$$t = \frac{v-u}{a} = \frac{15}{3} = 5 \text{ s}$$

To find the second time, consider the decelerating part of the motion. $u = 27$, $v = 15$, $a = -6.75$:

$$t = \frac{v-u}{a} = \frac{15-27}{6.75} = 1\frac{7}{9} \text{ s}$$

But this represents the time *after* $t = 9$ s. Hence:
Ans: 5 s and $10\frac{7}{9}$ s.

2.6 Problem: LC OL 1982

Consider three points on a line, p, q and r, along the line in that order. A car travelling towards p at a steady speed of 15 m/s, accelerated at a constant rate between p and q. At q its speed was 25 m/s. This speed was maintained as far as r.
If $|pr| = 980$ *m and the time from p to q was 40 seconds, draw a time-velocity graph of the motion and hence, or otherwise, calculate the acceleration.*

2.6.1 Solution

To draw the time-velocity graph note it has the rough shape of Fig 1.

By the theorem, the area under the graph must equal to the distance travelled. Let acceleration from 15 m/s to 25 m/s take T s. Using, for the time $t = 0$ to $t = T$

$$s = \left(\frac{u+v}{2}\right)t = \left(\frac{15+25}{2}\right)T = 20T$$

$$\begin{aligned} \therefore 980 &= 20T + 25(40 - T) \\ \Rightarrow 980 &= 20T + 1000 - 25T \\ \Rightarrow 5T &= 20 \\ \Rightarrow T &= 4 \text{ s} \end{aligned}$$

Therefore the time-velocity graph is Fig 2.

Using

$$\begin{aligned} a &= \frac{v-u}{t} \\ \Rightarrow a &= \frac{25-15}{4} = \frac{5}{2} \text{ m/s}^2. \end{aligned}$$

2.7 Problem: LC OL 1981

Define uniform acceleration in a straight line. A particle starts from rest with uniform acceleration 2 m/s^2. After how many seconds will its speed be 30 km/hr?
How far from its starting point will the particle be when its speed is 60 km/hr? The particle is then brought to rest in 2 m. Calculate the deceleration.

2.7.1 Solution

Uniform acceleration in a straight line is motion in a single direction with constant acceleration; acceleration is the rate of change of acceleration with respect to time.

$$30\frac{\text{km}}{\text{hr}} = 30\frac{1000}{3600}\text{ m/s} = \frac{25}{3}\text{ m/s}$$

Using

$$t = \frac{v-u}{a} = \frac{25/3}{2} = \frac{25}{6}\text{ s}$$

$$60\text{ km/hr} = \frac{50}{3}\text{ m/s}$$

Using

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow s &= \frac{v^2-u^2}{2a} \\ \Rightarrow s = \frac{(50/3)^2}{4} = \frac{2500/9}{4} &= \frac{625}{9} \\ \Rightarrow s &= \frac{625}{9}\text{ m} \end{aligned}$$

Again using

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow a &= \frac{v^2-u^2}{2s} \\ \underset{s=2,v=0}{\Rightarrow} a = \frac{-2500/9}{4} &= -\frac{625}{9}\text{ m/s}^2 \\ \Rightarrow d &= \frac{625}{9}\text{ m/s}^2 \end{aligned}$$

2.8 Problem: LC OL 1980

p and q are points 162 m apart. A body leaves p with initial speed 5 m/s and travels toward q with uniform acceleration 3 m/s^2. At the same instant another body leaves q and travels towards p with initial speed 7 m/s and uniform acceleration 2 m/s^2. After how many seconds do they meet and what, then, is the speed of each body?

2.8.1 Solution

The particles meet after T s when the sum of the distance travelled by the particle at p-particle and the distance travelled by the q-particle is 162 m:

$$\text{particles meet } \Leftrightarrow s_p + s_q = 162 \tag{15}$$

Using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \Rightarrow s_p &= 5T + \frac{3}{2}T^2 \\ \underset{\frac{1}{2}2=1}{\Rightarrow} s_q &= 7T + T^2 \end{aligned}$$

Hence solve for[1] T

$$\begin{aligned} s_p + s_q &= 162 \\ \Rightarrow 12T + \frac{5}{2}T^2 &= 162 \\ \Rightarrow 24T + 5T^2 &= 324 \\ \Rightarrow 5T^2 + 24T - 324 &= 0 \\ \Rightarrow 5T^2 + 54T - 30T - 324 &= 0 \\ \Rightarrow T(5T + 54) - 6(5T + 54) &= 0 \\ \Rightarrow (T - 6)\underbrace{(5T + 54)}_{\text{see footnote}} &= 0 \\ \Rightarrow T &= 6 \text{ s} \end{aligned}$$

The velocities of the p and q-particles after t s, using:

$$\begin{aligned} v &= u + at \\ \Rightarrow v_p = 5 + 3t &= 23 \text{ m/s at } t = 6 \\ \Rightarrow v_q = 7 + 2t &= 19 \text{ m/s at } t = 6 \end{aligned}$$

[1] ignore $t < 0$; refers to time when distance between them was 324 m

2.9 Problem: LC HL 2009

- *A particle is projected vertically upwards from a point p. At the same instant a second particle is let fall vertically from a point q directly above point p. The particles meet at a point r between them after 2s*
  The particles have equal speeds when they meet at r
  Prove that $|pr| = 3|rq|$
- *A train accelerates uniformly from rest to a speed v m/s with uniform acceleration f m/s^2.*
  It then declerates uniformly to rest with uniform retardation $2f$ m/s^2.
  The total distance travelled is d metres.
  - *Draw a speed-time graph for the motion of the train*
  - *If the average speed for the whole journey is $\sqrt{d/3}$, find the value of f.*

2.9.1 Solution

- Let v_1 be the speed of the particle projected from p and v_2 the speed of the particle dropped from q. After 2 s, using

$$\begin{aligned} v &= u + at \\ v_1 &= u - 2g \\ v_2 &= 2g \\ \Rightarrow u - 2g &= 2g \\ \Rightarrow u &= 4g \end{aligned}$$

  Let s_1 and s_2 be the distance travelled by the particles. After 2s, using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ s_1 &= 4g(2) + \frac{1}{2}(-g)(4) \\ \Rightarrow s_1 &= 6g \text{ , and} \\ s_2 &= (0)(2) + \frac{1}{2}g(4) = 2g \end{aligned}$$

  But after 2 s, $s_1 = |pr|$ and $s_2 = |rq|$. Hence as $s_1 = 3s_2$; $|pr| = 3|rq|$. □

-
  - See Fig 1.
  - The average speed is given by:

$$\bar{v} = \frac{\text{total distance}}{\text{total time}} = \frac{d}{T} \tag{16}$$

    where $T = t_1 + t_2$ is the total time. Hence

$$\frac{d}{T} = \sqrt{\frac{d}{3}}$$
$$\Rightarrow T = d\sqrt{\frac{3}{d}} = \sqrt{3d} \tag{17}$$

Also the total distance is equal to the area under the time-velocity curve, in this case the triangle of width T and height v:

$$d = \frac{1}{2}vT \tag{18}$$

Since the motion is acceleration from rest immediately followed by declaration to rest:

$$d : a = t_1 : t_2 \tag{19}$$

Hence

$$\begin{aligned} 2f : f &= t_1 : t_2 \\ \Rightarrow t_1 : t_2 &= 2 : 1 \\ \Rightarrow t_1 : t_2 &= \frac{2}{3} : \frac{1}{3} \\ \Rightarrow t_1 &= \frac{2}{3}T \end{aligned}$$

Using

$$\begin{aligned} v &= u + at \\ v &= 0 + ft_1 \\ \Rightarrow v &= \frac{2}{3}fT \end{aligned}$$

Hence using this and (17) in (18):

$$\begin{aligned} d &= \frac{1}{2}vT \\ \Rightarrow d &= \frac{1}{2}\frac{2}{3}f\sqrt{3d}.\sqrt{3d} \\ \Rightarrow d &= \frac{1}{3}f3d = fd \\ \Rightarrow f &= 1 \text{ m/s}^2 \end{aligned}$$

2.10 Problem: LC HL 2008

- *A ball is thrown vertically upwards with an initial velocity of 39.2 m/s. Find*
  - *the time taken to reach the maximum height*
  - *the distance travelled in 5 s*

- *Two particles P and Q, each having constant acceleration, are moving in the same direction along parallel lines. When P passes Q the speeds are 23 m/s and 5.5 m/s, respectively. Two minutes later Q passes P, and Q is then moving at 65.5 m/s. Find*
  - *the acceleration of P and the acceleration of Q*
  - *the speed of P when Q overtakes it*
  - *the distance P is ahead of Q when they are moving with equal speeds*

2.10.1 Solution

- – The ball reaches the maximum when $v = 0$. Using

$$v = u + at$$
$$v = 39.2 - gt$$
$$\Rightarrow t_{s=\max} = \frac{39.2}{g} = 4 \text{ s}$$

  – After 5 s, using

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s = 39.2(5) - \frac{1}{2}g(25)$$
$$\Rightarrow s = 4g(5) - \frac{25}{2}g = 20g - 12.5g = \frac{15}{2}g$$

- – Consider the motion of Q.

$$t = 120 \ \ (120 \text{ s} = 2 \text{ min})$$
$$v_Q = 65.5$$
$$u_Q = 5.5$$
$$a_Q = ?$$
$$s_Q = ?$$

  Using

$$a = \frac{v - u}{t}$$
$$a_Q = \frac{65.5 - 5.5}{120} = \frac{1}{2} \text{ m/s}^2$$

After 120 s, $s_P \stackrel{!}{=} s_Q$. Using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ s_P &= 23(120) + \frac{1}{2}a_P(120)^2 \\ s_Q &= 5.5(120) + \frac{1}{4}(120)^2 \\ &\Rightarrow s_P = s_Q \\ \Rightarrow \frac{1}{2}a_P(120)^2 &= 5.5(120) - 23(120) + \frac{1}{4}(120)^2 \\ \Rightarrow 240a_P &= 22 - 92 + 120 = 50 \\ \Rightarrow a_P &= \frac{5}{25} \text{ m/s}^2 \end{aligned}$$

– Using

$$\begin{aligned} v &= ua + at \\ \Rightarrow v_P &= 23 + \frac{5}{24}(120) = 48 \text{ m/s} \end{aligned}$$

– After t s, using

$$\begin{aligned} v &= u + at \\ v_P &= 23 + \frac{5}{24}t \\ v_Q &= 5.5 + \frac{1}{2}t \end{aligned}$$

For what t is $v_P = v_Q$?

$$\begin{aligned} 23 + \frac{5}{24}t &= 5.5 + \frac{1}{2}t \\ \Rightarrow \frac{7}{24}t &= 17.5 \\ \Rightarrow t = \frac{(24)(17.5)}{7} &= 60 \text{ s} \end{aligned}$$

Hence look at s_P and s_Q after 60 s, using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ s_P &= 23(60) + \frac{1}{2}\frac{5}{24}(60^2) = 1755 \\ s_Q &= 5.5(60) + \frac{1}{4}(60)^2 = 1230 \end{aligned}$$

Hence after 60 s, $s_P > s_Q$ by 525 m. That is P is 525 m ahead of Q when they are moving with equal speed.

2.11 Problem: LC HL 2007

- *A particle is projected vertically downwards from the top of a tower with speed u m/s. It takes the particle 4 s to reach the bottom of the tower. During the third second of its motion the particle travels 29.9 m. Find*
  - *the value of u*
  - *the height of the tower*
- *A train accelerates uniformly from rest with a speed v m/s. It continues at this speed for a period of time and then decelerates uniformly to rest. In travelling a total distance d metres the train accelerates through a distance pd metres and decelerates through a distance qd metres, where $p < 1$ and $q < 1$.*
  - *Draw a speed-time graph for the motion of the train*
  - *If the average speed of the train for the whole journey is*

$$\frac{v}{p+q+b},$$

find the value of b.

2.11.1 Solution

- – The particle moves under acceleration $a = g$. After 2 s and 3 s, using

$$v = u + at$$
$$v(2) = u + 2g$$
$$v(3) = u + 3g$$

From $t = 2$ to $t = 3$, the particle travels 29.9 m where $u = v(2)$ and $v = v(3)$; using

$$s = \left(\frac{u+v}{2}\right)t$$
$$\Rightarrow 29.9 = \left(\frac{2u+5g}{2}\right)$$
$$\Rightarrow 59.8 = 2u + 5g$$
$$\Rightarrow u = \frac{59.8 - 5g}{2} = 5.4 \text{ m/s}$$

– The height of the tower is s after 4 s. Using

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s = (5.4)4 + \frac{1}{2}g(16)$$
$$\Rightarrow s = 21.6 + 8g = 100 \text{ m}$$

- – See figure 2
  – Let T be the total time taken for the journey. Where d is the total distance travelled, the average speed $\bar{v}$ is given by:

$$\bar{v} = \frac{d}{T} \tag{20}$$

$$\begin{aligned} \frac{v}{p+q+b} &= \frac{d}{T} \\ \Rightarrow d &= T\left(\frac{v}{p+q+b}\right) \\ \Rightarrow pd + qd + bd &= Tv \\ \Rightarrow bd &= Tv - pd - qd \\ \Rightarrow b &= \frac{Tv - pd - qd}{d} \end{aligned} \tag{21}$$

Let t_1 be the time spent accelerating and t_2 the time spent decelerating. Examining the time-velocity graph, the distance travelled in t_1, the area of the triangle with perpendicular height v and base t_1 is:

$$\begin{aligned} \frac{1}{2}t_1 v &= pd \\ \Rightarrow t_1 &= \frac{2pd}{v} \end{aligned}$$

Similarly

$$t_2 = \frac{2qd}{v}$$

Let t_c be the time spent at constant speed. From the time-velocity graph, the distance travelled at constant speed is given by,

$$d - pd - qd = d(1-p-q)$$

Hence the distance travelled in t_c is the area under the curve:

$$\begin{aligned} vt_c &= d(1-p-q) \\ \Rightarrow t_c &= \frac{d(1-p-q)}{v} \end{aligned}$$

Now $T = t_1 + t_2 + t_c$;

$$\begin{aligned} T &= \frac{1}{v}(2pd + 2qd + d - pd - qd) \\ \Rightarrow T &= \frac{1}{v}(pd + qd + d) = \frac{d}{v}(p+q+1) \end{aligned}$$

Substituting into (21):

$$b = \frac{d(p+q+1) - pd - qd}{d} = p + q + 1 - p - q = 1.$$

2.12 Problem: LC HL 2006

- *A lift starts from rest. For the first part of its descent is travels with uniform acceleration f. It then travels with uniform retardation $3f$ and comes to rest. The total distance travelled is d and the total time taken is t.*
  - *Draw a speed-time graph for the motion*
  - *Find d in terms of f and t*
- *Two trains P and Q, each of length 79.5 m, moving in opposite directions along parallel lines, meet at o, when their speeds are 15 m/s and 10 m/s respectively.*
  The acceleration of P is 0.3 m/s^2 and the acceleration of Q is 0.2 m/s^2. It takes the trains t seconds to pass each other.
  - *Find the distance travelled by each train in t seconds.*
  - *Hence, or otherwise, calculate the value of t*
  - *How long does it take for 2/5 of the length of train Q to pass the point o?*

2.12.1 Solution

- – See Figure 3
  - Since the motion is uniform acceleration from rest followed immediately by uniform deceleration from rest:

$$t_1 : t_2 = 3f : f$$
$$\Rightarrow t_1 : t_2 = \frac{3}{4} : \frac{1}{4}$$
$$\Rightarrow t_1 = \frac{3}{4}t$$

Using

$$v = u + at$$
$$\Rightarrow v = ft_1$$
$$\Rightarrow v = f\left(\frac{3}{4}t\right) = \frac{3}{4}ft$$

The total distance d is the area under the graph:

$$d = \frac{1}{2}vt$$
$$\Rightarrow d = \frac{1}{2}\left(\frac{3}{4}ft\right)t$$
$$\Rightarrow d = \frac{3}{8}ft^2$$

- – Using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \Rightarrow s_P &= 15t + \frac{1}{2}\frac{3}{10}t^2 = 15t + \frac{3}{20}t^2 \\ \Rightarrow s_Q &= 10t + \frac{1}{2}\frac{1}{5}t^2 = 10t + \frac{1}{10}t^2 \end{aligned}$$

  – The trains pass each other when the distance they travel adds up to twice their length: 159 m;

$$\begin{aligned} 159 &= 15t + \frac{3}{20}t^2 + 10t + \frac{1}{10}t^2 \\ \Rightarrow 300t + 3t^2 + 200t + 2t^2 &= 3180 \\ \Rightarrow 5t^2 + 500t - 3180 &= 0 \\ \Rightarrow t^2 + 100t - 636 &= 0 \\ \Rightarrow t^2 + 106t - 6t - 636 &= 0 \\ \Rightarrow t(t + 106) - 6(t + 106) &= 0 \\ \Rightarrow (t + 106)(t - 6) &= 0 \\ \Rightarrow t &= 6 \text{ s} \end{aligned}$$

  The case $t = -106$ does not concern us.

  – In this case t needs to be found such that:

$$\begin{aligned} s_Q &= \frac{2}{5}79.5 = 31.8 \text{ m} \\ \Rightarrow 31.8 &= 10t + \frac{1}{10}t^2 \\ \Rightarrow 318 &= 100t + t^2 \\ \Rightarrow t^2 + 100t - 318 &= 0 \end{aligned}$$

  Using

$$\text{roots of } ax^2 + bx + c \text{ are } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{22}$$

$$\begin{aligned} t &= \frac{-100 \pm \sqrt{100^2 - 4(1)(-318)}}{2} \\ \Rightarrow t &= \frac{-100 \pm \sqrt{11272}}{2} \\ \Rightarrow t &= 3.085 \text{ or } -103.085 \end{aligned}$$

  Ignore $t < 0$.
  Ans: $t = 3.085$ s.

2.13 Problem: LC HL 2005 [Part (a)]

Car A and car B travel in the same direction along a horizontal straight road.
Each car is travelling at a uniform speed of 20 m/s.
Car A is at a distance d metres in front of car B.
At a certain instant car A starts to brake with a constant retardation of 6 m/s^2.
0.5 s later car B starts to brake with a constant retardation of 3 m/s^2.

Find

(i) *the distance travelled by car A before it comes to rest*

(ii) the minimum value of d for car B not to collide with car A

2.13.1 Solution

(i) With respect to car A, when at rest:

$$u = 20$$
$$v = 0$$
$$a = -6$$
$$s = ?$$

Using

$$v^2 = u^2 + 2as$$
$$\Rightarrow s = \frac{v^2 - u^2}{2a}$$
$$\Rightarrow s = \frac{-400}{-12} = \frac{100}{3} \text{ m}$$

(ii) Car A has a greater deceleration than car B and also begins it deceleration before car B; therefore car A comes to rest before car B does. Therefore when B stops it must have travelled $d + 100/3$ m to *just* avoid a collision. Now B travels at 20 m/s for half a second before decelerating. With respect to B decelerating, when at rest:

$$u = 20$$
$$v = 0$$
$$a = -3$$
$$t = ?$$

Using

$$t = \frac{v - u}{a}$$
$$\Rightarrow t = \frac{0 - 20}{-3} = \frac{20}{3} \text{ s.}$$

Now before decelerating B travels at constant speed for a half second. In this half second it travels:

$$s = \frac{1}{2}(20) = 10 \text{ m.}$$

Hence when stopped car B has travelled:

$$s = 10 + \left(20\left(\frac{20}{3}\right) - \frac{3}{2}\left(\frac{20}{3}\right)^2\right)$$
$$\Rightarrow s = \frac{230}{3}$$

and this must equal $d + 100/3$:

$$\frac{230}{3} = d + \frac{100}{3}$$
$$\Rightarrow d = \frac{130}{3} \text{ m}$$

2.14 Problem: LC HL 2004 [Part (a)]

A ball is thrown vertically upwards with an initial velocity of 20 m/s. One second later, another ball is thrown vertically upwards from the same point with an initial velocity of u m/s.
The balls collide after a further 2 seconds.

(i) Show that $u = 17.75$.

(ii) Find the distance *travelled by each ball before the collision, giving your answers correct to the nearest metre.*

2.14.1 Solution

(i) Let $s_1(t)$ be the height of the first particle and $s_2(t)$ be the height of the second particle. For the particles to collide after 3 s:

$$s_1(3) \stackrel{!}{=} s_2(3). \tag{23}$$

$$s_1(3) = 3(20) - \frac{1}{2}g(3^2)$$
$$\Rightarrow s_1(3) = 60 - \frac{9}{2}g$$

Particle 2 is motionless for one of these seconds and thus

$$s_2(3) = 2u - \frac{1}{2}g(4) = 2u - 2g$$
$$\Rightarrow 2u - 2g \stackrel{!}{=} 60 - \frac{9}{2}g$$
$$\Rightarrow 2u = 60 - \frac{5}{2}g$$
$$\Rightarrow u = \frac{60 - 5g/2}{2} = 17.75 \text{ m/s}.$$

(ii) Take the distance to mean total distance in the sense that if a particle travels up, stops, then falls down the total distance is the distance travelled going up *plus* the distance travelled going down. For the first particle the motion is upwards until $v = 0$. That is until, using

$$v = u + at$$
$$\Rightarrow 0 = 20 - gt$$
$$\Rightarrow t = \frac{20}{g} \simeq 2.041 \text{ s}$$

Hence the distance travelled *up* is given by, using:

$$s = \left(\frac{u+v}{2}\right)t$$
$$\Rightarrow s = \left(\frac{20+0}{2}\right)\frac{20}{g} = \frac{200}{g} \simeq 20.408 \text{ m}$$

The distance travelled on the way down is, using:

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s = \frac{1}{2}g\left(3 - \frac{20}{g}\right)^2$$
$$\Rightarrow s \simeq 4.9(0.920) = 4.508 \text{ m.}$$

Hence to the nearest metre the first particle travels 25 m.

For the second particle the motion is up until $v = 0$; using

$$v = u + at$$
$$\underset{17.75=71/4}{\Rightarrow} 0 = \frac{71}{4} - gt$$
$$\Rightarrow t = \frac{71}{4g} \simeq 1.811 \text{ s}$$

Hence the distance travelled *up* is given by, using:

$$s = \left(\frac{u+v}{2}\right)t$$
$$\Rightarrow s = \left(\frac{71}{2(4)}\right)\frac{71}{4g} = \simeq 16.075 \text{ m}$$

The distance travelled on the way down is, using:

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s = \frac{1}{2}g\left(2 - \frac{71}{4g}\right)^2$$
$$\Rightarrow s \simeq 4.9(0.0356) = 0.175 \text{ m.}$$

Hence to the nearest metre the first particle travels 16 m.

2.15 Problem: LC HL 2003

(a) *The points p, q and r all lie on a straight line.*
A train passes point p with speed u m/s. The train is travelling with uniform retardation f m/s^2. The train takes 10 s to travel from p to q and 15 s to travel from q to r, where $|pq| = |qr| = 125$ m.

(i) *Show that $f = 1/3$*

(ii) *The train comes to rest s metres after passing r.*
Find s, giving your answer correct to the nearest metre.

(b) *A man runs at constant speed to catch a bus.*
At the instant the man is 40 m away from the bus, it begins to accelerate uniformly from rest away from him.
The man just catches the bus 20 s later.

(i) *Find the constant speed of the man*

(ii) *If the constant speed of the man had instead been 3 m/s, show that the closest he gets to the bus is 17.5 m*

2.15.1 Solution

(a) (i) Considering the motion from p to q, using:

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow 125 = 10u - \frac{1}{2}f(100)$$
$$\Rightarrow 125 = 10u - 50f$$
$$\Rightarrow 25 = 2u - 10f \tag{24}$$

Now considering the motion from p to r:

$$250 = 25u - \frac{1}{2}f(625)$$
$$\Rightarrow 10 = u - \frac{25}{2}f$$
$$\Rightarrow u = 10 + \frac{25}{2}f \tag{25}$$

Plugging into (24):

$$25 = 20 + 25f - 10f$$
$$\Rightarrow 15 = 5f$$
$$\Rightarrow f = \frac{1}{3}$$

□

(ii) From (25),

$$u = 10 + \frac{25}{6} = \frac{85}{6}.$$

From p, the particle comes to rest when $v = 0$, using:

$$\begin{aligned} v^2 &= u^2 + 2as \\ \Rightarrow s &= \frac{v^2 - u^2}{2a} \\ \Rightarrow s = \frac{0^2 - (85/6)^2}{(-2/3)} &\simeq 301.042 \text{ m}. \end{aligned}$$

The particle travels 250 m from p to r hence travels $s = 51$ m after passing r before coming to rest.

(b) (i) If the man *just* catches the bus then when his constant speed u is equal to that of the bus v_b and he has travelled as far as the bus has *plus* the 40 m between them. That is if s_m is the distance travelled by the man and s_b the distance travelled by the bus the condition to *just* catch the bus is

$$u = v_b \text{ when } s_m = s_b + 40. \tag{26}$$

Let a be the acceleration of the bus. After 20 s, using:

$$\begin{aligned} v &= u + at \\ \Rightarrow v_b &= 20a \\ \underset{v_b \overset{!}{=} u}{\Rightarrow} a &= \frac{u}{20} \end{aligned}$$

Using, after 20 s,

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \Rightarrow s_m &= 20u \\ \Rightarrow s_b = \frac{1}{2}a(400) = 200a &= 10u \\ \underset{s_m \overset{!}{=} s_b + 40}{\Rightarrow} \quad 20u &= 10u + 40 \\ \Rightarrow u &= 4 \text{ m/s} \end{aligned}$$

(ii) If $u = 3$ m/s, the distance travelled by the man after t s is given by:

$$s_m = 3t$$

As $u = 4$ above, $a = 1/5$ m/s^2. With respect to the man, after t s, the bus has travelled a distance $s_b + 40$ away;

$$s_b + 40 = \frac{1}{2}\frac{1}{5}t^2 + 40 = \frac{t^2}{10} + 40.$$

Hence in terms of t, the distance between the man and bus is given by the distance travelled by the bus less the distance travelled by the man:

$$\frac{t^2}{10} + 40 - 3t. \tag{27}$$

To minimise this function differentiate with respect to t and solve equal to 0:

$$\frac{t}{5} - 3 = 0$$
$$\Rightarrow t_{\text{min}} = 15 \text{ s}$$

To show this is a min note the second derivative is

$$\frac{1}{5} > 0 \Rightarrow t = 15 \text{ a local minimum.}$$

Hence the minimum separation is:

$$\underset{t=15}{\Rightarrow} \frac{225}{10} + 40 - 45 = 17.5 \text{ m.}$$

□

2.16 Problem: LC HL 2002

(a) A stone is thrown vertically upwards under gravity with a speed of u m/s from a point 30 m above the horizontal ground.
The stone hits the ground 5 s later.

(i) Find the value of u

(ii) Find the speed with which the stone hits the ground.

(b) A particle, with initial speed u, moves in a straight line with constant acceleration.
During the time interval from 0 to t, the particle travels a distance p.
During the time interval from t to $2t$, the particle travels a distance q.
During the time interval from $2t$ to $3t$, the particle travels a distance r.

(i) Show that $2q = p + r$

(ii) Show that the particle travels a further distance $2r - q$ in the time interval from $3t$ to $4t$.

2.16.1 Solution

(a) (i) With respect to the point the stone was thrown with $s = -30$ after $t = 5$ s. The stone is under acceleration $-g$. Using

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow -30 = 5u - \frac{1}{2}g(25)$$
$$\Rightarrow 5u = \frac{25}{2}g - 30$$
$$\Rightarrow u = \frac{5}{2}g - 6 = 18.5 \text{ m/s}$$

(ii) Using

$$\begin{aligned} v &= u + at \\ \Rightarrow v &= 18.5 - g(5) \\ \Rightarrow v &= -30.5 \text{ m/s} \end{aligned}$$

Hence the stone hits the ground with *speed* $|v| = 30.5$ m/s.

(b) Considering the motion in the first t seconds, using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \Rightarrow p &= ut + \frac{1}{2}at^2 \qquad (28) \end{aligned}$$

In the first $2t$ seconds:

$$p + q = 2ut + \frac{4}{2}at^2 \qquad (29)$$

In the first $3t$ seconds:

$$p + q + r = 3ut + \frac{9}{2}at^2 \qquad (30)$$

Now $r = (30) - (29)$:

$$\begin{aligned} r &= 3ut + \frac{9}{2}at^2 - 2ut - \frac{4}{2}at^2 \\ \Rightarrow r &= ut + \frac{5}{2}at^2 \\ \underset{p=(28)}{\Rightarrow} p + r &= 2ut + 3at^2 \end{aligned}$$

Now $q = (29) - (28)$:

$$\begin{aligned} q &= 2ut + \frac{4}{2}at^2 - ut - \frac{1}{2}at^2 \\ \Rightarrow q &= ut + \frac{3}{2}at^2 \\ \Rightarrow 2q &= 2ut + 3at^2 = p + r. \end{aligned}$$

□

(ii) Suppose the particle travels a distance s further in the next second. Hence in the first $4t$ seconds:

$$\begin{aligned} p + q + r + s &= 4ut + \frac{16}{2}at^2 \\ \underset{p+q+r=(30)}{\Rightarrow} s &= ut + \frac{7}{2}at^2 \end{aligned}$$

Now

$$\begin{aligned} 2r - q &= 2ut + \frac{10}{2}at^2 - ut - \frac{3}{2}at^2 \\ \Rightarrow 2r - q &= ut + \frac{7}{2}at^2 = s \end{aligned}$$

□

2.17 Problem: LC HL 2001

(a) *Points p and q lie in a straight line, where* $|pq| = 1200$ *m.*
Starting from rest at p, a train accelerates at 1 m/s^2 *until it reaches the speed limit of 20 m/s. It continues at this speed of 20 m/s and then decelerates at 2 m/s*^{2}*, coming to rest at q.*

Find the time it takes the train to go from p to q.

Find the shortest time it takes the train to from rest at p to rest at q if there is no speed limit, assuming that the acceleration and deceleration remain unchanged at 1 m/s^2 *and 2 m/s*^{2}*, respectively.*

(b) A particle is projected vertically upwards with an initial velocity of u m/s and another particle is projected vertically upwards from the same point and with the same initial velocity T seconds later.

Show that the particles

(i) *will meet*

$$\left(\frac{T}{2} + \frac{u}{g}\right)$$

seconds from the instant of projection of the first particle

(ii) *will meet at a height of*

$$\frac{4u^2 - g^2T^2}{8g} \text{ metres.}$$

2.17.1 Solution

(a) Let s_a be the distance travelled whilst accelerating and s_d be the distance travelled while decelerating. Using

$$\begin{aligned} v^2 &= u^2 + 2as \\ s &= \frac{v^2 - u^2}{2a} \\ s_a = \frac{400}{2} &= 200 \text{ m} \\ s_b = \frac{-400}{-4} &= 100 \text{ m} \end{aligned}$$

Let s_c be the distance travelled at constant speed $v = 20$ and t_c be the time spent travelling at constant speed. The total distance travelled is 1200 m:

$$\begin{aligned} 1200 &= 200 + 100 + 20t_c \\ \Rightarrow t_c &= \frac{900}{20} = 45 \text{ s} \end{aligned}$$

To travel a distance from rest to rest in the shortest possible times implies acceleration followed by immediate deceleration such that if t_1 is the time spent accelerating and t_2 the time spent decelerating:

$$t_1 : t_2 = d : a \tag{31}$$

$$\Rightarrow t_1 : t_2 = 2 : 1$$
$$\Rightarrow t_1 : t_2 = \frac{2}{3} : \frac{1}{3}$$
$$\Rightarrow t_1 = 2T/3, \text{ and } t_2 = T/3$$

Also the maximum speed reached is given by, using

$$v = u + at$$
$$\Rightarrow v = \frac{2}{3}T$$

Now distance travelled is the area under the graph:

$$1200 \overset{!}{=} \frac{1}{2}vT$$
$$\Rightarrow 1200 = \frac{1}{2}\left(\frac{2}{3}T\right)T$$
$$\Rightarrow T^2 = \frac{3(2)(1200)}{2} = 3600$$
$$\Rightarrow T = 60 \text{ s}$$

(b) (i) If the particles meet at a time t after the first particle is emitted, then they will have equal heights at that time:

$$s_1(t) \overset{!}{=} s_2(t) \tag{32}$$

Using

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s_1(t) = ut - \frac{g}{2}t^2$$

The second particle is only in motion after a time T so in terms of t it is in motion for a time $t - T$:

$$s_2(t) \equiv s_2(t - T)$$
$$\Rightarrow s_2(t) = u(t - T) - \frac{g}{2}(t - T)^2$$
$$\underset{s_1 \overset{!}{=} s_2}{\Rightarrow} ut - \frac{g}{2}t^2 = ut - uT - \frac{g}{2}(t^2 - 2tT + T^2)$$
$$\Rightarrow uT = gtT - \frac{T^2 g}{2}$$
$$\Rightarrow t = \frac{uT}{gT} + \frac{T^2 g}{2gT}$$
$$\Rightarrow t = \left(\frac{T}{2} + \frac{u}{g}\right)$$

□

(ii) To find the height they meet at is to find s_1 or s_2 at the time t ($s_1(t) = s_2(t)$):

$$s_1\left(\frac{T}{2}+\frac{u}{g}\right) = u\left(\frac{T}{2}+\frac{u}{g}\right) - \frac{1}{2}g\left(\frac{T}{2}+\frac{u}{g}\right)^2$$

$$\Rightarrow s_1\left(\frac{T}{2}+\frac{u}{g}\right) = \frac{Tu}{2}+\frac{u^2}{g} - \frac{1}{2}g\left(\frac{T^2}{4}+\frac{Tu}{g}+\frac{u^2}{g^2}\right)$$

$$\Rightarrow s_1\left(\frac{T}{2}+\frac{u}{g}\right) = \frac{Tu}{2}+\frac{u^2}{g} - \frac{gT^2}{8} - \frac{Tu}{2} - \frac{u^2}{2g}$$

$$\Rightarrow s_1\left(\frac{T}{2}+\frac{u}{g}\right) = \frac{4Tug + 8u^2 - g^2T^2 - 4uTg - 4u^2}{8g}$$

$$\Rightarrow s_1\left(\frac{T}{2}+\frac{u}{g}\right) = \frac{4u^2 - g^2T^2}{8g}$$

□

2.18 Problem: LC HL 2000

(a) *A stone projected vertically upwards with an initial speed of u m/s rises 70 m in the first t seconds and another 50 m in the next t seconds.*

Find the value of u.

(b) A car, starting from rest and travelling from p to q on a straight level road, where $|pq| = 10\,000$ m, reaches its maximum speed 25 m/s by constant acceleration in the first 500 m and continues at this maximum speed for the rest of the journey.

A second car, starting from rest and travelling from q to p, reaches the same maximum speed by constant acceleration in the first 250 m and continues at this maximum speed for the rest of the journey.

(i) *If the two cars start at the same time, after how many seconds do the two cars meet?*
Find, also, the distance travelled by each car in that time.

(ii) *If the start of one car is delayed so that they meet each other at exactly halfway between p and q, find which car is delayed and by how many seconds.*

2.18.1 Solution

(a) Examining separately the motion in the first t seconds and in the first $2t$ seconds; using:

$$\begin{aligned}
s &= ut + \frac{1}{2}at^2 \\
\Rightarrow 70 &= ut - \frac{1}{2}gt^2 \quad (33)\\
\Rightarrow 120 &= 2ut - 2gt^2 \quad (34)\\
\underset{(33)}{\Rightarrow} 2ut &= 140 + gt^2 \quad (35)\\
\underset{(34)}{\Rightarrow} 120 &= 140 + gt^2 - 2gt^2 \\
\Rightarrow gt^2 &= 20 \\
\Rightarrow t &= \sqrt{\frac{20}{g}} \\
\underset{(35)}{\Rightarrow} 2u\sqrt{\frac{20}{g}} &= 140 + g\frac{20}{g} \\
\Rightarrow u = \sqrt{\frac{g}{20}(80)} = 8\sqrt{\frac{100g}{20}} &= 8\sqrt{5g} \quad (36)
\end{aligned}$$

(b) (i) The cars meet when

$$s_1(t) + s_2(t) = 10000 \text{ m} \quad (37)$$

How long does it take car 1 to accelerate to 25 m/s? Using

$$\begin{aligned}
s &= \left(\frac{u+v}{2}\right)t \\
\Rightarrow 500 &= \left(\frac{25}{2}\right)t_{a,1} \\
\Rightarrow t_{a,1} &= 40 \text{ s}
\end{aligned}$$

Similarly,

$$\begin{aligned}
250 &= \left(\frac{25}{2}\right)t_{a,2} \\
t_{a,2} &= 20 \text{ s}
\end{aligned}$$

Therefore the motion of the cars in terms of t after they take off (because the cars certainly don't meet in less than 40 s - $s_1(40)+s_2(40) = 500+250+20(25) = 1250$ m) is given by the distance whilst accelerating plus the distance travelled at constant speed 25 m/s for a time ($t-$the time spent accelerating):

$$\begin{aligned}
s_1(t) &= 500 + 25(t - 40) \quad (38)\\
s_2(t) &= 250 + 25(t - 20) \quad (39)
\end{aligned}$$

Now $(38) + (39) \stackrel{!}{=} 10000$:

$$\begin{aligned} \Rightarrow 750 + 25(2t - 60) &= 10000 \\ \Rightarrow 25(2t - 60) &= 9250 \\ \Rightarrow 2t - 60 &= 370 \\ \Rightarrow t &= 215 \text{ s} \end{aligned}$$

Also

$$\begin{aligned} s_1(215) &= 500 + 25(215 - 40) = 4875 \text{ m} \\ s_2(215) &= 250 + 25(215 - 20) = 5125 \text{ m} \end{aligned}$$

(ii) Car 1 travels 500 m in 40 s. At constant speed, $a = 0$, using:

$$\begin{aligned} s &= vt \\ \Rightarrow t &= \frac{s}{v} \\ \Rightarrow t = \frac{4500}{25} &= 180 \text{ s} \end{aligned}$$

So it takes car 1 $40 + 180 = 220$ s to travel 5000 m.
Similarly car 2 travels 250 m in 20 s. At constant speed, $a = 0$, using:

$$\begin{aligned} s &= vt \\ \Rightarrow t &= \frac{s}{v} \\ \Rightarrow t = \frac{4750}{25} &= 190 \text{ s} \end{aligned}$$

So it takes car 2 $20 + 190 = 210$ s to travel 5000 m. Hence if car 2 is delayed by 10 s, they will meet at half way.

2.19 Problem: LC HL 1999 [Part (b)]

A particle travels in a straight line with constant acceleration f for $2t$ seconds and covers 15 metres. The particle then travels a further 55 metres at constant speed in $5t$ seconds. Finally the particle is brought to rest by a constant retardation $3f$.

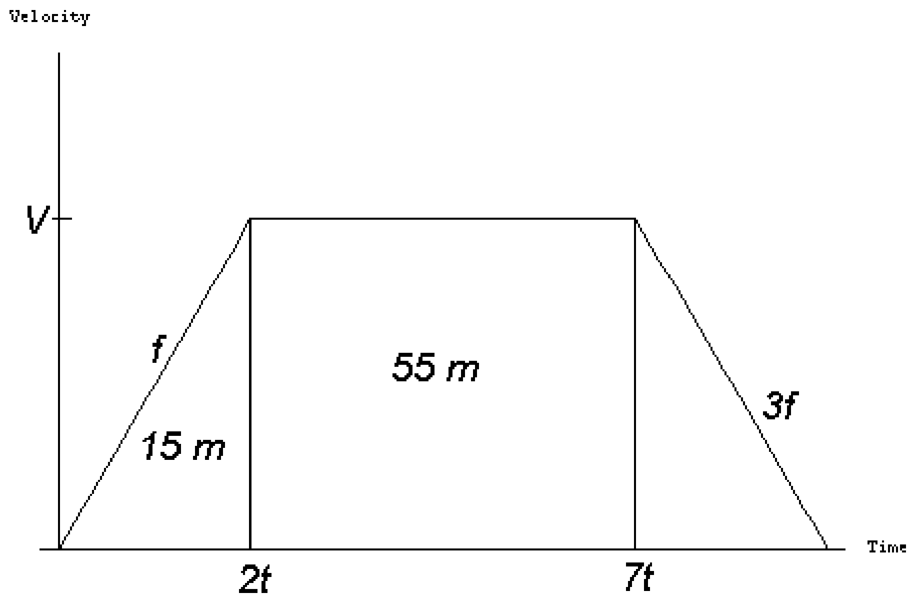
(i) *Draw a speed-time graph for the motion of the particle.*

(ii) *Find the initial velocity of the particle in terms of t.*

(iii) *Find the total distance travelled in metres, correct to two decimal places.*

(ii) First examining the constant speed motion, using:

$$\begin{aligned} s &= vt \\ \Rightarrow v = \frac{s}{t} = \frac{55}{5t} &= \frac{11}{t} \end{aligned}$$



(i)

Now using:

$$
\begin{aligned}
s &= \left(\frac{u+v}{2}\right)t \\
\Rightarrow u &= \frac{2s}{t} - v \\
\underset{t=2t}{\Rightarrow} u &= \frac{15}{t} - \frac{11}{t} \\
\Rightarrow u &= \frac{4}{t}
\end{aligned}
$$

(iii) Using

$$
\begin{aligned}
a &= \frac{v-u}{t} \\
\Rightarrow f &= \frac{\frac{11}{t} - \frac{4}{t}}{2t} \\
\Rightarrow f &= \frac{\frac{7}{t}}{2t} = \frac{7}{2t^2}
\end{aligned}
$$

Now for the decelerating motion, using:

$$
\begin{aligned}
v^2 &= u^2 + 2as \\
\Rightarrow s &= \frac{v^2 - u^2}{2a} \\
\Rightarrow s &= \frac{-\frac{121}{t^2}}{-6f} = \frac{121}{t^2 6\left(\frac{7}{2t^2}\right)} \\
\Rightarrow s &= \frac{121}{21} \simeq 5.762 \text{ m}
\end{aligned}
$$

Hence the total distance travelled, d, is:

$$d = 15 + 55 + 5.76 = 75.76 \text{ m}$$

2.20 Problem: LC HL 1998

(a) *A train accelerates uniformly from rest to a speed v m/s. It continues at this constant speed for a period of time and then decelerates uniformly to rest. If the average speed for the whole journey is $5v/6$, find what fraction of the whole distance is described at constant speed.*

(b) *Car A, moving with uniform acceleration $3b/20$ m/s^2 passes a point p with speed $9u$ m/s. Three seconds later car B, moving with uniform acceleration $2b/9$ m/s^2 passes the same point with speed $5u$ m/s. B overtakes A when their speeds are 6.5 m/s and 5.4 m/s respectively. Find*

(i) *the value u and the value b*

(ii) *the distance travelled from p until overtaking occurs.*

2.20.1 Remark

I think Q.1 in 1998 was particularly difficult. This is certainly the most difficult AM Q. 1 I've ever seen.

2.20.2 Solution

(a) Let s_1, s_2 and s_3 be the distances travelled at acceleration, constant speed and deceleration respectively. Similarly let t_1, t_2 and t_3 be the time spent at acceleration, constant speed and deceleration respectively. Hence as the area under the time-velocity graph is the distance travelled:

$$s_1 = \frac{1}{2}vt_1$$
$$s_2 = vt_2$$
$$s_3 = \frac{1}{2}vt_3$$

Using:

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}$$
$$\Rightarrow \frac{5v}{6} = \frac{\frac{1}{2}vt_1 + vt_2 + \frac{1}{2}vt_3}{t_1 + t_2 + t_3}$$
$$\Rightarrow 5(t_1 + t_2 + t_3) = 6\left(\frac{1}{2}t_1 + t_2 + \frac{1}{2}t_3\right)$$
$$\Rightarrow 5t_1 + 5t_2 + 5t_3 = 3t_1 + 6t_2 + 3t_3$$
$$\Rightarrow t_2 = 2t_1 + 2t_3$$

Now the fraction travelled at constant speed:

$$\begin{aligned}
&\frac{s_2}{s_1+s_2+s_3}\\
&=\frac{vt_2}{\frac{1}{2}vt_1+vt_2+\frac{1}{2}vt_3}\\
&=\frac{2t_2}{2t_2+(t_1+t_3)}\\
&\underset{(t_2=2t_1+2t_3)}{=}\frac{2t_2}{2t_2+\frac{1}{2}t_2}\\
&=\frac{4t_2}{4t_2+t_2}=\frac{4t_2}{5t_2}=\frac{4}{5}
\end{aligned}$$

(b) (i) In terms of a t after car A starts moving; using;

$$\begin{aligned}
v&=u+at\\
\Rightarrow v_A(t)&=9u+\left(\frac{3b}{20}\right)t
\end{aligned}$$

Car B is stationary for three of these seconds:

$$\begin{aligned}
v_B(t)&\equiv v_B(t-3)\text{ , where}\\
v_B(t-3)&=5u+\left(\frac{2b}{9}\right)(t-3)
\end{aligned}$$

For overtaking to occur after T seconds, $v_A(T)=5.4$ m/s, $v_B(T)=6.5$ m/s and

$$s_A(T)=:s:=s_B(T) \tag{40}$$

Now using;

$$\begin{aligned}
v^2&=u^2+2as\\
\Rightarrow v_A^2(T)\overset{!}{=}5.4^2&=81u^2+2\left(\frac{3b}{20}\right)s\\
\Rightarrow 29.16&=81u^2+\frac{3bs}{10} &(41)\\
\Rightarrow v_B^2(T)\overset{!}{=}6.5^2&=25u^2+2\left(\frac{2b}{9}\right)s\\
\Rightarrow 42.25&=25u^2+\frac{4bs}{9} &(42)
\end{aligned}$$

$$\begin{aligned}
40\times(41)=1166.4&=3240u^2+12bs\\
-27\times(42)=-1140.75&=-675u^2-12bs\\
\underset{40\times(41)+-27\times(42)}{\Rightarrow}25.65&=2565u^2\\
\Rightarrow u^2=0.01&=\frac{1}{100}\\
\Rightarrow u=\sqrt{\frac{1}{100}}=\frac{1}{\sqrt{100}}&=\frac{1}{10}\text{ m/s}
\end{aligned}$$

Now note $v_A(T) = 5.4$;

$$\begin{aligned} v_A(T) = 5.4 = 9u + \frac{3bT}{20} \\ \Rightarrow 5.4 = 0.9 + \frac{3bT}{20} \\ \Rightarrow 4.5 = \frac{3bT}{20} \\ \Rightarrow bT = 30 \end{aligned} \tag{43}$$

Similarly

$$\begin{aligned} v_B(T) = 6.5 = 5u + \frac{2b}{9}(T-3) \\ \Rightarrow 6.5 = 0.5 + \frac{2bT}{9} + \frac{6b}{9} \\ \Rightarrow 6 = \frac{2bT}{9} - \frac{6b}{9} \\ \Rightarrow 54 = 2bT - 6b \\ \Rightarrow 6b = 2(bT) - 54 \\ \underset{(43)}{\Rightarrow} 6b = 60 - 54 = 6 \\ \Rightarrow b = 1 \end{aligned}$$

(ii) Note the distance from overtaking is

$$s_1(T) =: s := s_2(T)$$

Taking (41):

$$\begin{aligned} 29.16 = 81u^2 + \frac{3bs}{10} \\ \Rightarrow s = (29.16 - 81u^2)\frac{10}{3b} \\ \Rightarrow s = (29.16 - 81(0.01))\frac{10}{3} \\ \Rightarrow s = 94.5 \text{ m} \end{aligned}$$

3 Projectiles

3.1 Problem: LC HL 2009 [Part (a)]

A straight vertical cliff is 200 m high.
A particle is projected from the top of the cliff.
The speed of projection is $14\sqrt{10}$ *m/s at an angle* α *to the horizontal.*
The particle strikes the level ground at a distance 200 m from the foot of the cliff.

(i) Find, in terms of α*, the time taken for the particle to hit the ground.*

(ii) Show that the two possible directions of projection are at right angles to each other.

3.1.1 Solution

(i) Let T be the time when $s_x = 200$ and $s_y = -200$. Using

$$\begin{aligned} s_x(t) &= u_x t \\ \Rightarrow 200 &= 14\sqrt{10}\cos\alpha T \\ \Rightarrow T &= \frac{200}{14\sqrt{10}}\sec\alpha\left(\times\frac{\sqrt{10}}{\sqrt{10}}\right) \\ \Rightarrow T &= \frac{10}{7}\sqrt{10}\sec\alpha \end{aligned}$$

(ii) Now $s_y(T) = -200$, using:

$$\begin{aligned} s_y(t) &= u_y t - \frac{1}{2}gt^2 \\ \Rightarrow -200 &= 14\sqrt{10}\sin\alpha\left(\frac{10}{7}\sqrt{10}\sec\alpha\right) - \frac{1}{2}g\left(\frac{1000}{49}\sec^2\alpha\right) \end{aligned}$$

Now $\sin\alpha\sec\alpha = \tan\alpha$ and

$$\sec^2\alpha \equiv 1 + \tan^2\alpha \tag{44}$$

$$\begin{aligned} \Rightarrow -200 &= 200\tan\alpha - \frac{g}{98}(1000(1+\tan^2\alpha)) \\ \Rightarrow -200 &= 200\tan\alpha - \frac{1}{10}(1000(1+\tan^2\alpha)) \\ \Rightarrow -200 &= 200\tan\alpha - 100(1+\tan^2\alpha) \\ \Rightarrow -2 &= 2\tan\alpha - (1+\tan^2\alpha) \end{aligned}$$

Let $u := \tan\alpha$:

$$\begin{aligned} -2 &= 2u - 1 - u^2 \\ \Rightarrow u^2 - 2u - 1 &= 0 \end{aligned}$$

Using the formula for the roots of a quadratic:

$$\begin{aligned} u &= \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} \\ \underset{\sqrt{8}=\sqrt{4.2}=\sqrt{4}\sqrt{2}}{\Rightarrow} \tan\alpha &= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \end{aligned}$$

Now if m_L and m_K are the slopes of lines L and K then

$$m_L \times m_L = -1 \Rightarrow L \perp K \tag{45}$$

Equivalently, two directions are at right angles if:

$$\begin{aligned} \tan\alpha_1 \times \tan\alpha_2 &= -1 \\ \Rightarrow \tan\alpha_1 \times \tan\alpha_2 &= (1+\sqrt{2})(1-\sqrt{2}) \\ \Rightarrow \tan\alpha_1 \times \tan\alpha_2 &= 1-2=-1 \end{aligned} \tag{46}$$

$\therefore$ the two directions are at right angles. □

3.2 Problem: LC HL 2008 [Part (b)]

A ball is projected from a point on the ground at a distance of a from the foot of a vertical wall of height b, the velocity of projection being u at angle 45° to the horizontal. If the ball just clears the wall prove that the greatest height reached is

$$\frac{a^2}{4(a-b)}.$$

3.2.1 Solution

In the first instance maximum height is s_y when $v_y = 0$. Using

$$\begin{aligned} v_y &= u_y - gt \\ \Rightarrow 0 &= u_y - gt_{\max} \\ \Rightarrow t_{\max} &= \frac{u_y}{g} \\ \Rightarrow s_{y_{\max}} &= u_y\left(\frac{u_y}{g}\right) - \frac{1}{2}g\left(\frac{u_y^2}{g^2}\right) \\ \Rightarrow s_{y_{\max}} &= \frac{u_y^2}{g} - \frac{1}{2}\frac{u_y^2}{g} = \frac{u_y^2}{2g} \end{aligned} \tag{47}$$

Now $u_y = u\sin 45° = u/\sqrt{2}$:

$$s_{y_{\max}} = \frac{u^2}{4g} \tag{48}$$

Now at a time, say T, $s_x(T) = a$ and $s_y(T) = b$. Using:

$$\begin{aligned}
s_x &= u_x t \\
\Rightarrow a &= u \cos 45^\circ T \\
\Rightarrow T &= \frac{\sqrt{2}a}{u} \\
\underset{s_y(T)=b}{\Rightarrow} b &= \frac{u}{\sqrt{2}}\left(\frac{\sqrt{2}a}{u}\right) - \frac{1}{2}g\left(\frac{2a^2}{u^2}\right) \\
\Rightarrow b &= a - \frac{ga^2}{u^2} \\
\Rightarrow a - b &= \frac{ga^2}{u^2} \\
\Rightarrow u^2 &= \frac{ga^2}{a-b} \\
\Rightarrow s_{y_{\max}} = \frac{u^2}{4g} &= \frac{ga^2}{4g(a-b)} \\
\Rightarrow s_{y_{\max}} &= \frac{a^2}{4(a-b)}
\end{aligned}$$

□

3.3 Problem: LC HL 2007 [Part (a)]

A particle is projected with a speed of $7\sqrt{5}$ m/s at an angle α to the horizontal. Find the two values of α that will give a range of 12.5 m.

3.3.1 Solution

The range is s_x when $s_y = 0$. Using

$$\begin{aligned}
s_y &= u_y t - \frac{1}{2}gt^2 \\
\Rightarrow t\left(u_y - \frac{g}{2}t\right) &= 0 \\
\Rightarrow t &= \frac{2u_y}{g} \\
\Rightarrow R = s_x(t) &= u_x\left(\frac{2u_y}{g}\right) \\
\Rightarrow R &= \frac{u^2}{g} 2\sin\alpha\cos\alpha \\
\underset{2\sin x\cos x = \sin 2x}{\Rightarrow} R &= \frac{u^2}{g}\sin 2\alpha \overset{!}{=} \frac{25}{2} \\
\Rightarrow \sin 2\alpha = \frac{25g}{2u^2} &= \frac{25g}{2(49)(5)} = \frac{1}{2} \\
\Rightarrow 2\alpha &= 30^\circ \text{ or } 150^\circ \\
\Rightarrow \alpha &= 15^\circ \text{ or } 75^\circ
\end{aligned}$$

3.4 Problem: LC HL 2006 [Part (a)]

A particle is projected from a point o with velocity $9.8\,\mathbf{i} + 29.4\,\mathbf{j}$ *m/s where* $\mathbf{i}$ *and* $\mathbf{j}$ *are unit perpendicular vectors in the horizontal and vertical directions, respectively.*

(i) Express the velocity and displacement of the particle after t seconds in terms of $\mathbf{i}$ *and* $\mathbf{j}$.

(ii) Find, in terms of t, the direction in which the particle is moving after t seconds.

(iii) Find the two times when the direction of the particle is at right angles to the line joining the particle to o.

3.4.1 Solution

(i) Noting first that $9.8 = g$ and $29.4 = 3g$. The velocity vector is given by:

$$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} \tag{49}$$

and using

$$\begin{aligned} v_x(t) &= u_x\,, \quad \text{and} \\ v_y &= u_y - gt \\ \Rightarrow \mathbf{v}(t) &= g\mathbf{i} + (3g - gt)\mathbf{j} \end{aligned}$$

The displacement vector is given by:

$$\mathbf{r}(t) = s_x(t)\mathbf{i} + s_y(t)\mathbf{j} \tag{50}$$

Using:

$$\begin{aligned} s_x(t) &= u_x t\,, \quad \text{and} \\ s_y(t) &= u_y t - \frac{1}{2}gt^2 \\ \Rightarrow \mathbf{r}(t) &= gt\mathbf{i} + \left(3gt - \frac{1}{2}gt^2\right)\mathbf{j} \end{aligned}$$

(ii) The direction the particle is travelling in is the slope of the tangent to the displacement at t; the tangent being given by the derivative:

$$\begin{aligned} \mathbf{r}'(t) &= v_x\mathbf{i} + v_y\mathbf{j} \\ \Rightarrow \text{direction after } t = \frac{\mathbf{j}\text{-component } \mathbf{r}'(t)}{\mathbf{i}\text{-component } \mathbf{r}'(t)} = \frac{v_y}{v_x} = \frac{3g - gt}{g} &= 3 - t \end{aligned}$$

(iii) The line joining the particle to o has slope:

$$\frac{s_y}{s_x}$$

For two lines L and K to be perpendicular the slopes m_L and m_K must satisfy

$$m_L \times m_L = -1. \tag{51}$$

Hence solve:

$$\begin{aligned}
\frac{s_y}{s_x} \times (3-t) &\overset{!}{=} -1 \\
\Rightarrow \frac{3gt - \frac{1}{2}gt^2}{gt}(3-t) &= -1 \\
\underset{\div gt}{\Rightarrow} \left(3 - \frac{1}{2}t\right)(3-t) &= -1 \\
\Rightarrow 9 - \frac{3}{2}t + \frac{1}{2}t^2 - 3t &= -1 \\
\Rightarrow 18 - 3t + t^2 - 6t &= -2 \\
\Rightarrow t^2 - 9t + 20 &= 0 \\
\Rightarrow t^2 - 4t - 5t + 20 &= 0 \\
\Rightarrow t(t-4) - 5(t-4) &= 0 \\
\Rightarrow (t-4)(t-5) &= 0
\end{aligned}$$

Ans: At $t = 4$s, and 5 s.

3.5 Problem: LC HL 2004: [Part (a)]

A particle is projected from a point on the horizontal floor of a tunnel with maximum height of 8 m. The particle is projected with an initial speed of 20 m/s inclined at an angle α to the horizontal floor.
Find, to the nearest metre, the greatest range which can be attained in the tunnel.

3.5.1 Solution

The range R is s_x when $s_y = 0$. Solving this gives:

$$R = \frac{u^2}{g}\sin 2\alpha$$

Next the angle of projection which gives the max height as the height of the tunnel, 8 m, is found. Max height is s_y when $v_y = 0$. Solving this gives:

$$\begin{aligned}
s_{y_{\max}} = \frac{u_y^2}{2g} &= \frac{u^2}{2g}\sin^2\alpha \\
\Rightarrow \frac{u^2}{2g}\sin^2\alpha &= 8 \\
\Rightarrow \sin^2\alpha &= \frac{16g}{400} \\
\Rightarrow \sin\alpha = \frac{4\sqrt{g}}{20} &= \frac{\sqrt{g}}{5} \\
\Rightarrow \alpha = \arcsin(\sqrt{g}/5) &\approx 38.763°
\end{aligned}$$

Now if $\alpha > \arcsin(\sqrt{g}/5)$ then max height is bigger than 8 m, so this motion is not in the tunnel as required. Hence $\alpha < \arcsin(\sqrt{g}/5)$. Now looking at the range:

$$R = \frac{u^2}{g}\sin 2\alpha$$

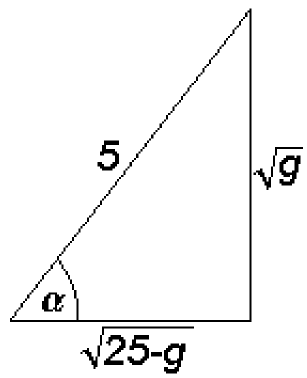
For $\theta \in [0, 45°]$, $\sin 2\theta$ is increasing, as

$$\frac{d}{d\theta}\sin 2\theta = 2\cos 2\theta > 0\,,\ \ \theta \in [0, 45°]$$

Hence as $0 \le \alpha \le \arcsin(\sqrt{g}/5)$, $\sin 2\alpha$ attains it maximum at $\alpha = \arcsin(\sqrt{g}/5)$:

$$R = \frac{20^2}{g} 2\sin\alpha\cos\alpha$$

Now the model triangle gives:



$$R = \frac{800}{g} 2\frac{\sqrt{g}}{5}\frac{\sqrt{25-g}}{5} = \frac{800}{25}\frac{\sqrt{g}}{g}\sqrt{15.2}$$
$$\Rightarrow R = \frac{32}{\sqrt{g}}(3.8987) = 39.85 \simeq 40 \text{ m}$$

3.6 Problem: LC HL 2003: [Part (a)]

A particle is projected from a point on level horizontal ground at an angle θ to the horizontal ground.
Find θ, if the horizontal range of the particle is five times the maximum height reached by the particle.

3.6.1 Solution

The range, R, is s_x when $s_y = 0$:

$$R = \frac{u^2}{g}\sin 2\theta \tag{52}$$

The maximum height, $s_{y_{\max}}$, of the particle is s_y when $v_y = 0$:

$$s_{y_{\max}} = \frac{u^2\sin^2\theta}{2g} \tag{53}$$

Hence θ is the angle such that $R = s_{y_{\max}}$:

$$\begin{aligned}
\frac{u^2}{g}\sin 2\theta &\overset{!}{=} \frac{5u^2\sin^2\theta}{2g} \\
\underset{u\neq 0}{\Rightarrow} 2\sin\theta\cos\theta &= \frac{5}{2}\sin^2\theta \\
\underset{\theta\neq 0}{\Rightarrow} 2\cos\theta &= \frac{5}{2}\sin\theta \\
\Rightarrow \frac{\sin\theta}{\cos\theta} &= \frac{4}{5} \\
\Rightarrow \tan\theta &= \frac{4}{5} \\
\Rightarrow \theta = \arctan\left(\frac{4}{5}\right) &\simeq 38.66^\circ
\end{aligned}$$

3.7 Problem: LC HL 2002: [Part (a)]

A particle is projected from a point on the horizontal ground with a speed of 39.2 m/s inclined at an angle α to the horizontal ground. The particle is at a height of 14.7 m above the horizontal ground at times t_1 and t_2 seconds, respectively.

(i) Show that

$$t_2 - t_1 = \sqrt{64\sin^2\alpha - 12}$$

(ii) Find the value of α for which $t_2 - t_1 = \sqrt{20}$.

3.7.1 Solution

(i) First note $14.7 = 3g/2$ and $39.2 = 4g$. t_1 and t_2 are times when $s_y = 14.7$; hence are the solutions of the quadratic equation:

$$\begin{aligned}
\frac{3}{2}g &= 4gt\sin\alpha - \frac{1}{2}gt^2 \\
\underset{\times 2, \div g}{\Rightarrow} 3 &= 8t\sin\alpha - t^2 \\
\Rightarrow t^2 - 8t\sin\alpha - 3 &= 0
\end{aligned}$$

Using the formula for the roots of a quadratic, letting t_1 be the '+' solution and t_2 be the '-' solution:

$$\begin{aligned}
t &= \frac{8 \pm \sqrt{64\sin^2\alpha - 12}}{2} \\
t_1 - t_2 = \left(\frac{8+\sqrt{64\sin^2\alpha-12}}{2}\right) &- \left(\frac{8-\sqrt{64\sin^2\alpha-12}}{2}\right) \\
\Rightarrow t_1 - t_2 = 4 + \frac{\sqrt{64\sin^2\alpha-12}}{2} &- 4 + \frac{\sqrt{64\sin^2\alpha-12}}{2} \\
\Rightarrow t_1 - t_2 &= \sqrt{64\sin^2\alpha - 12}
\end{aligned}$$

□

(ii) Hence α is such that:

$$\begin{aligned}
64\sin^2\alpha - 12 &= 20\\
\Rightarrow 64\sin^2\alpha &= 32\\
\Rightarrow \sin^2\alpha &= \frac{1}{2}\\
\Rightarrow \sin\alpha &= \frac{1}{\sqrt{2}}\\
\Rightarrow \theta &= 45^\circ
\end{aligned}$$

3.8 Problem: LC HL 2001: [Part (a)]

A player hits a ball with an initial speed of u m/s from a height of 1 m at an angle of 45° *to the horizontal ground. A member of the opposing team, 21 m away, catches the ball at a height of 2 m above the ground.*
Find the value of u.

3.8.1 Solution

$s_y = 1$ when $s_x = 21$. Suppose the catcher catches the ball at a time T, noting $\sin 45^\circ = 1/\sqrt{2} = \cos 45^\circ$:

$$\begin{aligned}
s_x &= \frac{u}{\sqrt{2}}T\\
\Rightarrow T &= \frac{21\sqrt{2}}{u}T
\end{aligned}$$

Now $s_y(T) = 1$, using:

$$\begin{aligned}
s_y &= u_y t - \frac{1}{2}gt^2\\
\Rightarrow 1 &= \frac{u}{\sqrt{2}}\frac{21\sqrt{2}}{u} - \frac{1}{2}g\frac{(21)^2(2)}{u^2}\\
\underset{\times u^2,\text{ cancelling}}{\Rightarrow} \quad u^2 &= 21u^2 - 21^2 g\\
\Rightarrow 20u^2 &= 21^2 g\\
\Rightarrow u^2 &= \frac{21^2}{20}g\\
\Rightarrow u = 21\sqrt{\frac{g}{20}} &= 14.7 \text{ m/s}
\end{aligned}$$

3.9 Problem: LC HL 2000: [Part (b)]

A particle is projected with a velocity u m/s at an angle β to the horizontal ground. Show that the particle hits the ground at a distance

$$\frac{u^2}{g}\sin 2\beta$$

from the point of projection. Find the angle of projection which gives maximum range.

3.9.1 Solution

Range, R, is s_x when $s_y = 0$;

$$\begin{aligned} s_y = u_y t - \frac{1}{2}gt^2 &\stackrel{!}{=} 0 \\ \Rightarrow t &= \frac{2u_y}{g} \end{aligned}$$

Now

$$\begin{aligned} s_x &= u_x t \\ \Rightarrow R &= u_x\left(\frac{2u_y}{g}\right) \\ \Rightarrow R &= \frac{2u^2\sin\beta\cos\beta}{g} \\ \Rightarrow R &= u^2\frac{2\sin\beta\cos\beta}{g} \\ \underset{2\sin x\cos x = \sin 2x}{\Rightarrow} \quad R &= \frac{u^2}{g}\sin 2\beta \end{aligned}$$

□

Taking u to be fixed, to maximise R vary β. The maximum value of sine is 1. This occurs when

$$\begin{aligned} 2\beta &= 90^\circ \quad (\sin 90^\circ = 1) \\ \Rightarrow \beta &= 45^\circ \text{ for maximum range} \end{aligned}$$

3.10 Problem: LC HL 2009: [Part(b)]

A plane is inclined at an angle $60°$ *to the horizontal. A particle is projected up the plane with initial speed* u *at an angle* θ *to the inclined plane. The plane of projection is vertical and contains the line of greatest slope.*
The particle strikes the plane at right angles.
Show that the range on the inclined plane is

$$\frac{4\sqrt{3}u^2}{13g}.$$

3.10.1 Solution

Let T be the time of flight. If the particle lands at right angles, $s_y(T) = 0$ and $v_x(T) = 0$. Using

$$v_x = u\cos\theta - g\sin 60°\, t$$
$$\Rightarrow T = \frac{u\cos\theta}{g\sin 60°}$$

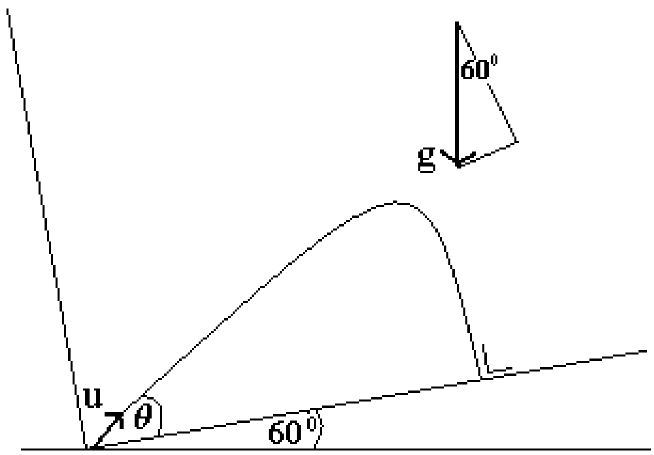


Figure 1: If the particle lands at right angles, the final velocity is entirely in the y-direction

Using

$$s_y = u\sin\theta\, t - \frac{1}{2}g\cos 60°\, t^2$$
$$\Rightarrow T = \frac{2u\sin\theta}{\cos 60°}$$
$$\Rightarrow \frac{u\cos\theta}{g\sin 60°} = \frac{2u\sin\theta}{\cos 60°}$$
$$\Rightarrow \tan\theta = \frac{1}{2}.\frac{\cos 60°}{\sin 60°} = \frac{1}{2}.\cot 60°$$
$$\Rightarrow \tan\theta = \frac{1}{2\sqrt{3}}$$

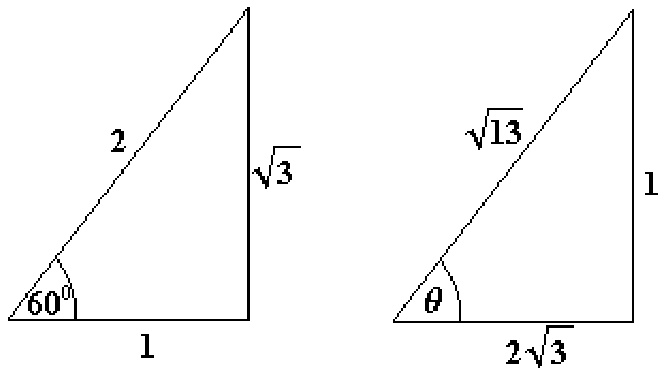


Figure 2: The model triangles for 60° and θ

Now range means $s_x(T)$. Using

$$
\begin{aligned}
s_x &= u_x t - \frac{1}{2} g_x t^2 \\
\Rightarrow R &= u\cos\theta\left(\frac{u\cos\theta}{g\sin 60^\circ}\right) - \frac{1}{2}\cancel{g\sin 60^\circ}\left(\frac{u^2\cos^2\theta}{g^{\cancel{2}}\sin^{\cancel{2}} 60^\circ}\right) \\
\Rightarrow R &= \frac{u^2\cos^2\theta}{2g\sin 60^\circ}(2-1) \\
\Rightarrow R &= \frac{u^2}{\cancel{2}g}.\frac{4(3)}{13}.\frac{\cancel{2}}{\sqrt{3}} \\
\Rightarrow R &= \frac{4\sqrt{3}u^2}{13g}
\end{aligned}
$$

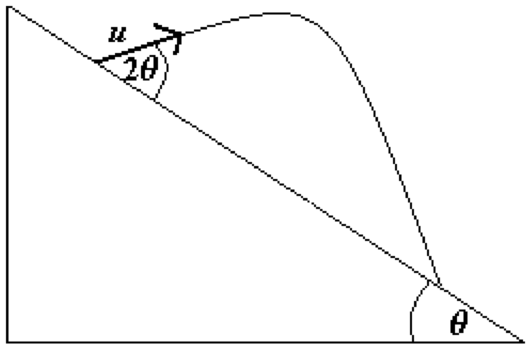
□

3.11 Problem: LC HL 2008: [Part(b)]

A particle is projected down an inclined plane with initial velocity u m/s. The line of projection makes an angle $2\theta°$ with the inclined plane and the plane is inclined at $\theta°$ to the horizontal. The plane of projection is vertical and contains the line of greatest slope.
The range of the particle on the inclined plane is $ku^2 \sin\theta/g$.
Find the value of k.

3.11.1 Solution

Range, R, means s_x when $s_y = 0$.



Using

$$s_y = u\sin 2\theta\, t - \frac{1}{2} g\cos\theta\, t^2$$
$$\Rightarrow T = \frac{2u\sin 2\theta}{g\cos\theta}$$
$$\underset{\sin 2x = 2\sin x\cos x}{\Rightarrow} \frac{4u\sin\theta}{g}$$

Using

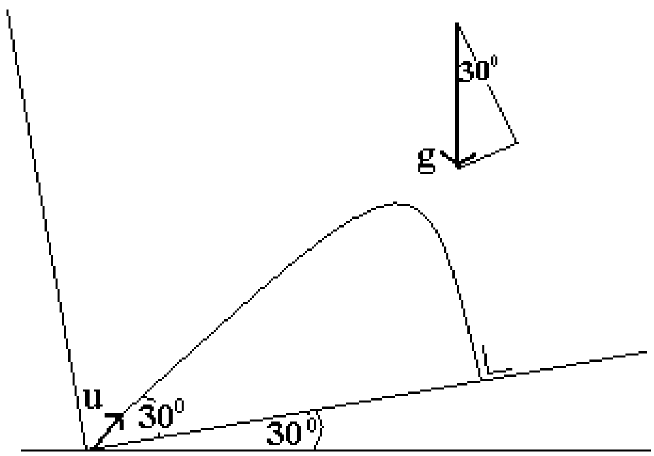
$$s_x = u_x t + \frac{1}{2} g_x t^2$$
$$\Rightarrow R = u\cos 2\theta\left(\frac{4u\sin\theta}{g}\right) + \frac{1}{\cancel{2}}.\cancel{g}\sin\theta\left(\frac{\overset{8}{\cancel{16}}u^2\sin^2\theta}{g^{\cancel{2}}}\right)$$
$$\Rightarrow R = \frac{4u^2\sin\theta}{g}(\cos 2\theta + 2\sin^2\theta)$$
$$\underset{\cos 2x = \cos^2 x - \sin^2 x}{\Rightarrow} R = \frac{4u^2\sin\theta}{g}\underbrace{(\cos^2\theta - \sin^2\theta + 2\sin^2\theta)}_{=\cos^2\theta + \sin^2\theta}$$
$$\underset{\cos^2 x + \sin^2 x = 1}{\Rightarrow} R = \frac{4u^2}{g}\sin\theta$$
$$\Rightarrow k = 4$$

3.12 Problem: LC HL 2006: [Part (b)]

A particle is projected up an inclined plane with initial speed u m/s. The line of projection makes an angle 30° with the plane and the plane is inclined at 30° to the horizontal. The plane of projection is vertical and contains the line of greatest slope.
Find in terms of u, the range of the particle on the inclined plane.

3.12.1 Solution

Range, R, means s_x when $s_y = 0$.



Using

$$s_y = u\sin 30^\circ\, t - \frac{1}{2} g\cos 30^\circ\, t^2$$
$$\Rightarrow T = \frac{2u}{g}\tan 30^\circ$$
$$\Rightarrow R = u\cos 30^\circ\left(\frac{2u}{g}\tan 30^\circ\right) - \frac{1}{\not{2}}\not{g}\sin 30^\circ . \frac{\overset{2}{\not{4}}u^2\tan^2 30^\circ}{g^{\not{2}}}$$
$$\Rightarrow R = \frac{2u^2\tan 30^\circ}{g}(\cos 30^\circ - \sin 30^\circ\ \tan 30^\circ)$$
$$\underset{\text{using Figure 2}}{\Rightarrow} R = \frac{\not{2}u^2}{g}.\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\not{2}} - \frac{1}{\not{2}}.\frac{1}{\sqrt{3}}\right)$$
$$\Rightarrow R = \frac{u^2}{\sqrt{3}\,g}\left(\frac{3-1}{\sqrt{3}}\right)$$
$$\Rightarrow R = \frac{2u^2}{3g}$$

3.13 Problem: LC HL 2005: [Part (b)]

A plane is inclined at an angle β to the horizontal. A particle is projected up the plane with initial speed u at an angle α to the inclined plane. The plane of projection is vertical and contains the line of greatest slope.

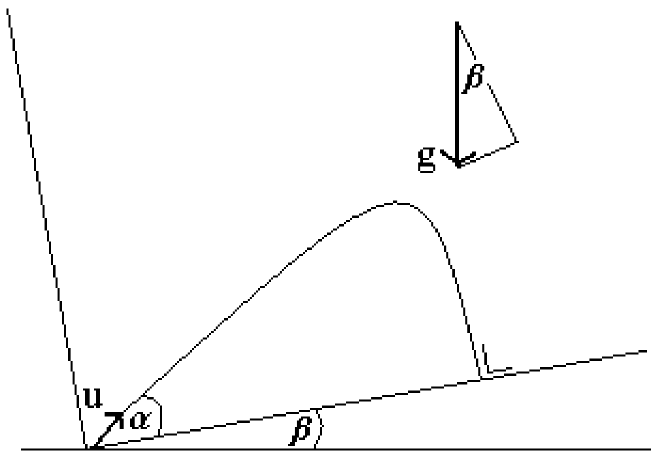
(i) Find the range of the particle on the inclined plane in terms of u, α and β.

(ii) Show that for a constant value of u the range is a maximum when

$$\alpha = 45^\circ - \frac{\beta}{2}$$

3.13.1 Solution

Range, R, means s_x when $s_y = 0$.



Using

$$s_y = u\sin\alpha\, t - \frac{1}{2} g\cos\beta\, t^2$$
$$\Rightarrow T = \frac{2u\sin\alpha}{g\cos\beta}$$
$$\Rightarrow R = u\cos\alpha\left(\frac{2u\sin\alpha}{g\cos\beta}\right) - \frac{1}{\cancel{2}}\cancel{g}\sin\beta\left(\frac{\overset{2}{\cancel{4}}u^2\sin^2\alpha}{g^{\cancel{2}}\cos^2\beta}\right)$$
$$\Rightarrow R = \frac{2u^2\sin\alpha}{g\cos^2\beta}(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$
$$\underset{\cos(x+y)=\cos x\cos y-\sin x\sin y}{\Rightarrow} R = \frac{u^2}{g\cos^2\beta} 2\sin\alpha\cos(\alpha+\beta)$$
$$\underset{2\sin x\cos y=\sin(x+y)+\sin(x-y)}{\Rightarrow} R = \frac{u^2}{g\cos^2\beta}(\sin(2\alpha+\beta) + \sin(-\beta))$$
$$\underset{\sin(-x)=-\sin x}{\Rightarrow} R = \frac{u^2}{g\cos^2\beta}(\sin(2\alpha+\beta) - \sin\beta)$$

To maximise R the only term which may be varied is $\sin(2\alpha + \beta)$. Sine has a maximum value of 1; namely $\sin 90° = 1$. Hence for $R_{\max}$;

$$\begin{aligned} 2\alpha + \beta &= 90° \\ \Rightarrow 2\alpha &= 90° - \beta \\ \Rightarrow \alpha &= 45° - \frac{\beta}{2} \end{aligned}$$

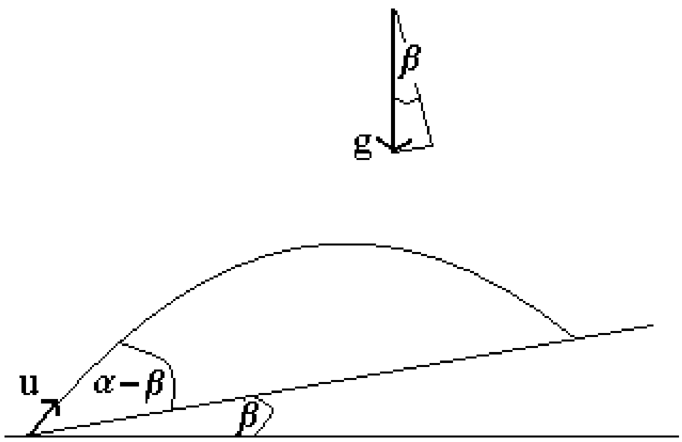
□

3.14 Problem: LC HL 2003: [Part (b)]

A particle is projected up the inclined plane with initial speed u m/s. The line of projection makes an angle α ***with the horizontal*** *and the inclined plane makes an angle β with the horizontal. (The plane of projection is vertical and contains the line of greatest slope). Find in terms of u, g, α and β, the range of the particle up the inclined plane.*

3.14.1 Solution

Range, R, means s_x when $s_y = 0$.



Using

$$\begin{aligned}
s_y &= u\sin(\alpha-\beta)\,t - \frac{1}{2}g\cos\beta\,t^2 \\
\Rightarrow T &= \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \\
\Rightarrow R &= u\cos(\alpha-\beta)\left(\frac{2u\sin(\alpha-\beta)}{g\cos\beta}\right) - \frac{1}{\cancel{2}}\cancel{g}\sin\beta\left(\frac{\overset{2}{\cancel{4}}u^2\sin^2(\alpha-\beta)}{g^{\cancel{2}}\cos^2\beta}\right) \\
\Rightarrow R &= \frac{2u^2\sin(\alpha-\beta)}{g\cos^2\beta}(\cos(\alpha-\beta)\cos\beta - \sin(\alpha-\beta)\sin\beta) \\
\underset{\cos(x+y)=\cos x\cos y-\sin x\sin y}{\Rightarrow} R &= \frac{u^2}{g\cos^2\beta}2\sin(\alpha-\beta)\cos\underbrace{(\alpha-\beta+\beta)}_{=\alpha} \\
\underset{2\sin x\cos y=\sin(x+y)+\sin(x-y)}{\Rightarrow} R &= \frac{u^2}{g\cos^2\beta}(\sin(\alpha-\beta+\alpha)+\sin(\alpha-\beta-\alpha)) \\
\underset{\sin(-x)=-\sin x}{\Rightarrow} R &= \frac{u^2}{g\cos^2\beta}(\sin(2\alpha-\beta)-\sin\beta)
\end{aligned}$$

3.15 Problem: LC HL 1997: [Part (b)]

A particle is projected from a point p with initial speed 15 m/s, down a plane inclined at an angle of $30°$ to the horizontal. The direction of projection is at right angles to the inclined plane. (The plane of projection is vertical and contains the line of greatest slope). Find

(i) the perpendicular height of the particle above the plane after t seconds and hence, or otherwise, show that the vertical height h of the particle above the plane after t seconds is

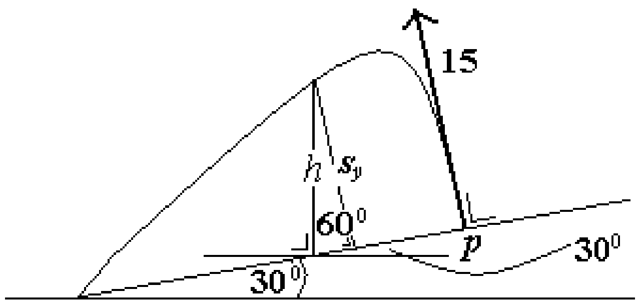
$$10\sqrt{3}\,t - 4.9t^2$$

(ii) the greatest vertical height it attains above the plane (i.e. the maximum value of h) correct to two places of decimals.

3.15.1 Solution

Using

$$\begin{aligned}
s_y &= u_y t - \frac{1}{2}g_y t^2 \\
\Rightarrow s_y(t) &= 15t - 4.9\cos 30°t^2 \\
\underset{\cos 30° \text{ from Fig 2}}{\Rightarrow} s_y(t) &= 15t - \frac{4.9\sqrt{3}}{2}t^2
\end{aligned}$$



Due to the geometry,

$$\begin{aligned} \sin 60^\circ &= \frac{s_y}{h} \\ \Rightarrow h &= \frac{s_y}{\sin 60^\circ} \\ \underset{\sin 60^\circ \text{ from Fig 2}}{\Rightarrow} h &= \frac{2}{\sqrt{3}}\left(15t - \frac{4.9\sqrt{3}}{2}t^2\right) \\ \Rightarrow h &= \frac{30}{\sqrt{3}}t - 4.9t^2 \\ \Rightarrow h &= 10\sqrt{3}t - 4.9t^2 \end{aligned}$$

To maximise $h(t)$ note that it is a concave down[2] quadratic and hence has a single local maximum when

$$\begin{aligned} \frac{dh}{dt} &= 0 \\ \Rightarrow 10\sqrt{3} - gt_{\text{max}} &= 0 \\ \Rightarrow t_{\text{max}} &= \frac{10\sqrt{3}}{g} \\ \Rightarrow h_{\text{max}} = h(t_{\text{max}}) = 10\sqrt{3}\left(\frac{10\sqrt{3}}{g}\right) - \frac{g}{2}\left(\frac{300}{g^2}\right) \\ \Rightarrow h_{\text{max}} &= \frac{300}{g} - \frac{1}{2}.\frac{300}{g} = \frac{150}{g} \\ \Rightarrow h_{\text{max}} &\simeq 15.31 \text{ m} \end{aligned}$$

[2]or *sad*

3.16 Problem: LC HL 2007: [Part(b)]

A plane is inclined at an angle 45° to the horizontal. A particle is projected up the plane with initial speed u at an angle θ to the ***horizontal****. The plane of projection is vertical and contains the line of greatest slope. The particle is moving horizontally when it strikes the inclined plane.*
Show that $\tan\theta = 2$.

3.16.1 Solution

If the particle lands horizontally then the landing angle is 45°.

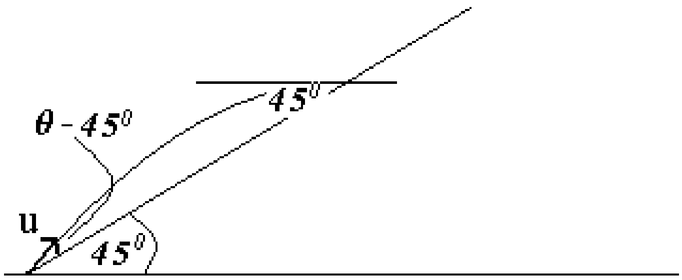


Figure 3: If the particle lands horizontally then as alternate angles the landing angle is equal to the angle the plane makes with the horizontal.

The landing angle is given by:

$$\tan l = \frac{-v_y}{v_x} \tag{54}$$

$$\Rightarrow \frac{-v_y}{v_x} = \tan 45^\circ = 1 \tag{55}$$

$$\Rightarrow -v_y = v_x \tag{56}$$

where v_y, v_x are the final speeds in the y- and x-directions. Using

$$\begin{aligned}
s_y(T) = 0 &= u\sin(\theta - 45^\circ)T - \frac{1}{2}g\cos 45^\circ.T^2 \\
\Rightarrow T &= \frac{2u\sin(\theta - 45^\circ)}{g\cos 45^\circ} \\
\Rightarrow v_y &= u\sin(\theta - 45^\circ) - \cancel{g\cos 45^\circ}.\frac{2u\sin(\theta - 45^\circ)}{\cancel{g\cos 45^\circ}} \\
\Rightarrow v_y &= -u\sin(\theta - 45^\circ) \\
\underset{\sin 45^\circ = 1/\sqrt{2} = \cos 45^\circ}{\Rightarrow} \quad v_x &= u\cos(\theta - 45^\circ) - \cancel{g\sin 45^\circ}\left(\frac{2u\sin(\theta - 45^\circ)}{\cancel{g\cos 45^\circ}}\right) \\
\Rightarrow v_x &= u(\cos(\theta - 45^\circ) - 2\sin(\theta)) \\
\underset{(56)}{\Rightarrow} \sin(\theta - 45^\circ) &= \cos(\theta - 45^\circ) - 2\sin(\theta - 45^\circ) \\
\Rightarrow 3\sin(\theta - 45^\circ) &= \cos(\theta - 45^\circ) \\
\Rightarrow \tan(\theta - 45^\circ) &= \frac{1}{3} \\
\underset{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}{\Rightarrow} \frac{\tan\theta - 1}{1 + \tan\theta} &= \frac{1}{3} \\
\Rightarrow 1 + \tan\theta &= 3\tan\theta - 3 \\
\Rightarrow 2\tan\theta &= 4 \\
\Rightarrow \tan\theta &= 2
\end{aligned}$$

□

3.17 Problem: LC HL 2004: [Part(b)]

*A particle is projected up an inclined plane with initial speed u m/s. The line of projection makes an angle α **with the horizontal** and the inclined plane makes an angle θ with the horizontal. (The plane of projection is vertical and contains the line of greatest slope.)*
If the particle strikes the inclined plane at right angles, show that

$$\tan\alpha = \frac{1 + 2\tan^2\theta}{\tan\theta}$$

3.17.1 Solution

If the particle strikes the plane at right angles $v_x(T) = 0$ where T is the time of flight.

$$\begin{aligned}
T &= \frac{2u\sin(\alpha-\theta)}{g\cos\theta} \\
\Rightarrow v_x(T) = u\cos(\alpha-\theta) - \cancel{g}\sin\theta\left(\frac{2u\sin(\alpha-\theta)}{\cancel{g}\cos\theta}\right) &= 0 \\
\underset{\times 1/u}{\Rightarrow} \cos(\alpha-\theta) &= 2\tan\theta\sin(\alpha-\theta) \\
\Rightarrow 1 &= 2\tan\theta\tan(\alpha-\theta) \\
\Rightarrow 1 &= 2\tan\theta\left(\frac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta}\right) \\
\Rightarrow 1 + \tan\alpha\tan\theta &= 2\tan\theta\tan\alpha - 2\tan^2\theta \\
\Rightarrow \tan\alpha\tan\theta &= 1 + 2\tan^2\theta \\
\Rightarrow \tan\alpha &= \frac{1+2\tan^2\theta}{\tan\theta}
\end{aligned}$$

□

3.18 Problem: LC HL 2002: [Part(b)]

*A particle is projected with speed u m/s at an angle θ **to the horizontal**, up a plane inclined at an angle β to the horizontal. (The plane of projection is vertical and contains the line of greatest slope). The particle strikes the plane at right angles,*

(i) *Show that* $2\tan\beta\tan(\theta-\beta) = 1$

(ii) *Hence, or otherwise, show that if $\theta = 2\beta$, the range of the particle up the inclined plane is $u^2/(g\sqrt{3})$*

3.18.1 Solution

(i) This is shown in four lines in the above solution. □

(ii) If $\theta = 2\beta$;

$$\begin{aligned}
2\tan\beta\tan(2\beta-\beta) &= 1 \\
\Rightarrow 2\tan^2\beta &= 1 \\
\Rightarrow \tan\beta &= \frac{1}{\sqrt{2}}
\end{aligned}$$

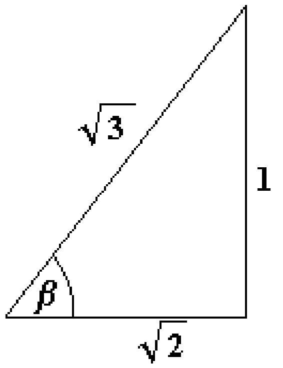


Figure 4: The Model Triangle for β.

Range is $s_x(T)$:

$$\begin{aligned}
R &= u\cos\beta.\frac{2u\sin\beta}{g\cos\beta} - \frac{1}{\cancel{2}}\cancel{g}\sin\beta.\frac{\overset{2}{\cancel{4}}u^2\sin^2\beta}{g^{\cancel{2}}\cos^2\beta} \\
\Rightarrow R &= \frac{2u^2\sin\beta}{g\cos^2\beta}(\cos^2\beta - \sin^2\beta) \\
\Rightarrow R &= \frac{2u^2}{g}.\frac{1}{\sqrt{3}}.\frac{3}{2}\left(\frac{2}{3} - \frac{1}{3}\right) \\
\Rightarrow R &= \frac{\cancel{2}u^2}{g\sqrt{3}}.\frac{\cancel{3}}{\cancel{2}}\left(\frac{1}{\cancel{3}}\right) \\
\Rightarrow R &= \frac{u^2}{g\sqrt{3}}
\end{aligned}$$

□

3.19 Problem: LC HL 2000: [Part(b)]

*A particle is projected at an angle $\alpha = \tan^{-1} 3$ **to the horizontal** up a plane inclined at an angle θ to the horizontal. (The plane of projection is vertical and contains the line of greatest slope). The particle strikes the plane at right angles.*
Find the two possible values for θ.

3.19.1 Solution

From Problem 3.17:

$$\begin{aligned}
\tan\alpha &= \frac{1 + 2\tan^2\theta}{\tan\theta} \\
\Rightarrow 3 &= \frac{1 + 2\tan^2\theta}{\tan\theta} \\
\Rightarrow 3\tan\theta &= 1 + 2\tan^2\theta \\
\Rightarrow 2\tan^2\theta - 3\tan\theta + 1 &= 0 \\
\Rightarrow 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 &= 0 \\
\Rightarrow 2\tan\theta(\tan\theta - 1) - 1(\tan\theta - 1) &= 0 \\
\Rightarrow (2\tan\theta - 1)(\tan\theta - 1) &= 0 \\
\Rightarrow \theta = \arctan(1/2) \simeq 26.565^\circ \text{ , or } 45^\circ&.
\end{aligned}$$

3.20 Problem: LC HL 1999 [Parts (a) & (b)(i)]

A particle is projected from a point p up an inclined plane with a speed of $4g\sqrt{2}$ m/s at an angle $\tan^{-1}(1/3)$ to the inclined plane. The plane is inclined at an angle θ to the horizontal. (The plane of projection is vertical and contains the line of greatest slope). The particle is moving horizontally when it strikes the plane at the point q.

(a) Find the two possible values for θ.

(b)(i) If $\tan\theta = 0.5$ then find the magnitude of the velocity with which the particle strikes the inclined plane at q.

3.20.1 Solution

(a) Let $\alpha := \tan^{-1}(1/3)$. As in Figure 18, the landing angle is θ as the particle lands horizontally. Hence

$$\tan\theta = \frac{-v_y(T)}{v_x(T)}$$

where T is the time of flight;

$$\begin{aligned}
T &= \frac{2u\sin\alpha}{g\cos\theta} \\
\Rightarrow s_y(T) &= -u\sin\alpha \text{ , and} \\
s_x(T) &= u\cos\alpha - \cancel{g}\sin\theta.\frac{2u\sin\alpha}{\cancel{g}\cos\theta} \\
\Rightarrow s_x(T) &= u\cos\alpha - 2u\tan\theta\sin\alpha \\
\Rightarrow \tan\theta &= \frac{\cancel{u}\sin\alpha}{\cancel{u}\cos\alpha - 2\cancel{u}\tan\theta\sin\alpha} \\
\Rightarrow \tan\theta\cos\alpha - 2\tan^2\theta\sin\alpha &= \sin\alpha \\
\underset{\times 1/\sin\alpha}{\Rightarrow} \frac{\tan\theta}{\tan\alpha} - 2\tan^2\theta &= 1 \\
\underset{\tan\alpha = 1/3}{\Rightarrow} 2\tan^2\theta - 3\tan\theta + 1 &= 0 \\
\Rightarrow 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 &= 0 \\
\Rightarrow 2\tan\theta(\tan\theta - 1) - 1(\tan\theta - 1) &= 0 \\
\Rightarrow (2\tan\theta - 1)(\tan\theta - 1) &= 0 \\
\Rightarrow \theta = \arctan(1/2) \simeq 26.565^\circ &\text{ , or } 45^\circ.
\end{aligned}$$

(b)(i) The final *speed* of the particle is given by:

$$|\mathbf{v}(T)| = \sqrt{v_x^2(T) + v_y^2(T)} \tag{57}$$

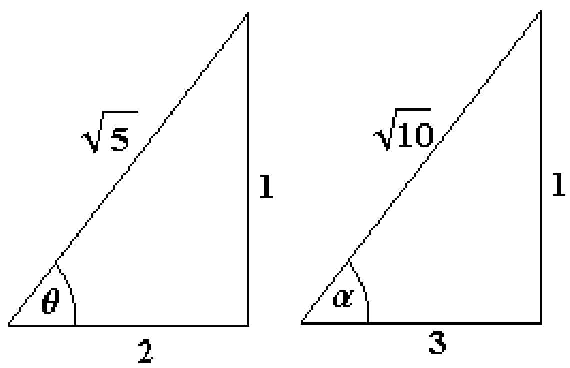


Figure 5: The model triangles for θ & α

Now from (a);

$$\begin{aligned}
v_y(T) &= -u\sin\alpha \\
\Rightarrow v_y(T) &= -4g\frac{\sqrt{2}}{\sqrt{10}} \\
\Rightarrow v_y^2(T) &= \frac{16g^2}{5} \text{ , and} \\
v_x(T) &= u\cos\alpha - 2u\tan\theta\sin\alpha \\
\Rightarrow v_x(T) &= 4g\sqrt{2}\frac{3}{\sqrt{10}} - 4g\sqrt{2}\frac{1}{\sqrt{10}} \\
\Rightarrow v_x(T) &= \frac{4g\sqrt{2}}{\sqrt{10}}(3-1) = \frac{8g}{\sqrt{5}} \\
\Rightarrow v_x^2(T) &= \frac{64g^2}{5} \\
\Rightarrow |\mathbf{v}(T)| &= \sqrt{\frac{80g^2}{5}} = \sqrt{16g^2} \\
\Rightarrow |\mathbf{v}(T)| &= 4g
\end{aligned}$$

3.21 Problem: LC HL 1999: [Part (b)]

A particle is projected down a slope which is inclined at $45°$ *to the horizontal. The particle is projected from a point on the slope and has an initial velocity of* $7\sqrt{2}$ *m/s at an angle* α *to the inclined plane. Find the value of* α *if*

(i) the particle first hits the slope after 2 seconds

(ii) the landing angle with the slope is $\tan^{-1}(1/3)$

3.21.1 Solution

(i) The particle hits the slope when $s_y = 0$:

$$\begin{aligned}
s_y &= u\sin\alpha t - \frac{1}{2}g\sin 45°t^2 \\
\Rightarrow 0 &\overset{!}{=} 7\sqrt{2}\sin\alpha.(2) - \frac{1}{\not{2}}g\sin 45°(2)^{\not{2}} \\
\Rightarrow 0 &= 14\sqrt{2}\sin\alpha - \frac{2}{\sqrt{2}}g \\
\Rightarrow \sin\alpha &= \frac{2g}{28} = \frac{19.6}{28} \\
\Rightarrow \alpha &= \sin^{-1}\left(\frac{19.6}{28}\right) \simeq 44.4°
\end{aligned}$$

(ii) The landing angle l is given by:

$$\tan l = \frac{-v_y(T)}{v_x(T)} \tag{58}$$

where T is the time of flight and $v_y(T)$ and $v_x(T)$ are the final velocities in the y- and x-directions respectively. The time of flight is when $s_y = 0$:

$$
\begin{aligned}
s_y = u\sin\alpha t - \frac{1}{2}g\cos 45^\circ t^2 &\stackrel{!}{==} \\
\Rightarrow T &= \frac{2u\sin\alpha}{g\cos 45^\circ} \\
\Rightarrow v_y(T) = u\sin\alpha - \cancel{g\cos 45^\circ}&\left(\frac{2u\sin\alpha}{\cancel{g\cos 45^\circ}}\right) \\
\Rightarrow v_y(T) = u\sin\alpha - 2u\sin\alpha &= -u\sin\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
v_x(T) = u\cos\alpha + \cancel{g\sin 45^\circ}&\left(\frac{2u\sin\alpha}{\cancel{g\cos 45^\circ}}\right) \\
\Rightarrow v_x(T) &= u\cos\alpha + 2u\sin\alpha
\end{aligned}
$$

Now if $l = \tan^{-1}(1/3)$:

$$
\begin{aligned}
\Rightarrow \frac{1}{3} &= \frac{-v_y(T)}{v_x(T)} \\
\Rightarrow v_x(T) &= -3v_y(T) \\
\Rightarrow \cancel{u}\cos\alpha + 2\cancel{u}\sin\alpha &= 3\cancel{u}\sin\alpha \\
\Rightarrow \cos\alpha &= \sin\alpha \\
\Rightarrow \tan\alpha &= 1 \\
\Rightarrow \alpha &= 45^\circ
\end{aligned}
$$

3.22 Problem: F.A.M. Exercises 3.D 3(iii) [LC 1979]

A plane is inclined at an angle α to the horizontal. A particle is projected up the plane with a speed u at an angle θ to the plane. The plane of projection is vertical and contains the line of greatest slope.
Prove that the particle will strike the plane horizontally if

$$\tan\theta = \frac{\sin\alpha\cos\alpha}{(2 - \cos^2\alpha)}$$

3.22.1 Solution

Let l be the landing angle.

$$\tan l = \frac{-v_y(T)}{v_x(T)} \tag{59}$$

Suppose $\tan l = \tan\alpha$. As both l and α are greater than 0° and less than 90°, and $\tan^{-1}$ is one-to-one on $[0^\circ, 90^\circ]$;

$$
\begin{aligned}
\tan^{-1}(\tan l) &= \tan^{-1}(\tan\alpha) \\
\Rightarrow l &= \alpha
\end{aligned}
$$

which implies that the particle strikes horizontally. Now from before

$$\begin{aligned} v_y(T) &= -u \sin\theta \\ v_x(T) &= u\cos\theta - 2u\tan\alpha\sin\theta \\ \Rightarrow \tan l &= \frac{\sin\theta}{\cos\theta - 2\tan\alpha\sin\theta} \end{aligned}$$

Now if

$$\tan\theta = \frac{\sin\alpha\cos\alpha}{(2-\cos^2\alpha)}$$

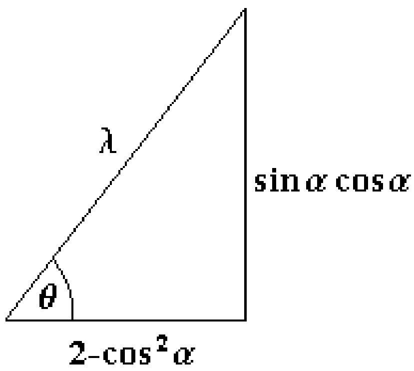


Figure 6: The model triangle for θ

Hence, where λ is the hypotenuse of the model triangle, by Pythagoras Theorem;

$$\begin{aligned} \lambda^2 &= \sin\alpha\cos^2\alpha + (2-\cos^2\alpha)^2 \\ \Rightarrow \lambda &= \sqrt{\sin\alpha\cos^2\alpha + (2-\cos^2\alpha)^2} \\ \Rightarrow \tan l &= \frac{\frac{\sin\alpha\cos\alpha}{\cancel{\lambda}}}{\frac{(2-\cos^2\alpha)}{\cancel{\lambda}} - \frac{2\tan\alpha\sin\alpha\cos\alpha}{\cancel{\lambda}}} \\ \underset{\tan\sin\cos=\sin^2}{\Rightarrow} \tan l &= \frac{\sin\alpha\cos\alpha}{2-\cos^2 - 2\sin^2\alpha} \\ \underset{\cos^2+\sin^2=1}{\Rightarrow} \tan l &= \frac{\sin\alpha\cos\alpha}{2\cos^2\alpha + \cancel{2\sin^2}\alpha - \cos^2\alpha - \cancel{2\sin^2}\alpha} \\ \Rightarrow \tan l &= \frac{\sin\alpha\cancel{\cos\alpha}}{\cos^{\cancel{2}}\alpha} = \frac{\sin\alpha}{\cos\alpha} \\ \Rightarrow \tan l &= \tan\alpha \end{aligned}$$

Hence the particle strikes the plane horizontally. □

3.22.2 Remark

Questions of this subtlety rarely come up. I wouldn't expect anything of this difficulty in LC 2010 Q.3(b). This question is taken from LC 1979. Consider these statements P & Q;

P. the particle strikes the plane horizontally

Q.

$$\tan\theta = \frac{\sin\alpha\cos\alpha}{(2-\cos^2\alpha)}$$

Ordinarily one is asked to show that $P \Rightarrow Q$ (read P *implies* Q). That is assuming P, prove Q; and this is straightforward. However in this example one is asked to show $Q \Rightarrow P$. Note the difference. If one proved $P \Rightarrow Q$ one would not receive full marks as the question has been fundamentally misunderstood. Note our approach here. It is shown for $l, \alpha \in [0°, 90°]$:

If $\tan l = \tan\alpha$, then $l = \alpha$

and $\tan l$ is computed; and shown to be equal to $\tan\alpha$. Thence $l = \alpha$ and implies that the particle lands horizontally.

4 Relative Velocity

4.1 Problem: F.A.M. Q.4.A.6

With O as the origin the position vectors of P, Q, S are $\mathbf{r}_P = \mathbf{i} - 2\mathbf{j}$, $\mathbf{r}_Q = -4\mathbf{i} + \mathbf{j}$, $\mathbf{r}_S = -3\mathbf{i} + 5\mathbf{j}$. Find $\mathbf{r}_{QP}$, the displacement of Q relative to P. Find in terms of $\mathbf{i}$ and $\mathbf{j}$ the position vector of T if $\mathbf{r}_{TS} = \mathbf{r}_{QP}$.

4.1.1 Solution

Using

$$\begin{aligned} \mathbf{r}_{AB} &= \mathbf{r}_A - \mathbf{r}_B \\ \Rightarrow \mathbf{r}_{QP} &= \mathbf{r}_Q - \mathbf{r}_P \\ \Rightarrow \mathbf{r}_{QP} &= -5\mathbf{i} + 3\mathbf{j} \end{aligned}$$

Let

$$\begin{aligned} \mathbf{r}_T &= x\mathbf{i} + y\mathbf{j} \\ \Rightarrow \mathbf{r}_{TS} &= \mathbf{r}_T - \mathbf{r}_S \\ \Rightarrow \mathbf{r}_{TS} &= (x+3)\mathbf{i} + (y-5)\mathbf{j} \end{aligned}$$

Now for $\mathbf{r}_{TS} = \mathbf{r}_{QP}$, comparing components:

$$\begin{aligned} -5\mathbf{i} + 3\mathbf{j} &\stackrel{!}{=} (x+3)\mathbf{i} + (y-5)\mathbf{j} \\ \Rightarrow \mathbf{r}_T &= -8\mathbf{i} + 8\mathbf{j} \end{aligned}$$

4.2 Problem: F.A.M. Q.4.A.7

A train is travelling on a straight track with velocity $30\mathbf{j}$ *and a car, visible from the train, is travelling on a straight road with velocity* $10\mathbf{i} + 6\mathbf{j}$ *where speeds are measured in m/s. Calculate the magnitude and direction of the car's velocity as it appears to a person sitting on the train.*

4.2.1 Solution

Using

$$\begin{aligned}\mathbf{V}_{CT} &= \mathbf{V}_C - \mathbf{V}_T \\ \Rightarrow \mathbf{V}_{CT} &= 10\mathbf{i} - 24\mathbf{j} \\ \Rightarrow |\mathbf{V}_{CT}| &= \sqrt{10^2 + 24^2} = \sqrt{100 + 576} \\ \Rightarrow |\mathbf{V}_{CT}| &= 26\,\text{m/s}\end{aligned}$$

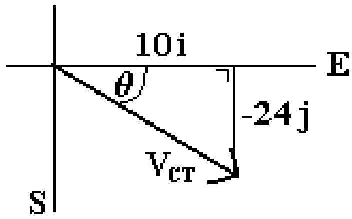


Figure 7: $\mathbf{V}_{CT}$ in the plane. Note θ. The direction of $\mathbf{V}_{CT}$ is E θ S.

Now

$$\begin{aligned}\tan\theta &= \frac{24}{10} = \frac{12}{5} \\ \Rightarrow \theta &= \tan^{-1}(12/5) \approx 67.38^\circ\end{aligned}$$

Hence the direction of $\mathbf{V}_{CT}$ is E 67.38° S.

4.3 Problem: F.A.M. Q.4.A.8

A particle P is 100 m due West of another particle Q. The velocity of P is $6\mathbf{i} + 2\mathbf{j}$ *m/s, and the velocity of Q is* $-4\mathbf{i} + 2\mathbf{j}$ *m/s. Show that P and Q are on collision course. How much time will pass before the collision occurs?*

4.3.1 Solution

Clearly $\mathbf{R}_{PQ} = -100\mathbf{i}$. Using

$$\begin{aligned}\mathbf{V}_{PQ} &= \mathbf{V}_P - \mathbf{V}_Q \\ \Rightarrow \mathbf{V}_{PQ} &= 10\mathbf{i}\end{aligned}$$

$\therefore$ the particles are on collision course. □

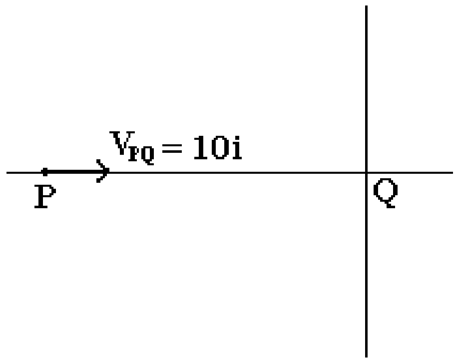


Figure 8: Particle P is due West of particle Q; in fact its position relative to particle Q is along the negative **i**-axis at $-100\mathbf{i}$ m. The velocity of particle P relative to Q is $+10\mathbf{i}$, back along the **i** axis towards particle Q. Therefore, they must be on collision course.

Using

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$
$$\Rightarrow t = \frac{100}{10} = 10 \text{ s}$$

4.4 Problem: F.A.M. Q.4.A.9

K is a particle which is 20 m due West of another particle T. Their velocities are $\mathbf{i} + 2\mathbf{j}$ m/s and $-2\mathbf{i} - 2\mathbf{j}$ m/s respectively. Find the velocity of K relative to T. Find the shortest distance between them in subsequent motion.

4.4.1 Solution

Now $\mathbf{R}_{KT} = -20\mathbf{i}$. Using

$$\mathbf{V}_{KT} = \mathbf{V}_K - \mathbf{V}_T$$
$$\Rightarrow \mathbf{V}_{KT} = 3\mathbf{i} + 4\mathbf{j}$$

Now consider the position of K relative to T, $\mathbf{R}_{KT}$:
Now $\tan\theta = 4/3$:
Now from Figure 9:

$$\sin\theta = \frac{d}{20} \stackrel{!}{=} \frac{4}{5}$$
$$\Rightarrow d = \frac{80}{5} = 16\,\text{m}$$

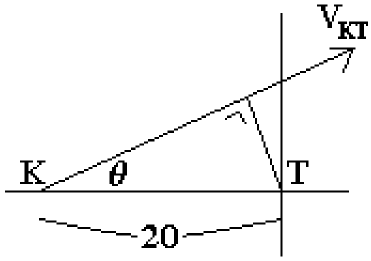


Figure 9: The shortest distance between K and T in the subsequent motion is the perpendicular distance, d, between T and the path of K relative to T.

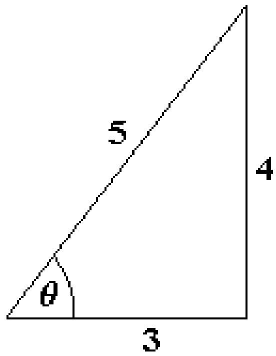


Figure 10: The model triangle for θ.

4.5 Problem: F.A.M. Q.4.A.10

A particle P is moving with velocity $-8\mathbf{i}+12\mathbf{j}$ while another particle Q is moving with velocity $7\mathbf{i}+4\mathbf{j}$. Find the velocity of P relative to Q. P is at the point $119\mathbf{i}$ when Q is at the origin O. Show the positions of P and Q on a diagram and show the path of P relative to Q. Calculate this distance of O from this path. What does this distance represent?

4.5.1 Solution

Using

$$\mathbf{V}_{PQ} = \mathbf{V}_P - \mathbf{V}_Q$$
$$\Rightarrow \mathbf{V}_{PQ} = -15\mathbf{i} + 8\mathbf{j}$$

Now from Figure 11:

$$\sin\theta = \frac{d}{119} \stackrel{!}{=} \frac{8}{17}$$
$$\Rightarrow d = 119\frac{8}{17} = 56$$

This distance represents the shortest distance between P and Q is subsequent motion.

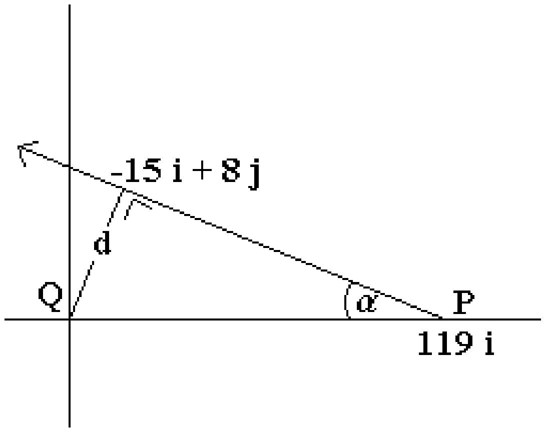


Figure 11: The position $\mathbf{R}_{PQ}$ and path $\mathbf{V}_{PQ}$ of P relative to Q. $\alpha = \tan^{-1}(8/15)$. d is the distance of O from the path.

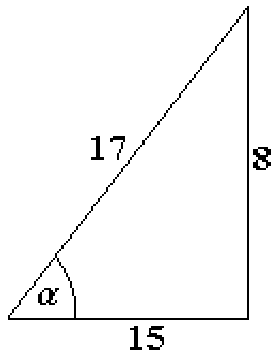


Figure 12: The model triangle for α. Sides 8 & 15 are given by $\alpha = \tan^{-1}(8/15)$. The hypotenuse is $\sqrt{8^2 + 15^2} = \sqrt{289} = 17$

4.6 Problem: F.A.M. Q.4.A.11

(a) If $|t\mathbf{i} + 3\mathbf{j}| = 5$*, find the value of* $t > 0$*,* $t \in \mathbb{R}$*.*

(b) (i) A ship K is 60 km due West of another ship M which is travelling with velocity $-2\mathbf{i} + 3\mathbf{j}$ *km/hr, find in terms of* $\mathbf{i}$ *and* $\mathbf{j}$ *its velocity if it is to intercept (collide with) M. (Hint: K will have to travel with the same* $\mathbf{j}$*-speed as ship M in order to keep on collision course.)*

(ii) When will collision occur?

4.6.1 Solution

(a) For any vector $\mathbf{a} := x\mathbf{i} + y\mathbf{j}$;

$$\begin{aligned} |\mathbf{a}| &= \sqrt{x^2 + y^2} \\ \Rightarrow |t\mathbf{i} + 3\mathbf{j}| &= \sqrt{t^2 + 9} \overset{!}{=} 5 \\ \Rightarrow t^2 + 9 &= 25 \\ \Rightarrow t^2 &= 16 \\ \Rightarrow t &= \pm 4 \\ \underset{t>0}{\Rightarrow} t &= 4. \end{aligned}$$

(b) (i) $\mathbf{R}_{KM} = -60\mathbf{i}$ hence for collision $\mathbf{V}_{KM} = k\mathbf{i}$ with $k > 0$. Let

$$\begin{aligned} \mathbf{V}_K = x\mathbf{i} + y\mathbf{j} \ &, \text{ with} \\ \sqrt{x^2 + y^2} &= 5 \end{aligned}$$

Using

$$\begin{aligned} \mathbf{V}_{KM} &= \mathbf{V}_K - \mathbf{V}_M \\ \Rightarrow \mathbf{V}_{KM} &= (x + 2)\mathbf{i} + (y - 3)\mathbf{j} \end{aligned}$$

For collision course, $y - 3 = 0$, $y = 3$; and $x + 2 > 0$, $x > -2$:

$$\begin{aligned} \sqrt{x^2 + 9} &= 5 \\ \Rightarrow x &= \pm 4 \\ \underset{x>-2}{\Rightarrow} x &= 4 \\ \Rightarrow \mathbf{V}_K &= 4\mathbf{i} + 3\mathbf{j} \end{aligned}$$

(ii) Now $\mathbf{V}_{KM} = 6\mathbf{i}$. Using

$$\begin{aligned} t &= \frac{\text{relative distance}}{\text{relative speed}} \\ \Rightarrow t &= \frac{60}{6} = 10 \text{ h} \end{aligned}$$

4.7 Problem: F.A.M. Q.4.A.12

A ship P is 3.4 km due West of another ship Q. P is moving with speed $5\sqrt{2}$ m/s in a NE direction. Q can travel at 13 m/s. If they are on collision course find the velocity of Q in terms of **i** *and* **j**. *When will collision occur?*

4.7.1 Solution

Let $\mathbf{V}_Q = x\mathbf{i} + y\mathbf{j}$ with

$$|\mathbf{V}_Q| = 13$$
$$\sqrt{x^2+y^2} \stackrel{!}{=} 13$$

$\mathbf{R}_{PQ} = -3400\,\mathbf{i}$, hence for P to collide with Q, it is required that:

$$\mathbf{V}_{PQ} = k\,\mathbf{i}\ ,\ k > 0 \tag{60}$$

Now $\mathbf{V}_P$:

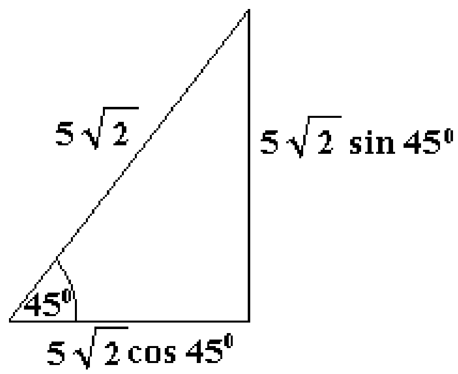


Figure 13: NE is the direction E 45° N as shown. As $\sin 45° = 1/\sqrt{2} = \cos 45°$, $\mathbf{V}_P = 5\,\mathbf{i}+5\,\mathbf{j}$.

Using

$$\mathbf{V}_{PQ} = \mathbf{V}_P - \mathbf{V}_Q$$
$$\Rightarrow \mathbf{V}_{PQ} = (5-x)\mathbf{i} + (5-y)\mathbf{j}$$

Hence for collision, $y = 5$. Now $|\mathbf{V}_P| = 13$:

$$\sqrt{x^2+5^2} = 13$$
$$\Rightarrow x^2 = 13^2 - 5^2 = 144$$
$$\Rightarrow x = \pm 12$$
$$\underset{(5-x)\stackrel{!}{>}0}{\Rightarrow} \quad x = -12$$
$$\Rightarrow \mathbf{V}_Q = -12\,\mathbf{i} + 5\,\mathbf{j}$$

Now $\mathbf{V}_{PQ} = 17\,\mathbf{i}$. Using

$$t = \frac{\text{relative distance}}{\text{relative speed}}$$
$$\Rightarrow t = \frac{\overset{200}{\cancel{3400}}}{\cancel{17}} = 200 \text{ s}$$

4.8 Problem: F.A.M. Q.4.A.13

Ship T is 100 km due West of ship Q. T is travelling at 10 km/hr in a direction E 30° S. Q is travelling at 20 km/hr in direction W 45° N. Find in terms of **i** *and* **j** *the velocity of T, the velocity of Q, and the velocity of T relative to Q. Find, also, the magnitude and direction of the velocity of T relative to Q and hence find the shortest distance between them in subsequent motion correct to one decimal place.*

4.8.1 Solution

Consider $\mathbf{V}_T$:

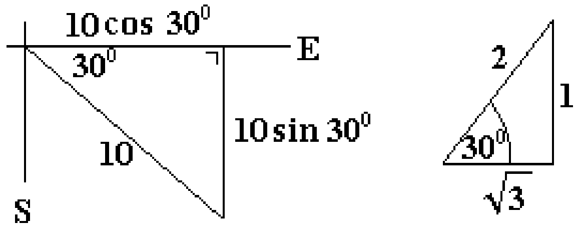


Figure 14: $\mathbf{V}_T$ and the model triangle for 30°. $\mathbf{V}_T = 10\cos 30°\,\mathbf{i} - 10\sin 30°\,\mathbf{j}$.

$$\mathbf{V}_T = 5\sqrt{3}\,\mathbf{i} - 5\,\mathbf{j} \tag{61}$$

Consider now $\mathbf{V}_Q$:

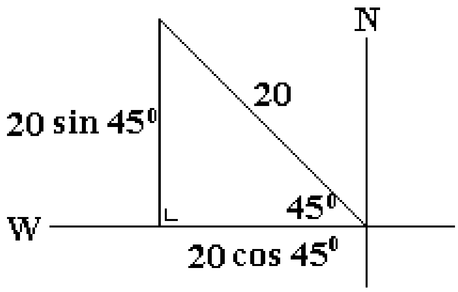


Figure 15: $\mathbf{V}_Q = -20\cos 45°\,\mathbf{i} + 20\sin 45°\,\mathbf{j}$.

$$\begin{aligned}
\mathbf{V}_Q &= -20\frac{1}{\sqrt{2}}\,\mathbf{i} + 20\frac{1}{\sqrt{2}}\,\mathbf{j} \\
\Rightarrow \mathbf{V}_Q &= -20\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\,\mathbf{i} + 20\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\,\mathbf{j} \\
\Rightarrow \mathbf{V}_Q &= -\frac{20}{2}\sqrt{2}\,\mathbf{i} + \frac{20}{2}\sqrt{2}\,\mathbf{j} \\
\Rightarrow \mathbf{V}_Q &= -10\sqrt{2}\,\mathbf{i} + 10\sqrt{2}\,\mathbf{j}
\end{aligned}$$

Using

$$\begin{aligned}
\mathbf{V}_{TQ} &= \mathbf{V}_T - \mathbf{V}_Q \\
\Rightarrow \mathbf{V}_{TQ} &= (5\sqrt{3} + 10\sqrt{2})\mathbf{i} + (-5 - 10\sqrt{2})\mathbf{j} \\
\Rightarrow \mathbf{V}_{TQ} &= 5(\sqrt{3} + 2\sqrt{2})\mathbf{i} - 5(1 + 2\sqrt{2})\mathbf{j}
\end{aligned}$$

For a general vector $\mathbf{a} = x\,\mathbf{i} + y\,\mathbf{j}$:

$$\begin{aligned}
|\mathbf{a}| &= \sqrt{x^2 + y^2} \\
\underset{(-x)^2 = x^2}{\Rightarrow} |\mathbf{V}_{TQ}| &= \sqrt{5^2(\sqrt{3} + 2\sqrt{2})^2 + 5^2(1 + 2\sqrt{2})^2} \\
\Rightarrow |\mathbf{V}_{TQ}| &= 5\sqrt{3 + 4\sqrt{6} + 8 + 1 + 4\sqrt{2} + 8} \\
\Rightarrow |\mathbf{V}_{TQ}| &= 5\sqrt{20 + 4\sqrt{2} + 4\sqrt{6}} \\
\underset{a\sqrt{b^2 x} = ab\sqrt{x}}{\Rightarrow} |\mathbf{V}_{TQ}| &= 10\sqrt{5 + \sqrt{2} + \sqrt{6}} \approx 29.772 \text{ km/hr}
\end{aligned}$$

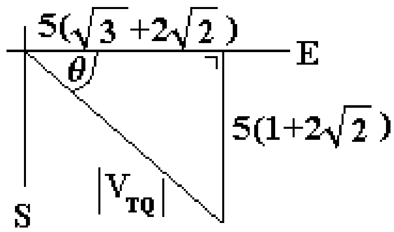


Figure 16: The direction of $\mathbf{V}_{TQ}$ is given by $\theta = \tan^{-1}((\cancel{5}(1 + 2\sqrt{2}))/(\cancel{5}(\sqrt{3} + 2\sqrt{2})))$

$$\tan\theta = \frac{1 + 2\sqrt{2}}{\sqrt{3} + 2\sqrt{2}} \approx 40.01^\circ$$

Hence the direction of $\mathbf{V}_{TQ}$ is given by E 40.01° S.

Now $\mathbf{R}_{TQ} = -100\,\mathbf{i}$.
Now from Figure 17;

$$\begin{aligned}
\sin 40.01 &= \frac{d}{100} \\
\Rightarrow d &= 100 \sin 40.01 \\
\Rightarrow d &= 100(0.64291) \approx 64.3 \text{ m}
\end{aligned}$$

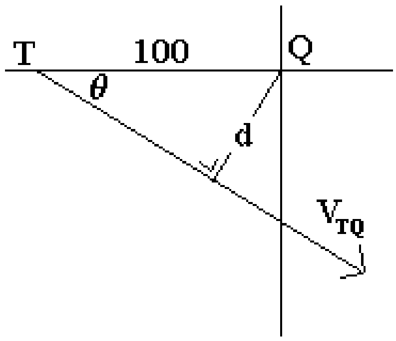


Figure 17: Ship T is initially 100 km West of ship Q. Relative to Q, T travels along $\mathbf{V}_{TQ}$; hence the shortest distance between T and Q in the subsequent motion is given by d.

4.9 Problem: LC HL 2009: [Part(a)]

(i) $\mathbf{V}_A = 15\,\mathbf{i}$ m/s and $\mathbf{V}_B = 20\,\mathbf{j}$ m/s. Using

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B$$
$$\Rightarrow \mathbf{V}_{AB} = 15\,\mathbf{i} - 20\,\mathbf{j}$$

When both cars are 800 m from the intersection $\mathbf{R}_A = -800\,\mathbf{i}$ and $\mathbf{R}_B = -800\,\mathbf{j}$. Using

$$\mathbf{R}_{AB} = \mathbf{R}_A - \mathbf{R}_B$$
$$\Rightarrow \mathbf{R}_{AB} = -800\,\mathbf{i} + 800\,\mathbf{j}$$

Now

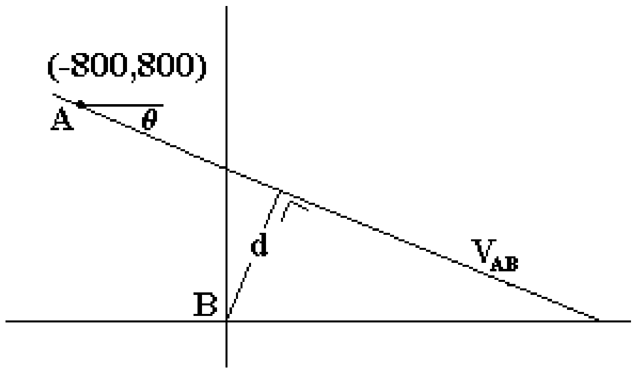


Figure 18: The position of A relative to B and the subsequent motion of A relative to B.

As a line, the slope of $\mathbf{V}_{AB}$ is $m = -\tan\theta = -4/3$. Also $(-800, 800) \in L := \mathbf{V}_{AB}$.

Using

$$\begin{aligned} L &\equiv y - y_1 = m(x - x_1) \\ \Rightarrow L &\equiv y - 800 = -\frac{4}{3}(x + 800) \\ \Rightarrow L &\equiv 3y - 2400 = -4x - 3200 \\ \Rightarrow L &\equiv 4x + 3y + 800 = 0 \end{aligned}$$

Using, where d is the perpendicular distance between a point $p(x_1, y_1)$ and the line $ax + by + c = 0$;

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \underset{x_1=0=y_1}{=} \frac{|c|}{\sqrt{a^2+b^2}}$$
$$\Rightarrow d = \frac{800}{5} = 160 \text{ m}$$

(ii) Using, where θ is the angle between two lines of slope m_1 and m_2, with respect to the angle α between the lines $\mathbf{R}_{AB}$ ($m_1 = -1$) and $\mathbf{V}_{AB}$ ($m_2 = -4/3$);

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2}$$
$$\Rightarrow \tan\theta = \pm\frac{-1 + 4/3}{1 + 4/3} = \pm\frac{-1/\not{3}}{7/\not{3}} \underset{\alpha\in[0,90°]}{=} \frac{1}{7}$$

Hence

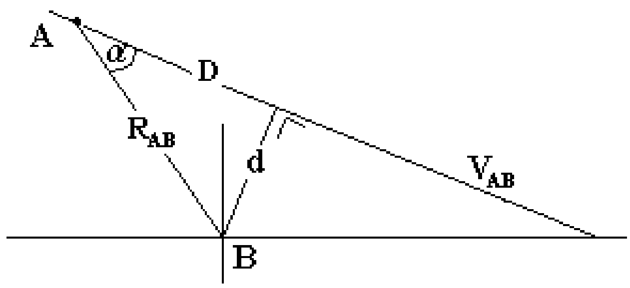


Figure 19: $\tan\alpha = d/D$

$$\tan\alpha = \frac{1}{7} = \frac{d}{D}$$
$$\Rightarrow D = 7d = 1120 \text{ m}$$

Using

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$
$$\Rightarrow \text{time} = \frac{1120}{|\mathbf{V}_{AB}|} \underset{|\mathbf{V}_{AB}|=25}{=} \frac{224}{5} \text{ s}$$

Hence in this time, using $s = ut$:

$$s_A = 15.\frac{224}{5} = 672 \text{ m , and}$$
$$s_B = 20.\frac{224}{5} = 896 \text{ m}$$

Hence A is 128 m from the intersection and B is 96 m from the intersection.

4.10 Problem: LC HL 2008: [Part(a)]

(i) $\mathbf{V}_C = 1.5\,\mathbf{i}$ m/s and $\mathbf{V}_D = 2\,\mathbf{j}$ m/s. Using

$$\mathbf{V}_{CD} = \mathbf{V}_C - \mathbf{V}_D$$
$$\Rightarrow \mathbf{V}_{CD} = 1.5\,\mathbf{i} - 2\,\mathbf{j}$$

(ii) When D passes the intersection $\mathbf{R}_{CD} = -100\,\mathbf{i}$ m. Consider

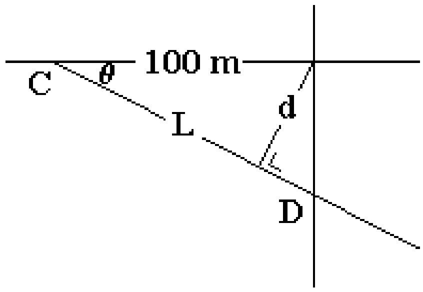


Figure 20: $\cos\theta = L/100$.

Now $\tan\theta = -4/3$

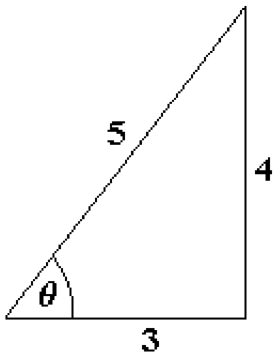


Figure 21: The model triangle for θ.

$$\cos\theta = \frac{3}{5} = \frac{L}{100}$$
$$\Rightarrow L = \frac{300}{5} = 60 \text{ m}$$

Using

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$
$$\Rightarrow \text{time} = \frac{60}{|\mathbf{V}_{CD}|} = \frac{60}{\sqrt{9/4+4}} = \frac{60}{\sqrt{25/4}} = 24 \text{ s}$$

Using

$$s = ut$$
$$\Rightarrow s_C = 1.5(24) = 36 \text{ m}$$
$$\Rightarrow 64 \text{ m from the intersection}$$

4.11 Problem: LC HL 2007: [Part(a)]

(i) $\mathbf{V}_B = -24\,\mathbf{i}$ km/hr and $\mathbf{V}_A = 32\,\mathbf{j}$ km/hr. Using

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B$$
$$\Rightarrow \mathbf{V}_{AB} = 24\,\mathbf{i} + 32\,\mathbf{j}$$

(ii) If ship B is 8 km NE of ship A, ship A is 8 km SW of ship B:

$$\mathbf{R}_{AB} = -8\cos 45^\circ\,\mathbf{i} - 8\sin 45^\circ\,\mathbf{j}$$
$$\Rightarrow \mathbf{R}_{AB} = -\frac{8}{\sqrt{2}}\,\mathbf{i} - \frac{8}{\sqrt{2}}\,\mathbf{j}$$
$$\underset{\times\sqrt{2}/\sqrt{2}}{\Rightarrow} \mathbf{R}_{AB} = -4\sqrt{2}\,\mathbf{i} - 4\sqrt{2}\,\mathbf{j}$$

Considered as a line, $V := \mathbf{V}_{AB}$ has slope $32/24 = 4/3$; and noting $(-4\sqrt{2}, -4\sqrt{2}) \in L$, using:

$$L \equiv y - y_1 = m(x - x_1)$$
$$\Rightarrow V \equiv y + 4\sqrt{2} = \frac{4}{3}(x + 4\sqrt{2})$$
$$\Rightarrow V \equiv 3y + 12\sqrt{2} = 4x{+}!6\sqrt{2}$$
$$\Rightarrow V \equiv y = \frac{4x + 4\sqrt{2}}{3}$$
$$\Rightarrow V \equiv \left\{ \left(x, \frac{4x + 4\sqrt{2}}{3} \right) : x \geq -4\sqrt{2} \right\} =: \{p_x : x \geq -4\sqrt{2}\}$$

Now consider the following:

V is the set of points $\{p_x\}$. The distance between ship A and ship B is 8 km when the

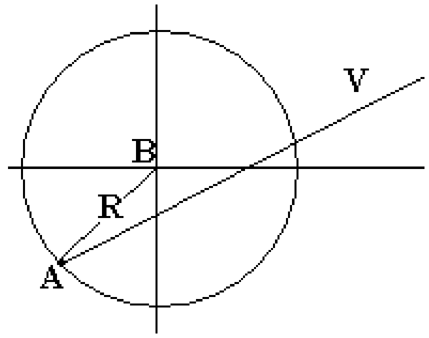


Figure 22: R denotes $\mathbf{R}_{AB}$. The circle represents a circle of radius 8 km about B. Twice ship A will be exactly 8 km from ship B; initially and again at a later time.

distance from p_x to $(0,0)$, d, is 8:

$$\begin{aligned} d = \sqrt{(x-0)^2 + \left(\frac{4x+4\sqrt{2}}{3} - 0\right)^2} &= |p_x| \stackrel{!}{=} 8 \\ \Rightarrow \sqrt{x^2 + \frac{16x^2 + 32\sqrt{2}\,x + 32}{9}} &\stackrel{!}{=} 8 \\ \Rightarrow x^2 + \frac{16x^2 + 32\sqrt{2}\,x + 32}{9} &= 64 \\ \Rightarrow 9x^2 + 16x^2 + 32\sqrt{2}\,x + 32 &= 576 \\ \Rightarrow 25x^2 + 32\sqrt{2}\,x - 544 &= 0 \end{aligned}$$

Using

$$\begin{aligned} x_{\pm} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \Rightarrow x_{\pm} &= \frac{-32\sqrt{2} \pm \sqrt{2048 + 54400}}{50} \\ \Rightarrow x_{\pm} &= \frac{-32\sqrt{2} \pm 237.588}{50} \\ \Rightarrow x &= 3.847 \text{ or } -5.657 \text{ km} \end{aligned}$$

The $x = -5.657$ km is the initial $(-5.657 \simeq -4\sqrt{2})$. Hence ship A travels $4\sqrt{2}+3.847 \simeq 9.504$ km in the **i**-direction. The speed of A relative to B in this direction is 24 km/hr. Using

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$

$$\Rightarrow \text{time} = \frac{9.504}{24} = 0.396 \text{ h} = 0.396(60 \text{ min}) = 23.76 \text{ min} \approx 24 \text{ min}$$

4.12 Problem: LC HL 2005: [Part(b)]

(i)

$$\mathbf{V}_A = -p\cos 45^\circ\,\mathbf{i} - p\sin 45^\circ\,\mathbf{j}$$
$$\underset{\times\sqrt{2}/\sqrt{2}}{\Rightarrow} \mathbf{V}_A = -\frac{p\sqrt{2}}{2}\,\mathbf{i} - \frac{p\sqrt{2}}{2}\,\mathbf{j}$$
$$\mathbf{V}_B = -8\,\mathbf{i}$$

Using

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B$$
$$\Rightarrow \mathbf{V}_{AB} = \left(8 - \frac{p\sqrt{2}}{2}\right)\mathbf{i} - \frac{p\sqrt{2}}{2}\,\mathbf{j} \overset{!}{=} -2\,\mathbf{i} - 10\,\mathbf{j}$$
$$\Rightarrow \frac{p\sqrt{2}}{2} = 10$$
$$\Rightarrow p = \frac{20}{\sqrt{2}} \underset{\times\sqrt{2}/\sqrt{2}}{=} \frac{20\sqrt{2}}{2} = 10\sqrt{2}$$

(ii) Initially

$$\mathbf{R}_A = 220\sqrt{2}\cos 45^\circ\,\mathbf{i} + 220\sin 45^\circ\,\mathbf{j} = 220\,\mathbf{i} + 220\,\mathbf{j}\ \text{, and}$$
$$\mathbf{R}_B = 136\,\mathbf{i}$$

Using

$$\mathbf{R}_{AB} = \mathbf{R}_A - \mathbf{R}_B$$
$$\Rightarrow \mathbf{R}_{AB} = 84\,\mathbf{i} + 220\,\mathbf{j}$$

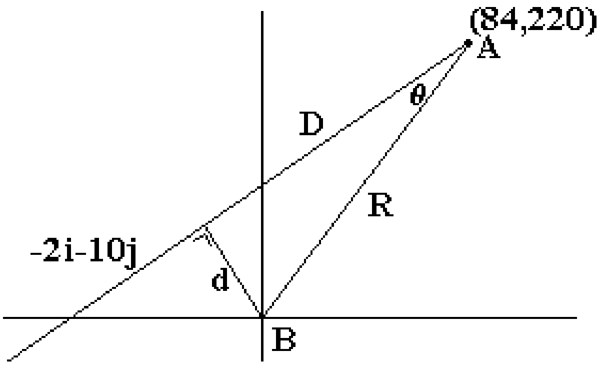


Figure 23: R denotes $\mathbf{R}_{AB}$. D is the distance A has to travel relative to B up to the instant the cars are d-close together. θ is the angle between V and R.

As a line $\mathbf{V}_{AB} =: V$ has slope $-10/-2 = 5$, $(84, 220) \in V$ and using

$$L \equiv y - y_1 = m(x - x_1)$$
$$\Rightarrow V \equiv y - 220 = 5(x - 84)$$
$$\Rightarrow V \equiv 5x - y - 200 = 0$$

Using

$$d = \frac{|ax+by+c|}{\sqrt{a^2+b^2}} = \frac{|c|}{\sqrt{a^2+b^2}}$$
$$\Rightarrow d = \frac{200}{\sqrt{26}} \text{ m}$$

The slope of $\mathbf{R}_{AB} =: R$ as a line is of slope $220/84 = 55/21$. Using

$$\tan\theta = \pm\frac{m_1 - m_1}{1 + m_1 m_2}$$
$$\Rightarrow \tan\theta = \pm\frac{5 - 55/21}{1 + 275/21} = \pm\frac{50/21}{296/21} \underset{\theta\in[0,90^\circ]}{=} \frac{25}{148}$$

Now

$$\tan\theta = \frac{d}{D}$$
$$\Rightarrow D = \frac{d}{\tan\theta} = \frac{\overset{8}{\cancel{200}}}{\sqrt{26}}.\frac{148}{\cancel{25}}$$
$$\Rightarrow D = 592\sqrt{\frac{2}{13}}$$

Using

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$
$$\Rightarrow \text{time} = 592\sqrt{\frac{2}{13}}.\frac{1}{\sqrt{4+100}} = \frac{296}{13} \text{ s}$$

Using

$$s = ut$$
$$\Rightarrow s_A = 10\sqrt{2}\left(\frac{296}{13}\right) \simeq 322.006 \text{ m}$$

But A is initially $220\sqrt{2} \simeq 311.127$ m from the intersection; hence A is now $322 - 311.1 = 10.9$ m≈ 11 m from the intersection.

4.13 Problem: LC HL 2004: [Part(b)]

(i) Using

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B$$
$$\Rightarrow \mathbf{V}_{QP} = \mathbf{i} - 8\,\mathbf{j}$$

(ii) As a line $\mathbf{V}_{QP} =: V$ has slope -8. Initially $\mathbf{R}_{QP} = 20\,\mathbf{i} + 40\,\mathbf{j}$; hence $(20, 40) \in V$. Using

$$L \equiv y - y_1 = m(x - x_1)$$
$$\Rightarrow V \equiv y - 40 = -8(x - 20)$$
$$\Rightarrow V \equiv y - 40 = -8x + 160$$
$$\Rightarrow V \equiv 8x + y - 200 = 0$$

Using

$$d = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}} \underset{(x,y)=(0,0)}{=} \frac{|c|}{\sqrt{a^2 + b^2}}$$
$$\Rightarrow d = \frac{200}{\sqrt{65}} \approx 25 \text{ m}$$

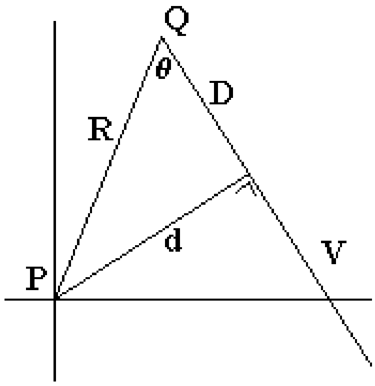
(iii) 

Figure 24: R denotes $\mathbf{R}_{QP}$. D is the distance Q has to travel relative to P up to the instant the particles are d-close together. θ is the angle between V and R.

Using, noting as a line $\mathbf{R}_{AB} =: R$ has slope $40/20 = 2$;

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2}$$
$$\Rightarrow \tan\theta = \pm\frac{-8 - 2}{1 - 16} \underset{\theta\in[0,90^\circ]}{=} \frac{2}{3}$$

From Figure 24;

$$\tan\theta = \frac{d}{D}$$
$$\Rightarrow D = \frac{d}{\tan\theta} = 25\frac{3}{2}$$

Using

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$
$$\Rightarrow \text{time} = \frac{75}{2}\frac{1}{\sqrt{65}} \approx 4.6 \text{ s}$$

4.14 Problem: LC HL 2003: [Part(b)]

•

$$\mathbf{V}_A = 7.5\,\mathbf{i}$$
$$\mathbf{V}_B = 10\cos 60^\circ\,\mathbf{i} + 10\sin 60^\circ\,\mathbf{j}$$
$$\Rightarrow \mathbf{V}_B = 5\,\mathbf{i} + 5\sqrt{3}\,\mathbf{j}$$
$$\Rightarrow \mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B$$
$$\Rightarrow \mathbf{V}_{AB} = 2.5\,\mathbf{i} - 5\sqrt{3}\,\mathbf{j}$$

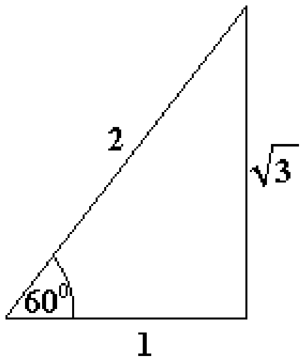


Figure 25: The model triangle for 60°

- Initially

$$\mathbf{R}_{AB} = -375\,\mathbf{i}$$

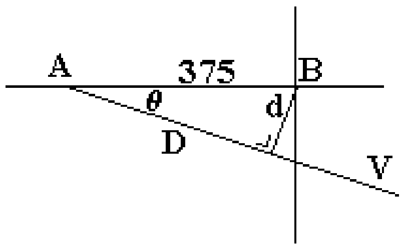


Figure 26: D is the distance A has to travel relative to B up to the instant the particles are d-close together.

From $\mathbf{V}_{AB} = 2.5\,\mathbf{i} - 5\sqrt{3}\,\mathbf{j}$, $\tan\theta = (\cancel{5}\sqrt{3})/(\cancel{5}/2) = 2\sqrt{3}$.

From Figures 27 and 26,

$$\cos\theta = \frac{1}{\sqrt{13}} = \frac{D}{375}$$
$$\Rightarrow D = \frac{375}{\sqrt{13}} \text{ m}$$

Using

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$
$$\Rightarrow \text{time} = \frac{375}{\sqrt{13}}.\frac{1}{\sqrt{2.5^2 + (5\sqrt{3})^2}}$$
$$\Rightarrow \text{time} = \frac{375/\sqrt{13}}{\sqrt{\frac{25}{4} + 75}}$$

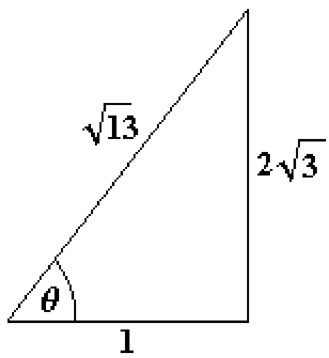


Figure 27: The model triangle for θ.

4.15 Problem: LC HL 2006: [Part(b)]

(i) Using

$$\begin{aligned} s &= ut \\ s_B &= 10(0.8) = 8\,\text{m} \\ s_A &= 10(0.4) = 4\text{ m} \end{aligned}$$

(ii) Consider the following diagram:

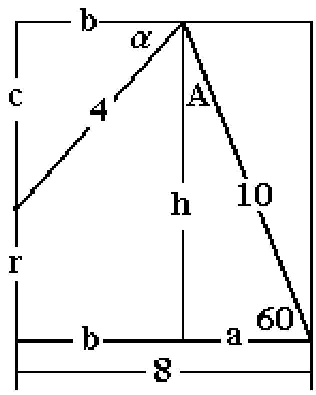


Figure 28: Straight away it should be clear $A = 30°$, $h = 10\sin 60°$ and $a = 10\cos 30°$.

Now $a = 10\sin 60° = 5$ m (see Figure 25). Hence $b = 3$ m and thus $\alpha = \arctan(\sqrt{7}/3)$.

(iii) From Figure 28, $c = \sqrt{7}$, $h = 10\sin 60° = 10\sqrt{3}/2 = 5\sqrt{3}$ and

$$r = h - c = 5\sqrt{3} - \sqrt{7}$$

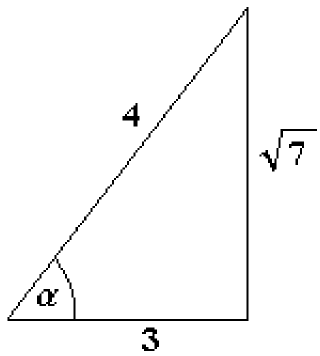


Figure 29: The model triangle for α.

Now

$$\mathbf{V}_G = -\frac{2}{5}\cos\alpha\,\mathbf{i} - \frac{2}{5}\sin\alpha\,\mathbf{j}$$
$$\Rightarrow \mathbf{V}_G = -\frac{\not{2}}{5}.\frac{3}{\not{2}_{2}}\,\mathbf{i} - \frac{\not{2}}{5}.\frac{\sqrt{7}}{\not{4}_{2}}\,\mathbf{j}$$
$$\Rightarrow \mathbf{V}_G = -\frac{3}{10}\,\mathbf{i} - \frac{\sqrt{7}}{10}\,\mathbf{j}$$
$$\mathbf{V}_B = -\frac{4}{5}\,\mathbf{i}$$

Using

$$\mathbf{V}_{GB} = \mathbf{V}_G - \mathbf{V}_B = \left(\frac{4}{5} - \frac{3}{10}\right)\mathbf{i} - \frac{\sqrt{7}}{10}\,\mathbf{j}$$
$$\Rightarrow \mathbf{V}_{GB} = \mathbf{V}_G - \mathbf{V}_B = \frac{1}{2}\,\mathbf{i} - \frac{\sqrt{7}}{10}\,\mathbf{j}$$

Now

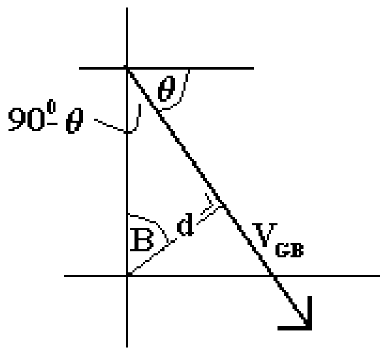


Figure 30: Clearly $B = \theta$ and so $d = (5\sqrt{3} - \sqrt{7})\cos\theta$.

$$\tan\theta = \frac{-\mathbf{j}\text{ component of }\mathbf{V}_{GB}}{\mathbf{i}\text{ component of }\mathbf{V}_{GB}} = \frac{\sqrt{7}}{10}.2 = \frac{\sqrt{7}}{5}$$

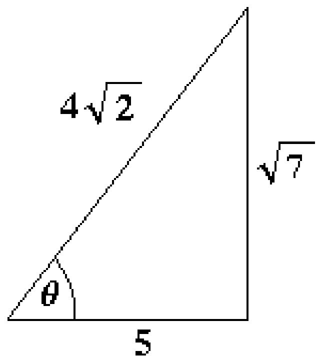


Figure 31: Model Triangle for θ.

$$d = (5\sqrt{3} - \sqrt{7}).\frac{5}{4\sqrt{2}} \approx 5.316 \text{ m}$$

4.16 Problem: LC HL 2005: [Part (a)]

To cross the river in the shortest time she must head straight across:

$$\mathbf{V}_{WC} = u\,\mathbf{j}$$
$$\mathbf{V}_C = v\,\mathbf{i}$$
$$\Rightarrow \mathbf{V}_W = \mathbf{V}_{WC} + \mathbf{V}_C$$
$$\Rightarrow \mathbf{V}_W = v\,\mathbf{i} + u\,\mathbf{j}$$

The time taken to cross is:

$$\frac{\text{distance in the } \mathbf{j} \text{ direction}}{\text{speed in the } \mathbf{j} \text{ direction}}$$
$$\Rightarrow 10 = \frac{d}{u}$$
$$\Rightarrow d = 10u$$

In order to cross by the shortest path she must head upstream at an angle θ as shown to counteract the current:

$$\mathbf{V}_{WC} = -u\sin\theta\,\mathbf{i} + u\cos\theta\,\mathbf{j}$$

To counteract the current:

$$u\sin\theta = v$$
$$\Rightarrow \sin\theta = \frac{v}{u}$$

Now using

$$t = \frac{\text{distance in the } \mathbf{j} \text{ direction}}{\text{speed in the } \mathbf{j} \text{ direction}}$$
$$\Rightarrow t = \frac{d}{u\cos\theta}$$
$$\Rightarrow t = (10u)\frac{\cancel{u}}{\cancel{u}\sqrt{u^2 - v^2}} = \frac{10u}{\sqrt{u^2 - v^2}}$$

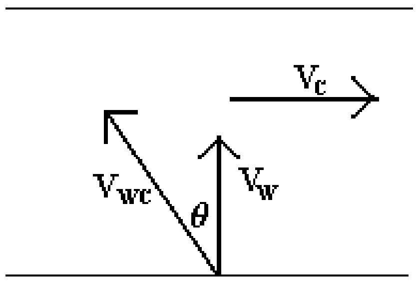


Figure 32: The woman must head upstream at a speed u at an angle θ to counteract the current.

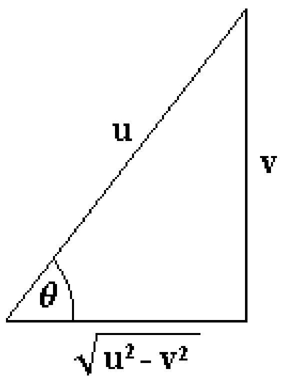


Figure 33: The model triangle for θ.

4.17 Problem: LC HL 2004: [Part (a)]

(i) If the wind blows *from* SE then it blows in a NW direction:

$$\begin{aligned} \mathbf{V}_W &= -18\cos 45^\circ\,\mathbf{i} + 18\sin 45^\circ\,\mathbf{j} \\ \Rightarrow \mathbf{V}_W &= -9\sqrt{2}\,\mathbf{i} + 9\sqrt{2}\,\mathbf{j} \end{aligned}$$

In order for the bird to fly due North, it must head at an angle θ against the wind as shown:

$$\begin{aligned} \mathbf{V}_{BW} &= 22\sin\theta\,\mathbf{i} + 22\cos\theta\,\mathbf{j} \\ \Rightarrow \mathbf{V}_B &= (22\sin\theta - 9\sqrt{2})\,\mathbf{i} + (22\cos\theta + 9\sqrt{2}) \end{aligned}$$

To counteract the current:

$$\begin{aligned} 22\sin\theta &= 9\sqrt{2} \\ \Rightarrow \sin\theta &= \frac{9\sqrt{2}}{22} \\ \Rightarrow \theta &= \arcsin(9\sqrt{2}/22) \approx 35^\circ \end{aligned}$$

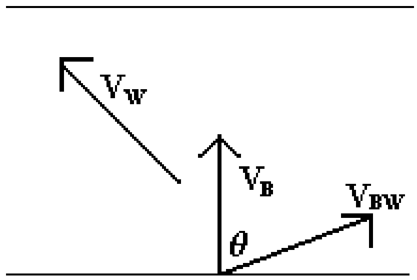
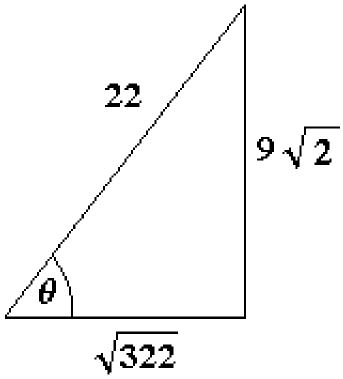


Figure 34: The bird must head into the wind at a speed u at an angle θ to counteract the wind.



(ii)

Figure 35: The model triangle for θ. Using Pythagoras Theorem, the adjacent side is $\sqrt{22^2 - 81(2)} = \sqrt{322}$

Now using

$$t = \frac{\text{distance in the } \mathbf{j} \text{ direction}}{\text{speed in the } \mathbf{j} \text{ direction}}$$
$$\Rightarrow t = \frac{250}{22\cos\theta + 9\sqrt{2}}$$
$$\Rightarrow t = \frac{250}{\sqrt{322} + 9\sqrt{2}} \simeq 8.1507 \approx 8.15 \text{ s}$$

5 Newton's Laws and Connected Particles

5.1 LC HL 2009: [Part (a)]

(i) The force diagrams are:

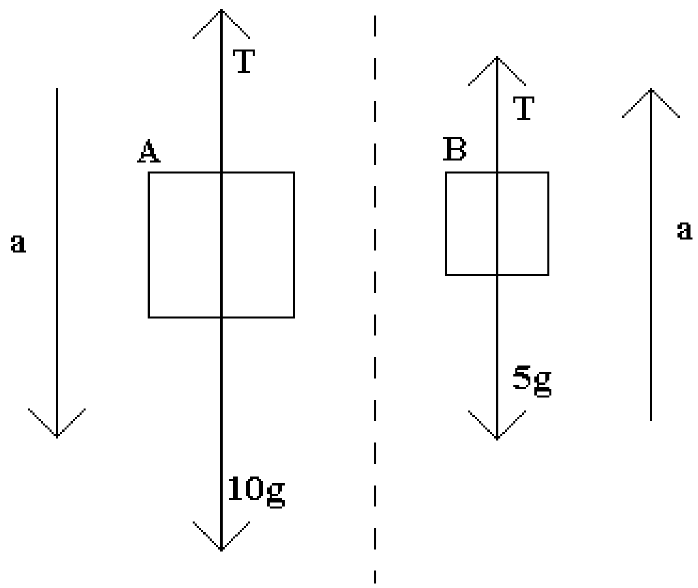


Figure 36: The force diagrams for particles A & B

Hence

$$10g - T = 10a \qquad (62)$$
$$T - 5g = 5a \qquad (63)$$
$$\Rightarrow 5g = 15a$$
$$\Rightarrow a = \frac{g}{3} \text{ m/s}^2 \qquad (64)$$

Using

$$v^2 = u^2 + 2as \qquad (65)$$
$$\Rightarrow v = \sqrt{u^2 + 2as}$$
$$\Rightarrow v = \sqrt{2\,(g/3)} = \sqrt{\frac{2g}{3}}$$

(ii) When particle A hits the ground, particle B is 1 m above ground and travelling at the same speed as particle A. When particle A hits the ground the string is no longer taut and particle B travels freely under gravity. Hence the speed of B is a maximum when $v = 0$:

$$s = \frac{v^2 - u^2}{2a}$$
$$\Rightarrow s = \frac{-2g/3}{-2g} = \frac{1}{3} \text{ m}$$

Hence particle B reaches a height 4/3 m above the ground.

5.2 LC HL 2008: [Part (a)]

The force diagrams are as follows:

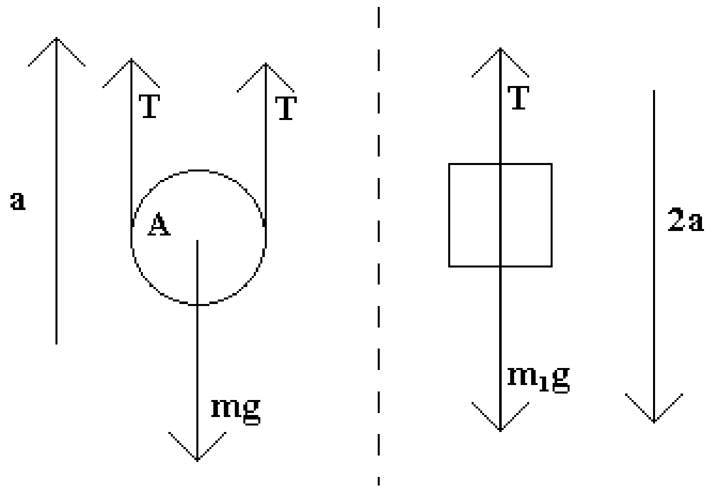


Figure 37: Note that there are two tensions acting on pulley A. Also the acceleration of the particle is twice that of pulley A because if pulley A is raised a distance x, the particle will be lowered a distance $2x$.

Hence

$$2T - mg = ma \quad (66)$$

$$m_1 g - T = 2m_1 a \quad (67)$$

$$\Rightarrow 2m_1 g - 2T = 4m_1 a$$

$$\Rightarrow 2m_1 g - mg = 4m_1 a + ma$$

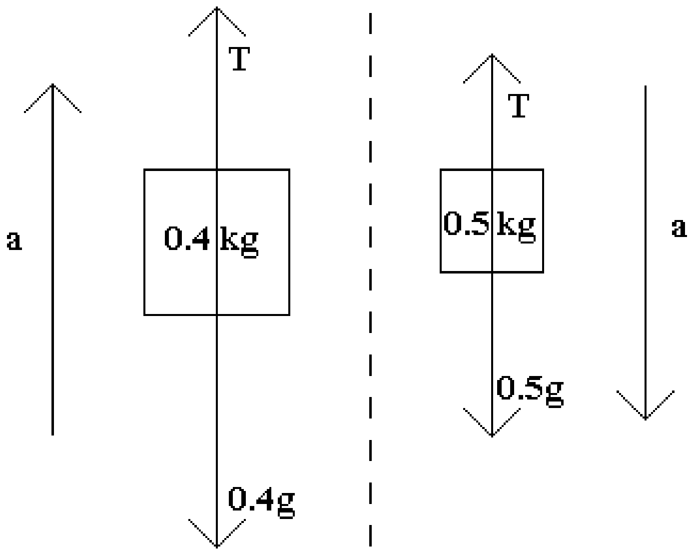
$$\Rightarrow (2m_1 - m)g = a(m + 4m_1)$$

$$\Rightarrow a = \frac{(2m_1 - m)g}{4m_1 + m} \quad (68)$$

□

5.3 LC HL 2006: [Part (a)]

(i) The force diagrams:



Hence

$$T - \frac{2}{5}g = \frac{2}{5}a \tag{69}$$
$$\frac{g}{2} - T = \frac{a}{2} \tag{70}$$
$$\Rightarrow \frac{9}{10}g = \frac{g}{10}$$
$$\Rightarrow a = \frac{g}{9} \text{ m/s}^2$$

(ii) First find the speed when the 0.5 kg mass strikes the horizontal surface. Using

$$v^2 = u^2 + 2as \tag{71}$$
$$\Rightarrow v = \sqrt{u^2 + 2as} = \sqrt{\frac{2g}{9}} \text{ m/s}$$

Once the 0.5 kg particle strikes the horizontal surface, the 0.4 kg particle acts as a projectile from this point and the string returns to taut when the 0.4 kg particle returns back down to this height. Now the constant acceleration is $-g$, the initial speed is $\sqrt{2g/9}$ and by symmetry the final speed is $-\sqrt{2g/9}$. Now using

$$v = u + at$$
$$\Rightarrow t = \frac{v - u}{a}$$
$$\Rightarrow t = \frac{-\sqrt{\frac{2g}{9}} - \sqrt{\frac{2g}{9}}}{-g}$$
$$\Rightarrow t = \frac{2\sqrt{\frac{2g}{9}}}{\sqrt{g^2}} = \sqrt{\frac{8}{9g}} \approx 0.30 \text{ s}$$

5.4 LC HL 2005: [Part (a)]

(i) The force diagrams are:

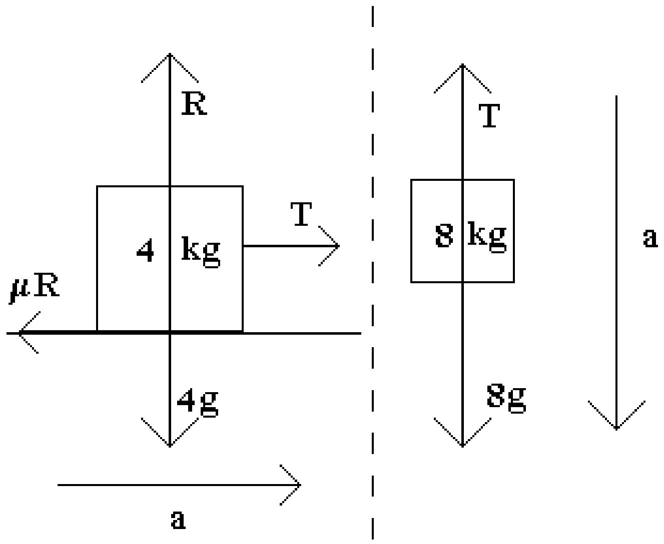


Figure 38: Note $R = 4g$ and hence $\mu R = g$.

Hence

$$T - g = 4a \quad (72)$$
$$8g - T = 8a \quad (73)$$
$$\Rightarrow 7g = 12a$$
$$\Rightarrow a = \frac{7}{12}g \text{ m/s}^2$$
$$\Rightarrow T = g + 4a = g + \frac{7}{3}g = \frac{10}{3}g \text{ N}$$

(ii) The force diagram is:

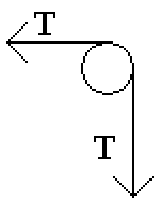


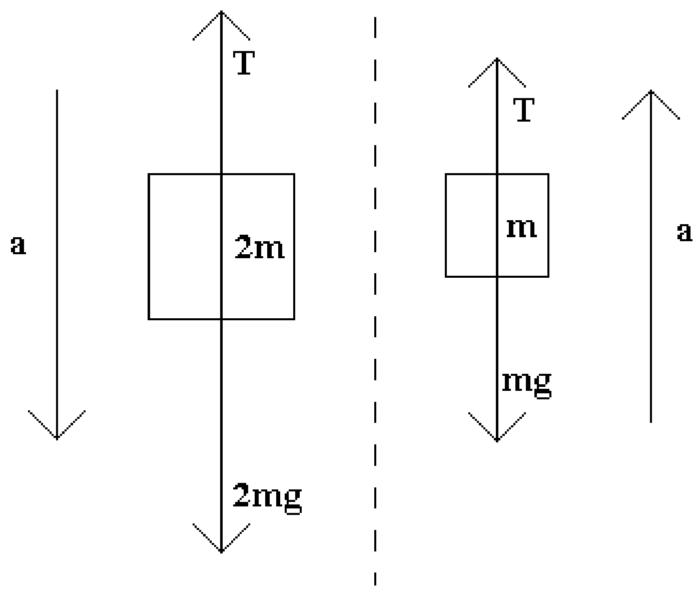
Figure 39: Note $R = 4g$ and hence $\mu R = g$.

Now to add these put them in an **i-j** basis:

$$\mathbf{F} = -T\,\mathbf{i} - T\,\mathbf{j}$$
$$\Rightarrow \mathbf{F} = -\frac{10}{3}g\,\mathbf{i} - \frac{10}{3}g\,\mathbf{j}$$

5.5 LC HL 2004: [Part (a)]

(i) The force diagrams are:



Hence

$$2mg - T = 2ma \qquad (74)$$
$$T - mg = ma \qquad (75)$$
$$\Rightarrow mg = 3ma$$
$$\Rightarrow a = \frac{g}{3}\ \text{m/s}^2$$

(ii) When the speed of the first particle is v, using

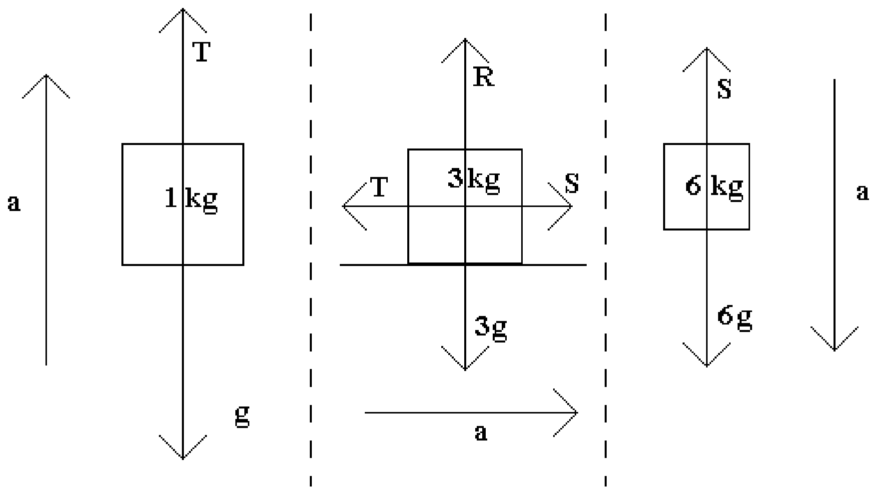
$$v^2 = u^2 + 2as$$
$$\Rightarrow s = \frac{v^2 - u^2}{2a}$$
$$\Rightarrow s = \frac{3v^2}{2g}$$

However the second particle will travel the same distance *up* as the first particle travels this distance *down*:

$$\therefore \text{vertical separation} = \frac{3v^2}{g} \qquad (76)$$

5.6 LC HL 2003: [Part (a)]

The force diagrams are:



Hence

$$\begin{aligned} T - g &= a && (77)\\ S - T &= 3a && (78)\\ 6g - S &= 6a && (79)\\ \Rightarrow 5g &= 10a \\ \Rightarrow a &= \frac{g}{2} \text{ m/s}^2 && (80) \end{aligned}$$

5.7 LC HL 2000: [Part (a)]

The force diagrams are:

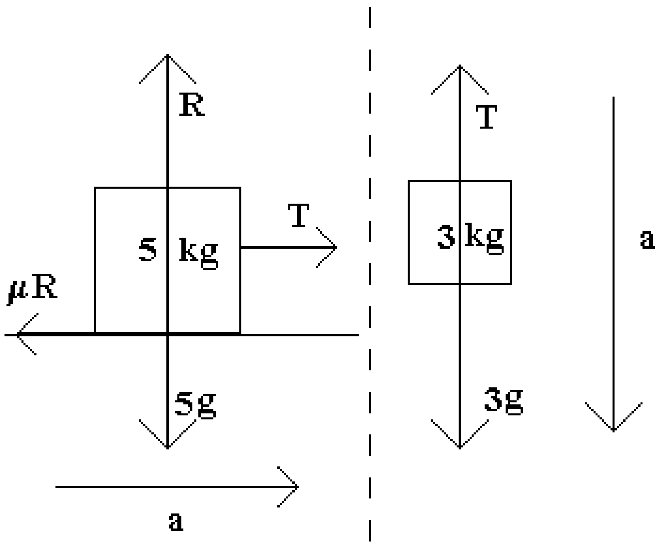


Figure 40: Note $R = 5g$ and hence $\mu R = g$.

Hence

$$T - g = 5a \quad (81)$$
$$3g - T = 3a \quad (82)$$
$$\Rightarrow 2g = 8a$$
$$\Rightarrow a = \frac{g}{4}$$

Using

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s = \frac{1}{2}\left(\frac{g}{4}\right)4 = \frac{g}{2}\text{ m}$$