Leaving Cert Applied Maths - Higher Level

J.P. McCarthy

February 18, 2010

Contents

Chapter 1

Introduction to Applied Maths

1.1 What is Applied Maths?

Applied maths could be defined as the use of mathematics in studying natural phenomena. The branch of applied maths studied at Leaving Cert level is *Newtonian mechanics*. Mechanics is the study of systems under the action of *forces*. Newtonian mechanics is concerned with systems that can be adequately described by Newton's Laws of Motion. Systems that aren't adequately described by Newtonian mechanics include systems with speeds approaching the speed of light, systems of extremely small particles and systems with a large number of particles. Hence Leaving Cert Applied Maths is the study of simple macro-systems that have moderate speeds.

1.2 Leaving Cert Applied Maths

Hence, applied maths is essentially a further study of the mathematics of chapters 6 through 13 in *Real World Physics*; following which more specific and involved questions than those of Leaving Cert Physics may be posed and answered. The emphasis in applied maths is more on problem-solving than anything and reflecting this, the need for rote-learning is almost non-existent. The course itself could be presented on one A4 sheet. The skills required for the course therefore include:

- Interpretation of a physical question
- Strategic solution of problem; e.g. setting up equations, knowledge that two simultaneous equations in two unknowns is solvable, etc.
- Competency in algebra
- Competency where required in trigonometry, vectors, differentiation and integration

The final chapter however, differential equations, contains new mathematical material; namely differential equations. This question in Leaving Cert is usually purely mathematical - part (b) deals with applications.

The course summary outlines which chapters will be covered in this course. The textbook available is *Fundamental Applied Maths* (1986) by Oliver Murphy. The notes will comprise a self-contained course however the notes will follow the content of this book. Also a copy of the LC applied maths exam papers should be bought as practising of exam questions should comprise your personal study. The LC exam is two and a half hours; six questions out of ten must be answered and as this is a condensed course, only six questions shall be covered. Anyone who is strong in higher level maths and is self-motivated can achieve great success in applied maths - especially in conjunction with LC Physics. There is then a three for the price of two and a half effect as applied maths will help your physics, physics will help your applied maths and maths will help your applied maths. The statistics over the last number of years are that typically over 90% of students taking applied maths do higher level, and of these about a quarter achieve an A, over half achieve an A or B and nearly two-thirds achieve an honour. 90% of students pass. So with strong maths and good, effective application to the task at hand, good grades are available in a one year course.

1.3 Introduction to Topics

To give a further flavour of the content of the course, an example of a basic question from each of the six chapters is presented.

1.3.1 Accelerated Linear Motion

Consider a car travelling at 40 m s^{-1}. While slowing down[1], the car covers a distance of 100 m s^{-1}. Find

(i) the time taken,

(ii) the deceleration,

(iii) the distance car would take to stop from 40 m s^{-1} if its deceleration were doubled.

1.3.2 Relative Velocity

A destroyer is 500 km due West of a frigate. The destroyer is travelling at 10 km/hr in a direction E 30° N. The frigate is travelling at $5\sqrt{2}$ km/hr in a NW direction. Find the velocity of the frigate relative to the destroyer. Show that they are on a collision course and find out when they will collide.

1.3.3 Projectiles

A particle is projected from a point on a horizontal plane, with initial speed 35 m/s at an angle to the horizontal where $\tan\alpha = 4/3$. Find

(i) its initial velocity in terms of $\mathbf{i}$ and $\mathbf{j}$,

(ii) its displacement vector (or position vector) at $t = 3$,

(iii) the times when its height above the horizontal plane is 36.4 m.

1.3.4 Newton's Laws and Connected Particles

An Eskimo pulls a sled from rest. The sled and the luggage on it have total mass 800 kg. The Eskimo pulls it by means of a horizontal light inextensible rope. There is no friction and the tension in the rope is 100 N in magnitude. Find the acceleration. Find, also, the distance covered in the first 10 seconds of motion.

1.3.5 Impacts and Collisions

A van of mass 1000 kg and moving at 16 m/s on a straight road collides with a car of mass 600 kg moving at 20 m/s along a perpendicular road. When the two collide, they become entangled. Find the speed of the joint mass immediately after impact.

1.3.6 Differential Equations

Find a function $y = f(x)$ such that

$$\frac{dy}{dx} = 2y,$$

and $y = 1$ when $x = 0$.

[1] in this chapter acceleration is always constant

1.4 Required Mathematics

Apart from Differential Equations[2], the applied maths course is broadly partitioned into problems that require numerical answers and problems that require an algebraic derivation. Broadly speaking, the numerical problems require the solution of simultaneous equations and such; and the algebraic problems require algebraic manipulation, knowledge of trigonometric identities and such.

1.4.1 Algebra

A competent and full working knowledge of algebra, quadratic & cubic equations, simultaneous equations and inequalities is required. Two simple and often useful algebraic facts are presented:

The Roots of a Product

If x, $y \in \mathbb{R}$ then

$$x.y = 0 \Rightarrow x = 0 \text{ or } y = 0. \tag{1.1}$$

Example
The trajectory of a particle projected at a speed of u from a point on level horizontal ground at an angle θ to the horizontal is given by:

$$\mathbf{r} = \{(u\cos\theta)t\}\mathbf{i} + \left\{(u\sin\theta)t - \frac{1}{2}gt^2\right\}\mathbf{j}. \tag{1.2}$$

The particle is not airborne when the **j**-component is zero:

$$\begin{aligned}(u\sin\theta)t - \frac{1}{2}gt^2 &= 0,\\ \Rightarrow t\left((u\sin\theta) - \frac{1}{2}gt\right) &= 0.\end{aligned}$$

Hence the projectile is not airborne when $t = 0$ or

$$\begin{aligned}(u\sin\theta) - \frac{1}{2}gt &= 0,\\ \Rightarrow t &= \frac{2u\sin\theta}{g}.\end{aligned}$$

These two times represent when the particle is projected and when the particle returns to the ground.

Simultaneous Equations

Suppose two equations are constructed:

$$\begin{aligned}a_1x + b_1y &= c_1\\ a_2x + b_2y &= c_2\end{aligned}$$

where x and y are variables and a_1, a_2, b_1, b_2, c_1 and c_2 are constants. A quick manipulation shows

$$\begin{aligned}y &= -\frac{a_1}{b_1}x + c_1,\\ y &= -\frac{a_2}{b_2}x + c_2.\end{aligned}$$

So therefore each of the equations represents the line with slope $-a_i/b_i$ and y-axis intercept c_i. Two lines intersect as long as they are not parallel; that is as long as:

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}. \tag{1.3}$$

In LC Applied Maths this will not occur hence the system will be solvable; that is there is a solution, (x, y), that is the point of intersection of the two lines, and satisfies both equations. i.e. if two linear equations in two unknowns can be constructed, the system can be solved for the two unknowns.

[2]D.E.s introduces new mathematics, requiring knowledge of differentiation and integration

1.4.2 Trigonometry

The following aspects of trigonometry must be known intimately. Most of these facts are presented on page 9 of the Maths Tables.

Angle Measurement

In L.C. and more advanced maths, angles are often measured in radians rather than degrees. Consider a circle of radius r. One radian is defined as that angle such that the arc-length subtended by the angle, s is equal to the radius r. Therefore, θ, in radians is given by:

$$\theta = \frac{s}{r}. \tag{1.4}$$

Therefore $360^\circ \Leftrightarrow 2\pi r/r = 2\pi$. Therefore

$$\pi \text{ rad} = 180^\circ \tag{1.5}$$
$$1^\circ = \frac{\pi}{180} \text{ rad} \tag{1.6}$$
$$1 \text{ rad} = \frac{180^\circ}{\pi} \tag{1.7}$$

Right Angle Trigonometry

$$\sin\theta := \frac{o}{h} \tag{1.8}$$
$$\cos\theta := \frac{a}{h} \tag{1.9}$$
$$\tan\theta := \frac{o}{a} \tag{1.10}$$
$$\csc\theta := \frac{1}{\sin\theta} \tag{1.11}$$
$$\sec\theta := \frac{1}{\cos\theta} \tag{1.12}$$
$$\cot\theta := \frac{1}{\tan\theta} \tag{1.13}$$

The Unit Circle

The circle of radius 1 is the unit circle. The points on the unit circle are $(\cos\theta, \sin\theta)$ where θ is the anti-clockwise angle made between the positive part of the x-axis and the radius to the point. With this knowledge some facts may be proved geometrically. The typical example is the proof of:

$$\cos^2\theta + \sin^2\theta = 1.$$

To derive the formula for $\cos(A-B)$, where $A > B$, the L.C. textbook measures the distance between $(\cos A, \sin A)$ and $(\cos B, \sin B)$, rotates the unit circle through an angle B and measures the distance between $(\cos(A-B), \sin(A-B))$ and $(1,0)$ and equating... complicated. This proof is much simpler.
Theorem:

$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Proof: Let $\mathbf{a}$ be the vector $(\cos A, \sin A) \equiv \cos A\mathbf{i} + \sin A\mathbf{j}$; $\mathbf{b}$ the vector $(\cos B, \sin B) \equiv \cos B\mathbf{i} + \sin B\mathbf{j}$. Then $\mathbf{a}\cdot\mathbf{b}$ has two representations:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta, \tag{1.14}$$
$$\mathbf{a}\cdot\mathbf{b} = x_a x_b + y_a y_b; \tag{1.15}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and x_a is the $\mathbf{i}$-component of $\mathbf{a}$, y_a is the $\mathbf{j}$-component of $\mathbf{a}$, etc. Clearly

$$\begin{aligned} \theta &= A - B, \\ x_a &= \cos A \\ x_b &= \cos B \\ y_a &= \sin A \\ y_b &= \sin B \end{aligned}$$

Of course $|\mathbf{a}| = 1 = |\mathbf{b}|$ as $\mathbf{a}$ and $\mathbf{b}$ are on the unit circle. Equating (1.14) and (1.15):

$$\begin{aligned} |\mathbf{a}||\mathbf{b}|\cos\theta &= x_a x_b + y_a y_b, \\ \Rightarrow 1.1\cos(A-B) &= \cos A\cos B + \sin A\sin B, \\ \Rightarrow \cos(A-B) &= \cos A\cos B + \sin A\sin B. \end{aligned}$$

□

Fundamental Identities

$$1 + \tan^2\theta = \sec^2\theta \tag{1.16}$$
$$1 + \cot^2\theta = \csc^2\theta \tag{1.17}$$
$$\sin(-\theta) = -\sin\theta \tag{1.18}$$
$$\cos(-\theta) = \cos\theta \tag{1.19}$$
$$\tan(-\theta) = -\tan\theta \tag{1.20}$$
$$\sin(90^\circ - \theta) = \cos\theta \tag{1.21}$$
$$\cos(90^\circ - \theta) = \sin\theta \tag{1.22}$$
$$\tan(90^\circ - \theta) = \cos\theta \tag{1.23}$$

Sine Law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.24}$$

Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{1.25}$$

Addition and Subtraction Formulae

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{1.26}$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B \tag{1.27}$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{1.28}$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{1.29}$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \tag{1.30}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \tag{1.31}$$

Double-Angle Formulae

$$\sin 2A = 2\sin A\cos A \tag{1.32}$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A} \tag{1.33}$$

$$\cos 2A = \cos^2 A - \sin^2 A \tag{1.34}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} \tag{1.35}$$

Triple Angle Formulae

$$\sin 3A = 3\sin A - 4\sin^3 A \tag{1.36}$$

$$\cos 3A = 4\cos^3 A - 3\cos A \tag{1.37}$$

Half-Angle Formulae

$$\sin^2\left(\frac{A}{2}\right) = \frac{1-\cos A}{2} \tag{1.38}$$

$$\cos^2\left(\frac{A}{2}\right) = \frac{1+\cos 2A}{2} \tag{1.39}$$

Changing Products to Sums and Sums to Products

$$\cos A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2} \tag{1.40}$$

$$\sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2} \tag{1.41}$$

$$\sin A\sin B = \frac{\cos(A-B)-\cos(A+B)}{2} \tag{1.42}$$

$$\cos A\sin B = \frac{\sin(A+B)-\sin(A-B)}{2} \tag{1.43}$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{1.44}$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{1.45}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{1.46}$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{1.47}$$

1.4.3 Coordinate Geometry

The following facts from coordinate geometry must be known.

The Line

$$|ab| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{1.48}$$
$$m_{ab} = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.49}$$
$$\text{midpoint of } ab = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \tag{1.50}$$
$$L \equiv y - y_1 = m(x - x_1) \tag{1.51}$$
$$d = \text{distance from } p(x_1, y_1) \text{ to } L : ax + by + c = 0 :$$
$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \tag{1.52}$$
$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} \tag{1.53}$$
$$\text{internal division of a line in ratio } m : n;$$
$$p = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right) \tag{1.54}$$

1.4.4 Vectors

Vectors: A More Technical Approach

A vector is often defined as an object in space that has both magnitude and direction. Vectors may be added together, they may be scaled (by a real number usually), there is a zero vector, there are negative vectors. A set of vectors, together with addition and scalar multiplication is what is called a *vector space.* Directed line segments in two-dimensional Euclidean space - the vectors presented as Leaving Cert level - call them geometric vectors - together with the parallelogram law (addition) and scalar multiplication - comprises a vector space in this technical sense. The *dimension* of the geometric vectors is two - that is two pieces of information are required to specify an individual vector; namely the length of the vector and the direction. It is a fact that every vector space of the same dimension is equivalent. Hence the geometric vectors are the same as $\{\mathbb{R}^2, +, \cdot\}$; a two-dimensional space that is more than familiar to the Leaving Cert student.

Also a *norm* may be put on a vector space. If $\mathbf{v}$ is a vector, $|\mathbf{v}|$ is the norm of $\mathbf{v}$. This norm corresponds to the magnitude of a geometric vector.

The Vector Space $\{\mathbb{R}^2, +, \cdot\}$

Vectors are points on the plane, $\mathbf{v} = (x, y)$; where x, $y \in \mathbb{R}$ (hence the 2 in $\mathbb{R}^2$). Vector addition is done component-wise. This means that if $\mathbf{v} = (x_1, y_1)$ and $\mathbf{u} = (x_2, y_2)$, then

$$\mathbf{v} + \mathbf{u} = (x_1 + x_2, y_1 + y_2). \tag{1.55}$$

Scalar multiplication is also defined component-wise. Suppose $\mathbf{v} = (x, y) \in \mathbb{R}^2$, $k \in \mathbb{R}$:

$$k \cdot \mathbf{v} = (kx, ky). \tag{1.56}$$

In $\mathbb{R}^2$, the norm of a vector is simply its distance from the origin. From coordinate geometry, the distance from a point (x_1, y_1) to (x_2, y_2) is given by:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1.57}$$

Hence letting $\mathbf{v} = (x, y)$, $|\mathbf{v}|$ is the distance from $(0, 0)$ to (x, y):

$$|\mathbf{v}| = \sqrt{x^2 + y^2}.$$

The i-j Plane

i and **j** are the same thing as:

$$\mathbf{i} = (1, 0), \tag{1.58}$$
$$\mathbf{j} = (0, 1). \tag{1.59}$$

There is no link between **i** and $i = \sqrt{-1}$. Hence

$$x\,\mathbf{i} + y\,\mathbf{j} = (x, y). \tag{1.60}$$

The mapping between the geometric vectors - specified by $(|\mathbf{v}|, \theta)$ - and the **i-j** basis is as follows. Suppose $\mathbf{v}$ is a geometric vector with length $|\mathbf{v}|$ and direction θ. Then f : {geometric vectors} $\to$ {**i-j** basis} is given by

$$f(\mathbf{v}) = (|\mathbf{v}|\cos\theta, |\mathbf{v}|\sin\theta)$$

This mapping is obvious if we consider that a vector of magnitude x in the direction θ is equivalent to a RAT with hypotenuse length x and angle θ. Let $o :=$ length of side opposite to θ; $a :=$ length of side adjacent to θ. Then

$$\sin\theta = \frac{o}{x},$$
$$\Rightarrow o = h\sin\theta = x\sin\theta.$$

and

$$\cos\theta = \frac{a}{h},$$
$$\Rightarrow a = h\cos\theta = x\cos\theta.$$

Similarly the mapping between the **i-j** basis and the geometric vectors is as follows. Suppose $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ is a vector in the **i-j** basis. Then g: {**i-j** basis} $\to$ {geometric vectors} is given by:

$$g(\mathbf{v}) = (|\mathbf{v}|, \theta) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right) \tag{1.61}$$

Primarily in LC Applied Maths the **i-j** basis is used; $\mathbf{v} = x\mathbf{i} + y\mathbf{j} \equiv (x, y)$; where x, $y \in \mathbb{R}$. Vector addition is done component-wise. This means that if $\mathbf{v} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{u} = x_2\mathbf{i} + y_2\mathbf{j}$, then

$$\mathbf{v} + \mathbf{u} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j}. \tag{1.62}$$

Scalar multiplication is also defined component-wise. Suppose $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, $k \in \mathbb{R}$:

$$k\mathbf{v} = kx\mathbf{i} + ky\mathbf{j}. \tag{1.63}$$

In the **i-j** basis, the norm of a vector is simply its distance from the origin. From coordinate geometry, the distance from a point $x_1\mathbf{i} + y_1\mathbf{j}$ to $x_2\mathbf{i} + y_2\mathbf{j}$ is given by:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1.64}$$

Hence letting $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, $|\mathbf{v}|$ is the distance from $(0, 0)$ to $x\mathbf{i} + y\mathbf{j}$:

$$|\mathbf{v}| = \sqrt{x^2 + y^2}.$$

i and **j** are *orthogonal unit vectors*. *Orthogonal* means they are at right angles to each other. *Unit vectors* mean they have magnitude or norm 1. The principal tool for solving problems at LC Applied Maths is suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors with $\mathbf{i}$ components $f_u(t)$ and $f_v(t)$; and $\mathbf{j}$-components $g_u(t)$ and $g_v(t)$ say, then to solve $\mathbf{u} = \mathbf{v}$ we solve the simultaneous equations:

$$f_u(t) = f_v(t), \tag{1.65}$$

$$g_u(t) = g_v(t). \tag{1.66}$$

i.e. two vectors in the $\mathbf{i}$-$\mathbf{j}$ basis are equal iff their $\mathbf{i}$ and $\mathbf{j}$ components are equal.

To find the unit vector in the direction of a vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ there are two common methods. Let $\hat{\mathbf{v}}$ denote such a unit vector:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j} \tag{1.67}$$

$$\hat{\mathbf{v}} = \cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\mathbf{i} + \sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\mathbf{j} \tag{1.68}$$

This second method uses the fact that the unit circle is composed of the points $(\cos A, \sin A)$ where A is the direction of $\mathbf{v}$; found by (1.61).

Chapter 2

Accelerated Linear Motion

2.1 Introduction

In this chapter linear motion with piece-wise constant acceleration is studied. *Motion* is a change in the position of a particle. *Linear* motion is motion in one dimension. In the context of questions this means backwards and forwards, up and down, etc. To understand constant acceleration a number of ideas must be introduced.

2.1.1 Definition

Fix a y-axis in space such that the point $1 = 1$ m. If a particle is at position y with respect to this axis, the *displacement* of the particle is given by y m.

Example

A particle 10 m above Earth has displacement 10 m with respect to the vertical axis with $y = 0$ at Earth's surface.

2.1.2 Definition

Consider a particle in motion. Fix a y-axis in space and a x-axis in time such that $1 = 1$ s, such that at $t = 0$ s the particle has displacement $y = 0$ m. Define $s := s(t)$, the *distance* as the displacement of the particle after t s.

Example

Suppose Usain Bolt runs the 100m in 10 s. Then $s(10) = s$ after 10 s is 100 m.

2.1.3 Definition

Consider a particle under motion. Suppose at a time t_1 the particle is at position a and at time $t_1 + \Delta t$ the particle is at position b. The *average velocity* of the particle from a to b, $\bar{v}_{[ab]}$ is:

$$\bar{v}_{[ab]} = \frac{\text{change in displacement}}{\text{time taken}} \tag{2.1}$$

Hence,

$$\Rightarrow \bar{v}_{[ab]} = \frac{s(t_1 + \Delta t) - s(t_1)}{t_1 + \Delta t - t_1}$$
$$\Rightarrow \bar{v}_{[ab]} = \frac{\Delta s}{\Delta t}$$

where $\Delta s := s(t_1 + \Delta t) - s(t_1)$ is usually the information given. The average velocity is measured in units of m s^{-1}.

Example

A particle travels 500 m in 4 s. What is its average velocity?
Solution: It must be inferred that average here refers to the average over 4 s as the displacement changes by 500 m. It must be inferred from this question, therefore, that $\Delta s = 500$ m and $\Delta t = 4$ s. Hence

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{500}{4}\text{m s}^{-1} = 125 \text{ m s}^{-1}.$$

2.1.4 Definition

A particle under motion from a to b in a time T is said to have *Constant Velocity* on ab if the distance $s = s(t)$ has form:

$$s = ut, \quad \forall t \leq T. \tag{2.2}$$

Example

A cyclist cycling at constant velocity 10 m s^{-1} travels a distance $10t$ in t s.

2.1.5 Definition

The *instantaneous velocity* or *velocity* of a particle is $v := v(t)$:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{2.3}$$

Example

A penny dropped down a well has distance with respect to the depth of the well

$$s = \frac{1}{2}gt^2.$$

where g is a constant. What is v after 4 s?
Solution:

$$\begin{aligned} v &= \lim_{\Delta t \to 0} \frac{s(4+\Delta t) - s(4)}{\Delta t} \\ \Rightarrow v &= \lim_{\Delta t \to 0} \frac{\frac{1}{2}g(4+\Delta t)^2 - \frac{1}{2}g(4)^2}{\Delta t} \\ \Rightarrow v &= \frac{1}{2}g\left(\lim_{\Delta t \to 0} \frac{16 + 8\Delta t + (\Delta t)^2 - 16}{\Delta t}\right) \\ \Rightarrow v &= \frac{1}{2}g\left(\lim_{\Delta t \to 0} \frac{8\Delta t + (\Delta t)^2}{\Delta t}\right) \\ \Rightarrow v &= \frac{1}{2}g\left(\lim_{\Delta t \to 0}(8+\Delta t)\right) = \frac{1}{2}g(8) = 4g \text{ m s}^{-1} \end{aligned}$$

Remark

If a particle has velocity v m s^{-1} it does not mean that the particle will travel v m in the following second. It means that if the particle had constant velocity v m s^{-1} the particle would travel v m in the following second. In this sense a particle has constant velocity over ab if the instantaneous velocity at any point within ab is equal to the average velocity over ab.

Remark

It will be seen in Calculus that

$$v = \frac{ds}{dt}. \tag{2.4}$$

Remark

In this chapter the terms velocity and speed are interchangeable as there is essentially only one direction. The velocity defined here is the speed. Velocity is *speed in a given direction*. The question may be posed at LC: Define velocity and speed. These are the appropriate definitions in this case.

Definition

Speed is the rate of change in distance with respect to time.
Velocity is speed in a given direction.

2.1.6 Definition

Suppose a particle moves from a to b in t seconds with velocity u at a and v at b. The *average acceleration* of the particle from a to b, $\bar{a}_{[ab]}$ is:

$$\bar{a}_{[ab]} = \frac{\text{change in velocity}}{\text{time taken}} \tag{2.5}$$

Hence,

$$\Rightarrow \bar{a}_{[ab]} = \frac{v-u}{t}$$
$$\Rightarrow \bar{a}_{[ab]} = \frac{v-u}{t}$$

The average acceleration is measured in m s^{-2}.

Example

A particle accelerates from rest to 12 m s^{-1} in 3 s. What is its average acceleration?
Solution: It must be inferred from this question that *from rest* implies $u = 0$. Hence

$$\bar{a} = \frac{12}{3} = 4\text{m s}^{-2}.$$

2.1.7 Definition

A particle under motion from a to b in a time T is said to have *Constant Acceleration* on ab if the velocity $v = v(t)$ has form:

$$v = u + at, \quad \forall t \leq T. \tag{2.6}$$

where $u = v(0)$.

Example

A cyclist accelerating from a velocity of 2 m s^{-1} down a hill at a constant acceleration 1 m s^{-2} accelerates to a velocity of

$$v = 2 + 1(3) = 5 \text{ m s}^{-1}$$

after 3 s.

2.1.8 Definition

The *instantaneous acceleration* or *acceleration* of a particle is $a := a(t)$:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \tag{2.7}$$

where $\Delta v = v(t + \Delta t) - v(t)$

Example

A penny dropped down a well has velocity

$$v = gt.$$

where g is a constant. What is the acceleration of the penny after t s?
Solution:

$$a = \lim_{\Delta t \to 0} \frac{(t + \Delta t)g - tg}{\Delta t}$$
$$\Rightarrow a = \lim_{\Delta t \to 0} \frac{tg + (\Delta t)g - tg}{\Delta t}$$
$$\Rightarrow a = \lim_{\Delta t \to 0} \frac{g\Delta t}{\Delta t} = g \text{ m s}^{-2}$$

Hence $a = g$ m s^{-2} at all times.

Remark

In general negative velocity is not considered. If it is velocity $-v$ refers to a speed v in a direction opposite to that of v. Negative acceleration, however, is considered. Negative acceleration is *slowing down*: *deceleration* (or *retardation*). Hence $-a$ may be referred to as an acceleration of $-a$ or a deceleration (or retardation) of a.

Remark

For the remainder of this chapter acceleration will always be constant on intervals ab. The acceleration can change from ab to bc but will be constant on ab and bc respectively. That is the instantaneous acceleration at any point in ab is equal to the average acceleration over ab, etc. This is *piece-wise constant* acceleration.

Remark

It will be seen in Calculus that

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \tag{2.8}$$

2.2 The Time-Velocity Graph

The time-velocity graph is the graph:

$$\{(t, v(t)) \, : \, t \in \mathbb{R}_+\} \tag{2.9}$$

For constant acceleration on an interval ab;

$$v = u + at$$

Compare with $y = mx + c$;

$$\Rightarrow m \sim a$$

Therefore the slope of the time-velocity graph at a time t is the acceleration. Also $c \sim u$ and the velocity axis intercept is $u = v(0)$. In this chapter the time-velocity graph will be piece-wise linear reflecting the piece-wise constant accelerations.

2.2.1 A Fundamental Theorem

What is the area under the time-velocity graph from $t = a$ to $t = b$. Start by subdividing the region into n strips S_1, S_2, ..., S_n of equal width T. The width of the interval $[a, b]$ is $b - a$ so the width of each of the n strips is

$$T = \frac{b - a}{n}.$$

Approximate the ith strip S_i by a rectangle with width T and height $v(t_i)$, which is the value of v at the left endpoint so that $t_{i+1} = t + T$. In particular $t_1 = a$ and $t_{n+1} = b$. Then the area of the ith rectangle is $v(t_i)\,T$. The area of the original shaded region is approximated by the sum of these rectangles:

$$A \approx v(t_1)\,T + v(t_2)\,T + \cdots + v(t_n)\,T. \tag{2.10}$$

This approximation becomes better and better as the number of strips increases, that is, as $n \to \infty$. Therefore the area of the shaded region is given by the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} [v(t_1)\,T + v(t_2)\,T + \cdots v(t_n)\,T]. \tag{2.11}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_i)\,T$$

Theorem

The area under the time-velocity graph from $t = a$ to $t = b$ is the distance travelled in the time from $t = a$ to $t = b$.

Proof

Let A be the area under the time-velocity graph from a to b.

$$v(t) = \lim_{\Delta t\to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t},$$

$$\Rightarrow A = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_i)\,T,$$

$$\Rightarrow A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\lim_{\Delta t\to 0} \frac{s(t_i+\Delta t) - s(t_i)}{\Delta t}\right] T.$$

Note that $n \to \infty$ and $\Delta t \to 0$ implies $T \to \Delta t$. This is because $T \underset{n\to\infty}{\to} 0$. Hence let $T \sim \Delta t$;

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\lim_{\Delta t\to 0} s(t_i+\Delta t) - s(t_i)\right] \frac{\Delta t}{\Delta t},$$

$$\Rightarrow A = \lim_{n\to\infty\to 0} \sum_{i=1}^{n} \lim_{\Delta t\to 0} [s(t_i+\Delta t) - s(t_i)].$$

Note $t_i + \Delta t = t_{i+1}$:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [s(t_{i+1}) - t(x_i)].$$

Note $\sum_i [s(t_{i+1}) - s(t_i)]$ forms a telescoping sum:

$$\begin{aligned}\sum_{i=1}^{n} [s(t_{i+1}) - s(t_i)] = s(t_2) - s(t_1) &+ \\ s(t_3) - s(t_2) &+ \\ s(t_4) - s(t_3) &+ \\ &\vdots \\ s(t_{n+1}) - s(t_n)&;\end{aligned}$$

where all but $s(t_{n+1}) - s(t_1)$ cancels. But $t_{n+1} = b$ and $t_1 = a$. Hence $s(t_{n+1}) = s(b)$ and $s(t_1) = s(a)$. Hence

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [s(t_{i+1}) - s(t_i)] = \lim_{n\to\infty} (s(t_{n+1}) - s(t_1)),$$

$$\Rightarrow A = s(b) - s(a).$$

□

Example

A particle travelling at constant velocity 4 m s^{-1} for 6 s travels 24 m as the area under the time-velocity graph is:

$$A = 4(6) = 24 \text{ m}$$

Remark

It will be seen in Calculus that the area under the graph of a function $f(x)$ from $x = x_1$ to $x = x_2$ is given by a number:

$$\int_{x_1}^{x_2} f(x)\,dx \tag{2.12}$$

Therefore the area under the time-velocity graph from $t = a$ to $t = b$:

$$\int_a^b v\,dt = \int_a^b \frac{ds}{dt}\,dt = \int_a^b ds = s(b) - s(a).$$

2.3 The Fundamental Equations

Let a particle move from the origin at $s = 0$ at a time $t = 0$ at a velocity of $u = v(0)$ with constant acceleration a.

2.3.1 Acceleration

$$a = \frac{v-u}{t} \tag{2.13}$$

Proof: When travelling at constant acceleration the instantaneous acceleration a equals the average acceleration $\bar{a}$.

$$a = \bar{a}$$
$$\Rightarrow a = \frac{v-u}{t}.$$

□

2.3.2 Velocity 1

$$v = u + at \tag{2.14}$$

Proof: Multiplying out (2.13) gives the result. □

2.3.3 Distance 1

$$s = \left(\frac{u+v}{2}\right)t \tag{2.15}$$

Proof: By Theorem 2.2.1 and the geometry of the time-velocity graph for a constant acceleration motion:

$$s = ut + \frac{1}{2}(v-u)t$$
$$\Rightarrow s = \left(u + \frac{1}{2}v - \frac{1}{2}u\right)t$$
$$\Rightarrow s = \left(\frac{u+v}{2}\right)t$$

□

Example

A car accelerates from a velocity of 10 m s^{-1} to 20 m s^{-2} in 4 s. How far does it travel in this time?
Solution:

$$s = \left(\frac{10+20}{2}\right)(4) \text{ m} = 60 \text{ m}.$$

2.3.4 Distance 2

$$s = ut + \frac{1}{2}at^2 \tag{2.16}$$

Proof: Substitute (2.14) into (2.15)

$$\begin{aligned} s &= \left(\frac{u+u+at}{2}\right)t \\ \Rightarrow s &= \left(\frac{2u+at}{2}\right)t \\ \Rightarrow s &= \left(u+\frac{1}{2}at\right)t \\ \Rightarrow s &= ut+\frac{1}{2}at^2 \end{aligned}$$

Example

A penny is dropped down a well. How far down has it fallen after 8 s in terms of g the *acceleration due to gravity*?
Solution: Once again it must be inferred that $u = 0$:

$$s = (0)(8) + \frac{1}{2}g(8)^2 = 32g \text{ m}$$

2.3.5 Velocity 2

$$v^2 = u^2 + 2as \tag{2.17}$$

Proof: From (2.13):

$$\begin{aligned} a &= \frac{v-u}{t} \\ \Rightarrow t &= \frac{v-u}{a}. \end{aligned}$$

Substitute into (2.15)

$$\begin{aligned} s &= \left(\frac{u+v}{t}\right)\left(\frac{v-u}{a}\right) \\ \Rightarrow s &= \frac{v^2-u^2}{2a} \\ \Rightarrow 2as &= v^2-u^2 \\ \Rightarrow v^2 &= u^2+2as \end{aligned}$$

□

Example

A car decelerates from u m s^{-1} to 5 m s^{-1} with an acceleration -3 m s^{-2} in 4 m. Calculate u.
Solution:

$$v^2 = u^2 + 2as$$
$$\Rightarrow u^2 = v^2 - 2as$$
$$\Rightarrow u = \sqrt{v^2 - 2as} = \sqrt{25 - 2(-3)(4)} = \sqrt{49} = 7 \text{ m s}^{-1}.$$

2.3.6 Summary

Consider the five variables $\{u, v, a, t, s\}$. It is a fact that given three of these variables the other two can be found using the equations (2.13-2.17). This is how the most basic problems are solved.

Remark

These equations are only valid for SI units. The relevant SI units are:

quantity	quantity symbol	SI unit	unit symbol
time	t	second	s
distance	s	metre	m
velocity	v	metres per second	m s^{-1}
acceleration	a	metre per second per second	m s^{-2}

Example

Convert 10 km/hour to SI units.
Solution: 1 km = 1000 m. 1 hour = 60 minutes. 1 minute = 60 s. Hence 1 hour = 60 × 60 s:

$$10 \text{ km/hour} = 10\frac{\text{km}}{\text{hour}} = 10\frac{1000}{3600} \text{ m s}^{-1} = \frac{25}{9} \text{ m s}^{-1}$$

2.4 Accelerated Linear Motion: Physical Interpretation

2.4.1 The Basics

The physical interpretation of a problem is translating it into the theoretical framework; i.e. s, t, v, u and a. To recap:

(s) The distance travelled

(t) The time taken to travel s

(u) The initial velocity at $t = 0$, $v(0)$

(v) The velocity after t seconds

(a) The constant acceleration; be it positive or negative. Negative acceleration may be termed *deceleration* or sometimes *retardation*

2.4.2 Motion *From Rest* and *To Rest*

From Rest means that $u = 0$, the initial speed is zero. Another example of $u = 0$ is when a particle is dropped. When a particle is dropped the initial speed is 0 and the particle falls under the constant *acceleration due to gravity*, $g = 9.8$ m s^{-2}. Later in the course it will be seen why this acceleration is constant.

Similarly *To Rest* means that $v = v(t) = 0$ where t is the duration of the motion. Another example of $v = 0$ is at the maximum height of a particle thrown upwards. Suppose a particle is thrown up with initial velocity u. Again gravity acts as a constant acceleration towards Earth g. In this case the positive sense of the motion is up so therefore $a = -g$. Now

$$\begin{aligned} v &= u + at \\ \Rightarrow v &= u - gt \\ \underset{t=u/g}{\Rightarrow} v &= 0 \end{aligned}$$

For $t > u/g$, the particle has negative velocity; i.e. the particle is returning to Earth.

Example

A stone is thrown upwards with a speed 10 m s^{-1}. What is the maximum height of the stone in the subsequent motion in terms of g?
Solution: The stone has initial velocity, $u = 10$ m s^{-1}. The acceleration is the acceleration due to gravity, $g = 9.8$ m s^{-1}; in this case $a = -g$. The stone reaches its maximum height when $v = 0$:

$$\begin{aligned} v &= u - gt \\ \Rightarrow 0 &\overset{!}{=} 10 - gt \\ \Rightarrow t_{\max} &= \frac{10}{g} \end{aligned}$$

Hence

$$\begin{aligned} s_{\max} &= ut_{\max} + \frac{1}{2}at_{\max}^2 \\ \Rightarrow s_{\max} &= 10\frac{10}{g} - \frac{1}{2}g\frac{100}{g^2} = \frac{50}{g} \end{aligned}$$

2.4.3 Speed Limit and Max Speed

Suppose a particle accelerates from an initial velocity u to a maximum speed $v_{\max}$ with acceleration a. Suppose $v = v_{\max}$ after T seconds. The time-velocity graph of this motion comprises an initial increase in velocity of constant slope a. For $t > T$, the graph takes on the constant value $v_{\max}$ [see diagram]. What is the distance travelled after t seconds?

$t \le T$ When $t \le T$, the particle is under constant acceleration so the equations for accelerated linear motion may be used.

$$s = ut + \frac{1}{2}at^2, \;\; t \le T$$

$t > T$ When $t > T$ the particle is not under constant acceleration. $a = a$ for $t \le T$, $a = 0$ for $t > T$. However the area under the graph gives the distance travelled. The area under the graph from $t = 0$ to $t = T$ is just s after T seconds: $uT + aT^2/2$. The area under the curve from $t = T$ to t is just $v_{\max}(t - T)$ [see diagram].

$$s = uT + \frac{1}{2}aT^2 + v_{\max}(t - T), \;\; t > T$$

2.4.4 Aggregated Distances

Suppose a particle covers x m in the first t_1 s after it passes a signal post. Suppose further it covers y m in the next t_2 s. The question may be posed: how far will the particle travel in the next t_3 seconds? This problem pops up often in exam papers. To solve, the motion can be broken into two segments. There are two *knowns* in each of the two segments; namely s and t. Thence *two equations in two unknowns may be formed*. These simultaneous equations may be solved for the unknowns u and a. It remains to use this knowledge to find how far the particle will travel in the final segment.

There is a common trap one can fall into here solving this problem. Worked Example 2.5 will be solved using the naïve approach:

Example

A train covers 24 m in the first two seconds after it passes a signal post. It then covers 51 m in the next 3 seconds. Assuming that the train is accelerating uniformly, find out how far it will go in the next 3 seconds.
Naïve Approach: Let the initial velocity be u. Let the acceleration be u. Looking at the information for the first two seconds:

$$u = u$$
$$a = a$$
$$t = 2$$
$$s = 24$$

Using $s = ut + at^2/2$;

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow 24 = u(2) + \frac{1}{2}a(4)$$
$$\Rightarrow 24 = 2u + 2a$$
$$u + a = 12 \tag{2.18}$$

Now look at the information for the next three seconds:

$$u = u$$
$$a = a$$
$$t = 3$$
$$s = 51$$

Using $s = ut + at^2/2$;

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow 51 = u(3) + \frac{1}{2}a(9)$$
$$\Rightarrow 51 = 3u + \frac{9}{2}a$$
$$6u + 9a = 102 \tag{2.19}$$

Equations (2.18) & (2.19) are simultaneous equations in u and a. Solving yields:

$$u = 2, \quad a = 10 \tag{2.20}$$

However, this is *not* the correct answer. The train travelled $24 + 51 = 75$ m in the first 5s. But!

$$s = 2(5) + \frac{1}{2}(10)(5^2) = 135 \text{ m}$$

The error occurs when the information for the *next three seconds was examined*. In this second segment it is true to say $a = a$, $t = 3$ and $s = 51$. However it is *not* true to say $u = u$. For the second segment $u > u$ as the train has accelerated since passing the signal post at speed u. How is this problem resolved?
Correct Approach: Instead the information for the first *five* seconds (first segment *plus* second segment) is examined. Now it is true to say $u = u$. But, $t = 2 + 3 = 5$ and $s = 24 + 51 = 75$. Hence

$$u = u$$
$$a = a$$
$$t = 5$$
$$s = 75$$

Using $s = ut + at^2/2$;

$$
\begin{aligned}
s &= ut + \frac{1}{2}at^2 \\
\Rightarrow 75 &= u(5) + \frac{1}{2}a(25) \\
\Rightarrow 75 &= 5u + \frac{25}{2}a \\
\Rightarrow 150 &= 10u + 25a
\end{aligned}
$$

$$2u + 5a = 30 \tag{2.21}$$

Equations (2.18) & (2.21) are simultaneous equations in u and a. Solving yields:

$$u = 10, \quad a = 2 \tag{2.22}$$

Now to find the distance travelled in the next three seconds note the information is

$$
\begin{aligned}
s &=? \\
a &= 2 \\
t &= 3 \\
v &=? \\
u &=? \ (\neq u!!)
\end{aligned}
$$

Three unknowns. This is not solvable. However if the first *eight* seconds (first segment *plus* second segment *plus* third segment) are examined:

$$
\begin{aligned}
s &=? \\
a &= 2 \\
t &= 8 \\
v &=? \\
u &= 10
\end{aligned}
$$

Using $s = ut + at^2/2$;

$$s = 10(8) + \frac{1}{2}(2)(64) = 144 \text{ m}$$

Hence the distance travelled in the last three seconds is the distance travelled in the eight seconds *less* the distance travelled in the first five seconds:

$$\textbf{Ans} = 144 - 75 = 69 \text{ m}$$

2.4.5 Maximum Distance

Another notion of maximum distance is as follows. Suppose a particle has maximum acceleration a and maximum deceleration d. What is the maximum distance the particle can travel in T seconds if the particle begins and ends at rest? In the first instance it shouldn't be hard to see why the motion that satisfies this question is maximum acceleration from rest followed immediately by maximum deceleration to rest [see diagram].

Similarly, suppose a particle with maximum acceleration a and maximum deceleration d wants to travel a distance x m from rest to rest. What type of motion makes the time taken to cover x m a minimum? Again it is maximum acceleration from rest followed immediately by maximum deceleration to rest is the required motion. There is a helpful fact about this type of motion.

Theorem

Suppose a particle accelerates from rest to a velocity with acceleration a for a time t_1 and immediately decelerates to rest with a deceleration d for a time t_2 seconds. Then

$$t_1 : t_2 = d : a \tag{2.23}$$

Proof

Suppose after t_1 s the particle has velocity v. Then for the accelerating segment of the motion:

$$\begin{aligned} s &= ? \\ t &= t_1 \\ u &= 0 \\ v &= v \\ a &= a \end{aligned}$$

Hence using $v = u + at$:

$$v = 0 + at_1$$
$$v = at_1 \tag{2.24}$$

In the decelerating part of the motion:

$$\begin{aligned} s &= ? \\ t &= t_1 \\ u &= v \\ v &= 0 \\ a &= -d \end{aligned}$$

Hence

$$\begin{aligned} v &= u + at \\ \Rightarrow 0 &= v - dt_2 \end{aligned}$$
$$v = dt_2 \tag{2.25}$$

Setting (2.24)=(2.25):

$$\begin{aligned} at_1 &= dt_2 \\ \Rightarrow \frac{t_1}{t_2} &= \frac{d}{a} \\ \Rightarrow t_1 : t_2 &= d : a \end{aligned}$$

□

Example: Worked Example 2.10 (ii)

A car can accelerate with acceleration 2 m/s^2, and deceleration 3 m/s^2. Find the least possible time it takes to cover a distance of 375 m, from rest to rest.
Solution: Let T be the least possible time it takes for the car to cover a distance of 375 m. The motion that achieves this is maximum acceleration followed immediately by maximum deceleration. Let the time at maximum acceleration be T_1 and the time at maximum deceleration be T_2. Let $a = 2$, $d = 3$. From the theorem;

$$T_1 : T_2 = d : a \tag{2.26}$$

$$T_1 : T_2 = 3 : 2 = \frac{3}{5} : \frac{2}{5}$$
$$\therefore T_1 = \frac{3}{5}T \text{ and } T_2 = \frac{2}{5}T$$

In the accelerating part of the motion:

$$\begin{aligned} s &= ? \\ t = T_1 &= \frac{3}{5}T \\ u &= 0 \\ v &= ? \\ a &= 2 \end{aligned}$$

Hence using $v = u + at$;

$$\begin{aligned} v &= 0 + 2\left(\frac{3}{5}T\right) \\ \Rightarrow v &= \frac{6}{5}T \end{aligned}$$

Considering the velocity-time graph, the area under the graph must equal 375. The graph is a triangle, and the area of a triangle = (base)×(perpendicular height)/2;

$$\begin{aligned} \therefore \frac{1}{2}T\left(\frac{6}{5}T\right) &\overset{!}{=} 375 \\ \Rightarrow \frac{3}{5}T^2 &= 375 \\ \Rightarrow T^2 &= 625 \\ \Rightarrow T &= 25 \text{ s} \end{aligned}$$

Ans: 25 s.

2.4.6 Overtaking and Crashing

Suppose a car A travelling at constant velocity of u m s^{-1} passes a car B at rest and suppose further that car B accelerates uniformly as the car A passes. When does car B overtake car B? Car B overtakes car A when *the distance travelled by both cars is equal and the velocity of car B is greater than the velocity of car A.*

Similarly consider a two trains on a single track. Suppose train A travelling at constant velocity u is x m behind train B which accelerates from rest with acceleration a. The question may be posed, what acceleration must train B have to *just* avoid a crash with train A (in this chapter the lengths of cars and trains are ignored)? Clearly the trains will crash if train A is travelling faster than train B when train A catches up to train B. So to *just* avoid a crash, $v_B = v_A$ when train A catches train B. When does train A catch train B? When $s_A = x + s_B$. Let a be the acceleration that achieves this.
Luckily, $v_A = u$ a constant, so $v_B \overset{!}{=} u$ when $s_A = x + s_B$. When is $v_B = u$?

$$t = \frac{u}{a}$$

Now,

$$\begin{aligned} s_A &= x + s_B \\ \Rightarrow ut &= x + \frac{1}{2}at^2 \\ \Rightarrow u\frac{u}{a} &= x + \frac{1}{2}a\frac{u^2}{a^2} \\ \Rightarrow a &= \frac{u^2}{2x} \end{aligned}$$

Example

Suppose at the instant a car takes off with acceleration 2 m s^{-2} a bus travelling at constant speed 10 m s^{-1} overtakes the car . If the car reaches a maximum speed of 12 m s^{-1}, after how long can the car overtake the bus again?
Solution: $\therefore$ The car can overtake the bus when $s_c = s_b$ and $v_c > v_b$. Can the car overtake the bus before it reaches maximum speed? How long does the car take to reach maximum speed?

$$t = \frac{v-u}{a} = \frac{12}{2} = 6\,\text{s}$$

After 6 s,

$$\begin{aligned} s_c &= \frac{1}{2}2(36)\text{ m} = 36\text{ m} \\ s_b &= (10)(6)\text{ m} = 60\text{ m} \end{aligned}$$

Hence overtaking takes place after $t = 6$ s. Suppose overtaking occurs after t seconds:

$$\begin{aligned} s_c &= 36 + 12(t-6) \\ s_b &= 10t \\ \underset{s_c=s_b}{\Rightarrow} 36 + 12t - 72 &= 10t \\ \Rightarrow 2t &= 36 \\ \Rightarrow t &= 18\text{ s} \end{aligned}$$

Also $v_c > v_b$ after 18 s.
Ans: The car can overtake the bus after 18 s.

Example (Ex. 2.C.23)

A passenger train, which is travelling at 80 m/s is 1,500 m behind a goods train which is travelling at 30 m/s in the same direction on the same track. At what rate must the passenger train decelerate to avoid a crash? (Ignore the length of the trains.)
Solution: To *just* avoid a crash, when the passenger train has velocity $v_p = 30$ m/s,

$$s_p \overset{!}{=} s_g + 1500 \tag{2.27}$$

To achieve this, let the deceleration of the train be given by $a =: -d$. Suppose $v_p = 30$ after T seconds;

$$\begin{aligned} v &= u + at \\ \Rightarrow 30 &= 80 - dT \\ \Rightarrow T &= \frac{50}{d} \end{aligned}$$

What is s_g after T seconds?

$$\begin{aligned} s_g &= ut + \frac{1}{2}at^2 \\ \Rightarrow s_g &= 30\left(\frac{50}{d}\right) + \frac{1}{2}(0)T^2 = \frac{1500}{d} \end{aligned}$$

What is s_p?

$$\begin{aligned} s_p &= 80\left(\frac{50}{d}\right) + \frac{1}{2}(-d)\frac{50^2}{d^2} \\ \Rightarrow s_p &= \frac{4000}{d} - \frac{1250}{d} = \frac{2750}{d} \end{aligned}$$

Now by (2.27)

$$\begin{aligned} \frac{2750}{d} &= 1500 + \frac{1500}{d} \\ \Rightarrow d &= \frac{5}{6}\text{ m/s}^2 \end{aligned}$$

Ans: 5/6 m/s^2.

2.5 Exercises 2.C (F.A.M.)

1. A cyclist accelerates from rest with acceleration 2 m/s^2 to a maximum speed of 10 m/s, and then continues at this speed. Show this motion on a time-speed graph and hence, or otherwise, find out how long it will take the cyclist to cover 100 m.

2. A car accelerates from rest for 15 s with a uniform acceleration of 1.5 m/s^2 and immediately decelerates with a uniform deceleration of 5 m/s^2 to rest. How long does the deceleration take? How far will the car go during the entire journey?

3. A car accelerates from rest at 2 m/s^2 to a maximum speed of 24 m/s. It immediately decelerates to rest covering a distance of 48 m during deceleration. Find:
   - the time used up during each part of the journey
   - the average speed throughout

4. A car starts from rest at a point p and moves with acceleration 4 m/s^2. As it starts, another car passes p moving in the same direction at a uniform speed of 20 m/s. Find at what time overtaking will occur and how far from p.

5. 2 cars travelling in the same direction side by side pass a traffic light simultaneously. The Metro is travelling at 10 m/s with acceleration 3 m/s^2; the Corsa is travelling with speed 20 m/s and acceleration 2 m/s^2. Write in terms of the time, t:
   - the velocity of each car at time t
   - the distance travelled by each car

   Hence find the time at which:
   - they are travelling at the same speed
   - they are side by side again

6. A car moves from rest with uniform acceleration of 1 m/s^2 followed by immediate deceleration of 3 m/s^2 to rest. The total time taken is 20 s. Find:
   - the time spent on each part of the journey
   - the maximum speed reached
   - the total distance covered

7. A monorail train leaves Tokyo Central where it had stopped and moves with acceleration 2 m/s^2 to a maximum speed and immediately decelerates at 7 m/s^2 to reach Kinahawa Station. If a journey takes 1.5 minutes find the distance between the two stations.

8. A train, which is accelerating uniformly, passes a point P at speed. In the next two seconds it travels a distance 18 m and in the two seconds after that covers a further 30 m. Find:
   - its acceleration
   - its speed as it passes P
   - the distance it will travel in yet another 2 seconds

9. A train travels 39 m, 37 m, 35 m in each of three consecutive seconds along a straight railway track. Show that this is consistent with motion under constant deceleration. Find out how much farther the train will travel before coming to rest.

10. A sprinter wins a race with constant acceleration throughout. During the race he passes four posts a, b, c, d in a straight line such that $|ab| = |bc| = |cd|$ =20 m. If the sprinter takes 5 s to go from a to b, and 3 s to go from b to c, find how long, to the nearest tenth of a second, it takes him to go from c to d.

11. A train starting from rest at P accelerates uniformly for 10 s, travels at a uniform speed for 32 s and decelerates for 6 s to rest at Q. Draw a rough time-velocity graph of this motion. If the distance $|PQ|$ is 1 km find the maximum speed reached and the distance travelled during each part of the journey.

12.
    - A train travels from rest at A to rest at B in one minute. It starts by accelerating uniformly for 12 s and finishes by decelerating uniformly for 8 s. In between it travels with uniform velocity. If the distance covered is 1 km, find the maximum speed reached.
    - As it was leaving A another train passed it moving in the same direction on a parallel track with a uniform speed of 14 m/s. After how long, and how far from A, will overtaking occur? (Neglect the length of the trains.)

13. An underground train accelerating uniformly passes 4 signals in a straight line, each 24 m apart. If it takes the train 2 s to go from the first to the second, 1 s to go from the second to the third, how long will it take to go from the third to the forth (correct to 3 decimal places).

14. A car can accelerate 1 m/s^2 and decelerate at 2 m/s^2. How long will it take to travel a distance of 300 m from rest to rest
    - subject to a 16 m/s speed limit
    - subject to no speed limit?

15. A car can move with acceleration $3a$ and declaration $5a$. Find, in terms of k, the time taken to cover a distance of $90ak^2$ from rest to rest
    - subject to a speed limit of $15ak$
    - subject to no speed limit?

16. Cyclists A and B leave the same spot from rest and travel in opposite directions. Cyclist A accelerates at 0.5 m/s^2, whilst cyclist B accelerates at 1 m/s^2. How far apart will they be after 10 s? After how long *more* will they be 108 m?

17. Two cars moving at speed in opposite directions pass a traffic light. The first one is travelling with speed 10 m/s and acceleration 1 m/s^2; the other is travelling with speed 20 m/s and deceleration 5 m/s^2. How far apart will they be when the second car comes to rest? At what time were they half this distance apart?

18. A car starting from rest and accelerating uniformly, travels a distance d in the first n seconds of motion and a distance k in the next n seconds. Prove that $k = 3d$.

19. At a certain moment in the Tour de France, Alberto is 22 m behind Gustav. Alberto is cycling at 12 m/s and accelerating at 1 m/s^2. Gustav, who has just fixed a puncture, starts from rest and accelerates with acceleration 2 m/s^2.
    - After how long will Alberto catch up with Gustav?
    - After how long more will Gustav overtake Alberto again?

20. A car has to travel a distance s on a straight road. The car has maximum acceleration a and maximum deceleration d. It starts and ends at rest.
    - Show that, if there is a speed limit of v m/s, the time taken to complete the journey is given by
    $$\frac{v}{2a} + \frac{v}{2b} + \frac{s}{v}$$
    - Show that is there is no speed limit, the time is given by
    $$\sqrt{2s\left(\frac{a+b}{ab}\right)}$$

21. LC HL 1981
A body starts from rest at p, travels in a straight line and then comes to rest at q which is 0.696 km from p. The time taken is 66 m. For the first 10 seconds it has uniform acceleration a_1. It then travels at a constant speed and is finally brought to rest by a uniform deceleration a_2 acting for 6 seconds. Calculate a_1 and a_2.
If the journey from rest at p to rest at q had been travelled with no interval of constant speed, but subject to a_1 for a time t_1 and followed by a_2 for time t_2, show that the time for the journey is $8\sqrt{29}$ seconds.

22.
    - A particle travels, starting with initial speed u, with uniform acceleration a. Show that the distance travelled during the nth second is $u + an - a/2$.
    - If the particle travels 17 m in the 2nd second of motion and 47 m in the 7th second of motion, how far will it go:
        - in the 10th second of motion?
        - in the nth second of motion?
    - During which two consecutive seconds will it cover 256 m?

23. A passenger train, which is travelling at 80 m/s is 1,500 m behind a goods train which is travelling at 30 m/s in the same direction on the same track. At what rate must the passenger train decelerate to avoid a crash? (Ignore the length of the trains.)

24. LC HL 1984
The driver of a car travelling at 20 m/s sees a second car 120 m in front, travelling in the same direction at a uniform speed of 8 m/s.
    - What is the least uniform retardation that must be applied to the faster car so as to avoid a collision?
    - If the actual retardation is 1 m/s^2, calculate:
        - the time interval in seconds for the faster the cat to reach a point 66 m behind the slower.
        - the shortest distance between the cars.

Chapter 3

Projectiles

3.1 Introduction

In Accelerated Linear Motion only motion is a single direction is studied. The first extension that may be made to this theory is to consider motion in *two* directions. In LC Applied Maths a particularly nice class of such motion is that of *projectiles.* A body thrown or projected with a particular initial speed and direction is a projectile. Its subsequent motion is determined entirely by external forces such as gravity and air resistance. In LC Applied Maths the effects of air resistance are ignored and this simplifies the problem considerably.

3.2 Background Theory

In Newton's Theory of Gravitation, two particles of mass m_1 and m_2, a distance r apart, attract each other with an equal but opposite force $\mathbf{F}$ given by:

$$\mathbf{F} = \frac{Gm_1m_2}{r^2} \tag{3.1}$$

where G is a universal constant. In the case of a projectile above Earth, each and every particle in Earth is attracted to the particle according to (3.1). However if the Earth is assumed to be a perfect sphere of uniform density, symmetry arguments prove that the aggregate gravitational affects of the multitudinous particles is equivalent to that of a single particle at the centre of Earth of mass equal to that of the earth. In that case, let M be the mass of earth, R the radius of Earth and consider a projectile of mass m a height r above Earth. It will be seen later that for a particle of mass m, the acceleration of the particle is given by:

$$m\mathbf{a} = \sum \mathbf{F} \tag{3.2}$$

where the sum is over all forces on the body. Hence consider the force on the projectile - what is ordinarily called the *weight* of the projectile. The only force acting on the projectile is that of gravity:

$$ma = \frac{GMm}{(R+r)^2}$$
$$\Rightarrow a = \frac{GM}{(R+r)^2}$$

Now suppose the restriction is made to projectiles at or near the Earth's surface such that

$$r \ll R \tag{3.3}$$

That is the radius of the Earth far, far exceeds that of the height of the projectile. Then $R + r \simeq R$ and

$$a = \frac{GM}{R^2}$$

G, M and R have known values and it happens that:

$$a = 9.8 \text{ m/s towards Earth} =: g \tag{3.4}$$

g is called the *acceleration due to gravity.* That is, under these assumptions, the nature of the projectile is insignificant and *all bodies fall to Earth with the same acceleration, namely g.* In LC Applied Maths, this is the only force on a projectile and all projectiles have the acceleration g.

3.3 Constructing the Model Equations

3.3.1 Introduction

Consider the motion of a projected particle. Let $\mathbf{r}(t)$ describe the position of the projectile in terms of the time after projection t. Note $\mathbf{r}(t)$ is a vector quantity, at each time t it describes a magnitude (the distance from the point of projection) and the direction (from the point of projection). As a vector quantity in two dimensions it can be decomposed in the i-j basis:

$$\mathbf{r}(t) = s_x(t)\mathbf{i} + s_y(t)\mathbf{j} \tag{3.5}$$

Where $s_x(t)$ and $s_y(t)$ are the distances travelled in the time t along the x and y-directions respectively.
Let $\mathbf{v}(t)$ describe the subsequent velocity of the particle in terms of the time after projection t. Note $\mathbf{v}(t)$ is also vector quantity, at each time t it describes a magnitude (the speed) and the direction. As a vector quantity in two dimensions it can also be decomposed in the i-j basis:

$$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} \tag{3.6}$$

where $v_x(t)$ and $v_y(t)$ are the speeds at the time t along the x and y-directions respectively.
Suppose it is projected with velocity $\mathbf{v}(0) =: \mathbf{u}$:

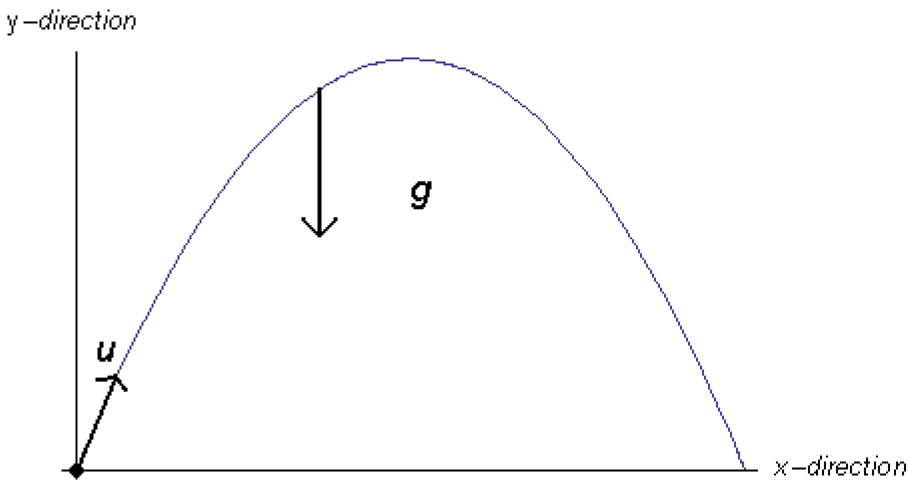


Figure 3.1: Note this is *not* a time-velocity graph

Definition

The initial speed in the x-direction is defined as:

$$u_x := v_x(0) \tag{3.7}$$

In LC Applied Math the primary focus is on particles that leave Earth with a velocity $\mathbf{v}(0)$, fly in the air and land. Once the particle has landed back to Earth, the subsequent motion does not generally concern us. In that vein, note that a particle has a height above the Earth of zero twice: once at the moment of projection and again upon landing.

Definition

The *Time of Flight*, T for a projectile is the time interval between initial projection and landing.
Equivalently T is the time when $s_y(T) = 0$.

Definition

The *range* of a projectile is $s_x(T)$.
Equivalently, s_x when $s_y = 0$.

Definition

The final speed in the x-direction is thus defined by

$$v_x := v_x(T) \tag{3.8}$$

3.3.2 Motion in the x-Direction

Since gravity acts in the y-direction only, there is *no* acceleration in the x-direction. The motion can be decomposed into two independent directions and hence consider the motion in the x-direction:

$$\begin{aligned} s &= s_x \\ t &= t \\ u &= u_x \\ v &= v_x \\ a &= 0 \end{aligned}$$

Using

$$\begin{aligned} v &= u + at \\ s &= ut + \frac{1}{2}at^2 \\ \Rightarrow v_x &= u_x \\ \Rightarrow s_x &= u_x t \end{aligned}$$

3.3.3 Motion in the y-Direction

Similarly the motion in the y-direction is independent. The introduction above shows us that in the y- or **j**- direction there is an acceleration of $-g$. In the case of the x-direction there is no acceleration and so the speed is constant. In the y-direction this is not the case. Using

$$\begin{aligned} v &= u + at \\ \Rightarrow v_y(t) &= u_y - gt \end{aligned}$$

Hence consider the motion in the y-direction:

$$\begin{aligned} s &= s_y \\ t &= t \\ u &= u_y \\ v &= v_y \\ a &= -g \end{aligned}$$

Using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \Rightarrow s_y &= u_y t - \frac{1}{2}gt^2 \end{aligned}$$

Remark

The maximum height is achieved when $v_y = 0$:

$$
\begin{aligned}
v_y &= u_y - gt \\
\Rightarrow t_{\text{max}} &= \frac{u_y}{t} \\
\Rightarrow s_{y_{\text{max}}} &= u_y \left(\frac{u_y}{g} \right) - \frac{1}{2} g \left(\frac{u_y}{g} \right)^2 \\
\Rightarrow s_{y_{\text{max}}} &= \frac{u_y^2}{2g}
\end{aligned}
$$

3.3.4 The Model Equations

The motion of a projectile is described by the following equations with the above notation:

$$v_x = u_x \tag{3.9}$$

$$s_x = u_x t \tag{3.10}$$

$$v_y = u_y - gt \tag{3.11}$$

$$s_y = u_y t - \frac{1}{2} g t^2 \tag{3.12}$$

3.3.5 Using the Equations

If given the initial velocity, $\mathbf{u}$, two pieces of information are given: the speed and the direction. Suppose $\mathbf{u} = (|\mathbf{u}|, \theta)$. This can be composed into x- ($\mathbf{i}$-) and y- ($\mathbf{j}$-) components easily. Thence consider the following diagram.

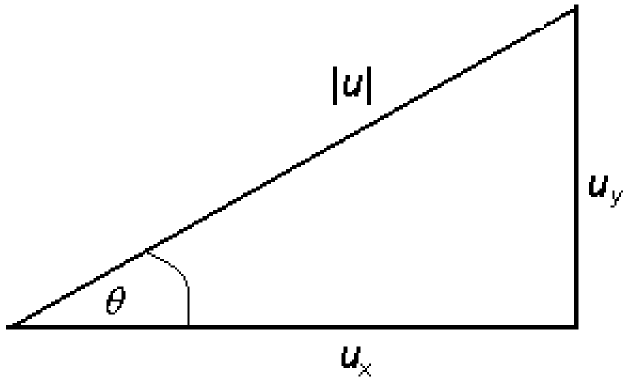


Figure 3.2: Decomposing the initial velocity $\mathbf{u} = \mathbf{v}(0)$ into its $\mathbf{i}$ and $\mathbf{j}$ components u_x and u_y

The speed of the initial velocity is $|\mathbf{u}|$ - the hypotenuse in the triangle:

$$\sin\theta = \frac{u_y}{|\mathbf{u}|}$$
$$\cos\theta = \frac{u_x}{|\mathbf{u}|}$$
$$\Rightarrow u_x = |\mathbf{u}|\cos\theta$$
$$\Rightarrow u_y = |\mathbf{u}|\sin\theta$$

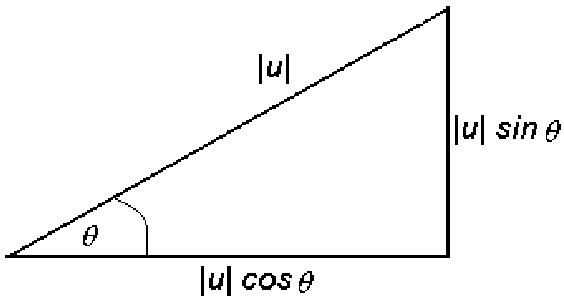


Figure 3.3: Decomposing the initial velocity $\mathbf{u} = \mathbf{v}(0)$ into as $\mathbf{u} = |\mathbf{u}|\cos\theta\,\mathbf{i} + |\mathbf{u}|\sin\theta\,\mathbf{j}$

3.4 Exercises 3. A HL (F.A.M.)

20. The greatest height reached by a particle above the horizontal plane from which it is fired is 3.6 m. It's range is 19.2 m. Find the tan of the angle of projection and the initial speed u.
    What would the maximum range of this particle be if the angle of projection can be changed but the initial speed remains fixed.

21. (a) If $\sin A = 0.5$, find two values of A where $0° < A < 360°$.
    (b) A particle is fired from a point on a horizontal plane with initial speed 28 m/s at an angle α to the plane.
    If its range is 40 m, find two possible angles of projection.

22. A particle is fired from a point on a horizontal plane with initial speed 10 m/s. If the greatest height reached above the plane is 2.5 m, find the sin of the angle of projection.

23. A particle is fired so as to have maximum possible range. Show that the angle of projection should be 45° and that the ratio of the greatest height to the range is 1:4.

25. A jet-fighter is flying horizontally at a *constant* height of 210 m with constant speed 140 m/s. As it passes over a gun, the gunner fires a bullet with speed $70\sqrt{5}$ m/s at an angle $\tan^{-1}(1/2)$ to the horizontal. If the plane of the bullet's flight is also the plane of the fighter's flight find when the bullet will strike the fighter.

26. A bird flies out of a tree exactly 5.6 m directly above a hunter's gun and flies at constant speed 28 m/s in the horizontal direction. If the speed of the bullets from the gun is 35 m/s, and the gun is fired just as the bird is leaving the tree show that:
    (i) the angle of projection must be $\tan^{-1}(3/4)$ if the bullet is to hit the bird.
    (ii) the time taken for the bullet to reach the bird is 2/7 s.

27. A projectile is fired from a point p on a horizontal plane with initial speed 35 m/s at an angle A to the plane. Show that if x and y are the horizontal and vertical distances of the particle from p, then

$$250y = 250x \tan A - (1 + \tan^2 A)x^2.$$

    If the projectile strikes a small target whose horizontal and vertical distances from p are 40 m and 20 m respectively, find 2 values for $\tan A$ and the time taken to reach the target in each case.

3.5 Projectiles on the Inclined Plane

3.5.1 Introduction

Suppose a projectile is fired with initial speed u at an angle α to the horizontal from the foot of a hill which is uniformly sloped at an angle θ to the horizontal:

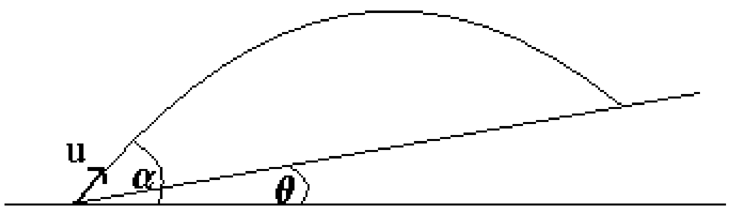


Figure 3.4: The Projectile in the Inclined Plane

It is true that our preceding analysis is valid here as long as the projectile is above the surface of the hill. However it becomes a cruel algebraic-book-keeping exercise to always check that the particle is above the surface of the hill, etc. Instead let the hill be the new x- or **i**-axis, and transform the equations appropriately. The new x axis is the old horizontal axis shifted and angle θ upwards, similarly the new y-axis is the old vertical axis shifted back an angle θ so that the new x- and y-axis are mutually perpendicular. So the new picture is:

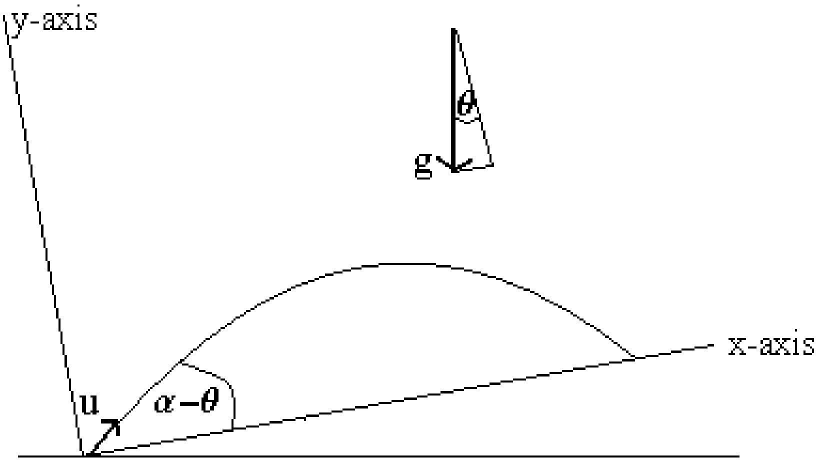


Figure 3.5: When the axes are transformed, the initial speed and gravity force must be too

3.5.2 The Initial Velocity

The initial velocity is decomposed as follows:

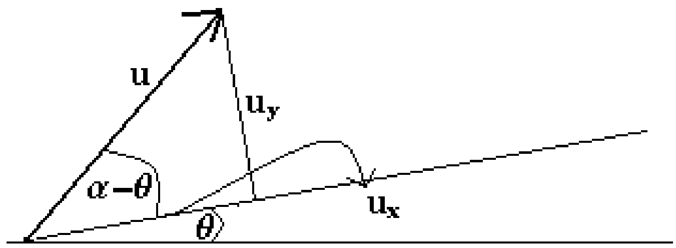


Figure 3.6: Decomposing the initial velocity in the x- and y-directions

Now

$$\begin{aligned}\cos(\alpha-\theta) &= \frac{u_x}{u}\\ \sin(\alpha-\theta) &= \frac{u_y}{u}\\ \Rightarrow u_x &= u\cos(\alpha-\theta)\\ \Rightarrow u_y &= u\sin(\alpha-\theta)\end{aligned}$$

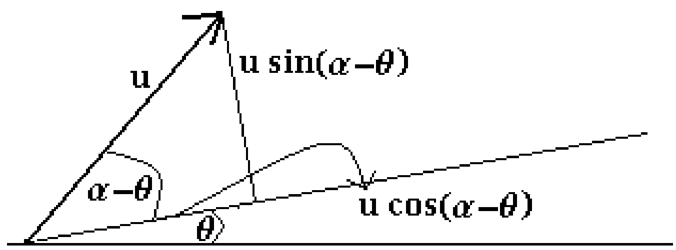


Figure 3.7: $\mathbf{u} := \mathbf{v}(0) = u\cos(\alpha-\theta)\mathbf{i} + u\sin(\alpha-\theta)\mathbf{j}$

3.5.3 The Acceleration due to Gravity

Due to the geometry of the transformed axes, the acceleration due to gravity, g, makes an angle θ with the y-axis:

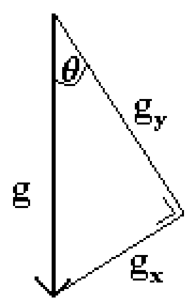


Figure 3.8: Decomposing the acceleration due to gravity in the x- and y-directions

Now

$$\cos\theta = \frac{g_y}{g}$$
$$\sin\theta = \frac{g_x}{g}$$
$$\Rightarrow g_x = g\sin\theta$$
$$\Rightarrow g_y = g\cos\theta$$

Note that in both directions, g_x and g_y is adverse in that it is a deceleration as the direction of the acceleration is opposite to that of the motion. So both will be negative as accelerations. Hence for the x- and y-motions respectively:

$$a_x = -g\sin\theta$$
$$a_y = -g\cos\theta$$

3.5.4 Motion in the x-Direction

Using

$$v = u + at$$
$$\Rightarrow v_x = u_x + a_x t$$
$$\Rightarrow v_x = u\cos(\alpha - \theta) - g\sin\theta\, t$$

Using

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s_x = u_x t - \frac{1}{2}gt^2$$
$$\Rightarrow s_x = u\cos(\alpha - \theta)\, t - \frac{1}{2}g\sin\theta\, t^2$$

3.5.5 Motion in the y-Direction

Similarly:

$$v_y = u\sin(\alpha - \theta) - g\cos\theta\, t$$

and

$$s_y = u\sin(\alpha - \theta)\, t - \frac{1}{2}g\cos\theta\, t^2$$

3.5.6 The Model Equations

The motion of a projectile in the inclined plane is modelled by the following equations with the above notation:

$$v_x = u\cos(\alpha - \theta) - g\sin\theta\, t \tag{3.13}$$
$$s_x = u\cos(\alpha - \theta)\, t - \frac{1}{2}g\sin\theta\, t^2 \tag{3.14}$$
$$v_y = u\sin(\alpha - \theta) - g\cos\theta\, t \tag{3.15}$$
$$s_y = u\sin(\alpha - \theta)\, t - \frac{1}{2}g\cos\theta\, t^2 \tag{3.16}$$

Chapter 4

Relative Velocity

4.1 Introduction

Suppose a VW Golf is travelling along a straight road at 20 m/s. It is overtaken by a Citröen Berlingo which is travelling at 30 m/s in the same direction. One second after the Berlingo overtakes the Golf, it will 10 m ahead. After two seconds it will be 20 m ahead, and after three seconds 30 m ahead. From the point of view of the driver of the Golf - or any other observer in the Golf - the velocity of the Berlingo *relative to the Golf* is 10 m/s. In vector notation

$$\mathbf{V}_G = 20\mathbf{i}$$
$$\mathbf{V}_B = 30\mathbf{i}$$
$$\Rightarrow \mathbf{V}_{BG} = 10\mathbf{i} = \mathbf{V}_B - \mathbf{V}_G$$

where $\mathbf{V}_{BG}$ is the velocity of the Berlingo relative to the Golf. There is an issue with confusion here: in LC Maths:

$$\mathbf{ab} = \mathbf{b} - \mathbf{a}$$

however this vector **ab** is the vector from **a** *to* **b**; and not quite the same object.

4.2 The Relative Velocity Vector

Consider two general vectors **a** and **b**. Indeed the vector from **a** to **b**, $\mathbf{ab} = \mathbf{b} - \mathbf{a}$:

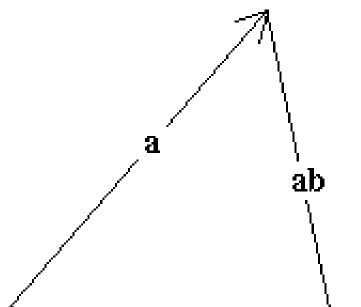


Figure 4.1: **ab** is the vector from **a** to **b** - not the position of **a** *relative* to **b**.

What this chapter is concerned with is not how to get from **a** to **b**; rather if **b** were fixed in space, where would **a** be? Where is **a** relative to **b**? This is where the $\mathbf{V}_{BG}$ comes from: *velocity of B relative to G*. It shouldn't be too difficult to see therefore that the vector that concerns relative velocity is $-\mathbf{ab}$ in the ordinary vector notation.

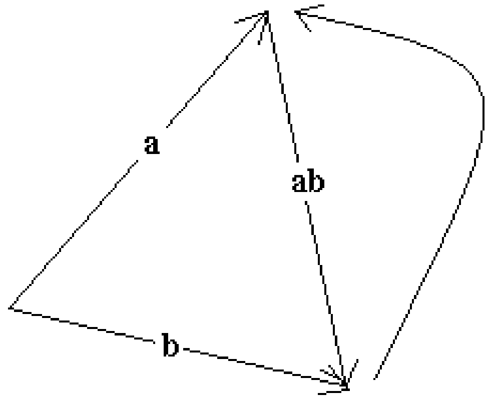


Figure 4.2: For relative motion, the position of **a** *were* **b** fixed is the important concept. If **b** were fixed; **a** would move away as the curved arrow shows - along the vector $-\mathbf{ab} = \mathbf{a} - \mathbf{b}$ in the ordinary vector notation.

4.2.1 Definition

If an object A has velocity $\mathbf{V}_A$, and an object B has velocity $\mathbf{V}_B$, then the *velocity of* A *relative to* B, $\mathbf{V}_{AB}$, is given by:

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B \tag{4.1}$$

4.2.2 Definition

If an object A has displacement (or position) $\mathbf{R}_A$, and an object B has displacement $\mathbf{R}_B$, then the *displacement of* A *relative to* B, $\mathbf{R}_{AB}$, is given by:

$$\mathbf{R}_{AB} = \mathbf{R}_A - \mathbf{V}_B \tag{4.2}$$

4.3 Compass Directions

As this chapter employs the navigational compass quite often, a short revision in order.

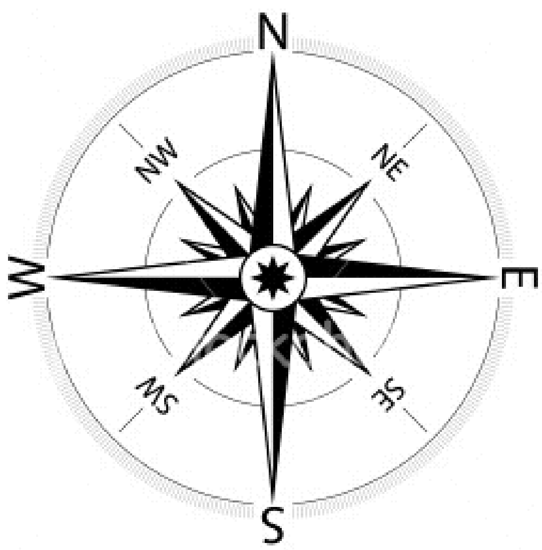


Figure 4.3: The navigational compass.

The directions on this compass are defined in the obvious way with respect to some axis. What occurs in LC Applied Maths are directions of the form E 20° N, etc. This is read East, 20° North. This means that the direction is as shown:

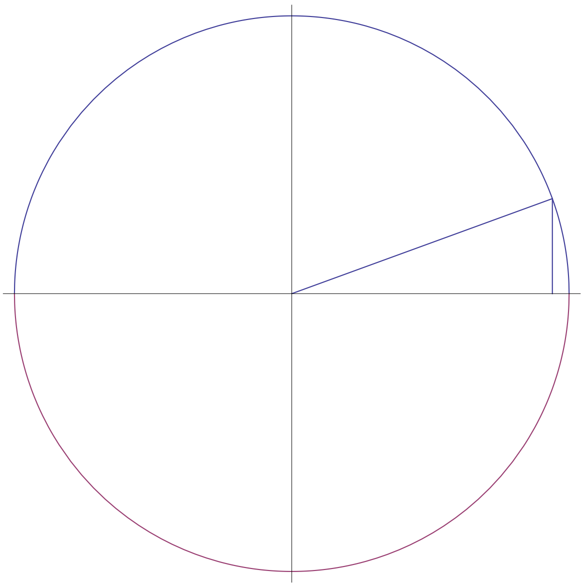


Figure 4.4: E 20° N on the navigational compass. The angle between the 'non-right' radius and the positive x-axis is 20°. Basically, start at East, and go up 20° towards North

Note the representations are in general non-commutative:

$$\mathrm{W}\,30^\circ\,\mathrm{S} \neq S\,30^\circ\,\mathrm{W} \tag{4.3}$$

and non-unique:

$$\mathrm{E}\,60^\circ\,\mathrm{N} = \mathrm{N}\,30^\circ\,\mathrm{E} \tag{4.4}$$

however in general, to confer with how latitude is defined, I understand in the case of (4.4) the first representation is used. That is the representation is started with East or West; rather than South or West. However note from Figure 4.3 how NW, NE, SW and SE do not follow this convention. Also SSW, etc. are defined in the obvious way.

4.4 Time in the Relative Picture

The time taken for an object A to travel a distance s from the origin o to a point P, travelling at a speed v is given by:

$$t = \frac{s}{v} \tag{4.5}$$

Does this simple form carry over into the relative picture? Does the same object, with respect to another object B, take the same time to cover the distance between the origin o and P with respect to B, o' and P' if travelling at the speed $\mathbf{V}_{AB}$? Worked Example 4.1 should convince that it does; that is

$$t = \frac{\text{relative distance}}{\text{relative speed}} \tag{4.6}$$

4.5 Collision Courses

Consider objects A and B and suppose further that object A is a distance s in some particular direction θ from B (given by $\mathbf{R}_{AB}$ a vector).

Suppose further that the objects have speeds $\mathbf{V}_A$ and $\mathbf{V}_B$. If the velocity of B relative to A; $\mathbf{V}_{BA} = \mathbf{V}_B - \mathbf{V}_A$ is found to lie across $\mathbf{R}_{AB}$ then the objects will collide.

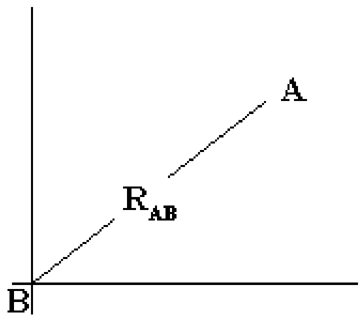


Figure 4.5: The position of A relative to B; $\mathbf{R}_{AB}$

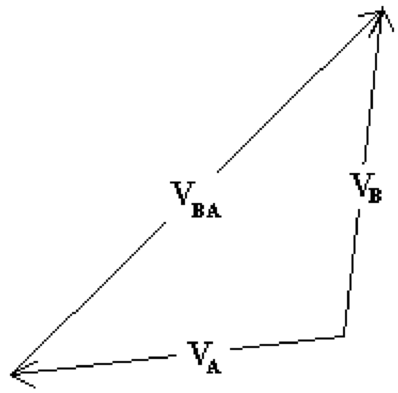


Figure 4.6: If the direction of $\mathbf{V}_{BA}$ is the same as $\mathbf{R}_{AB}$, then the objects will collide.

Worked Example 4.2 gives an good exposition of this. In general, the relative displacement will be along the **i**- or **j**-axes, which simplifies matters greatly.

4.6 Shortest Distance

Consider two objects A and B separated at some time by a displacement $\mathbf{R}_{AB}$. Suppose $\mathbf{V}_A$ and $\mathbf{V}_A$ are such that $\mathbf{V}_{AB}$ looks like:

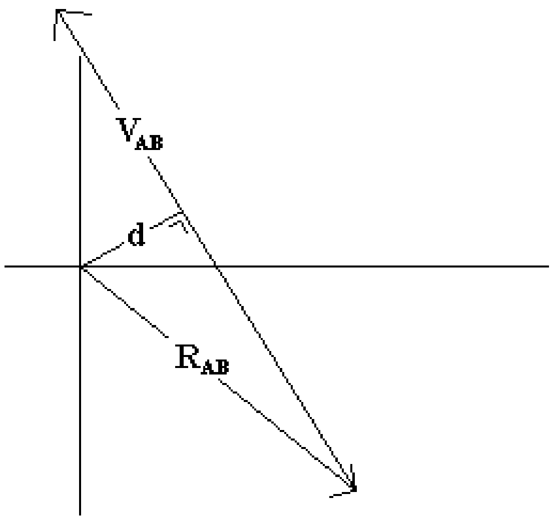


Figure 4.7: Initially A is at a displacement $\mathbf{R}_{AB}$ from B. Relative to B, A has constant velocity $\mathbf{V}_{AB}$. In the subsequent motion there is a time when A's distance from B is d - and this is the *shortest distance between A & B in the subsequent motion.*

There are two main methods of finding d. The first is through trigonometry, the second through coordinate geometry.

1. In Figure 4.7; drop a perpendicular from the x-axis to $\mathbf{R}_{AB}$, a little vertical line through the point where $\mathbf{V}_{AB}$ cuts the x-axis and label as shown:

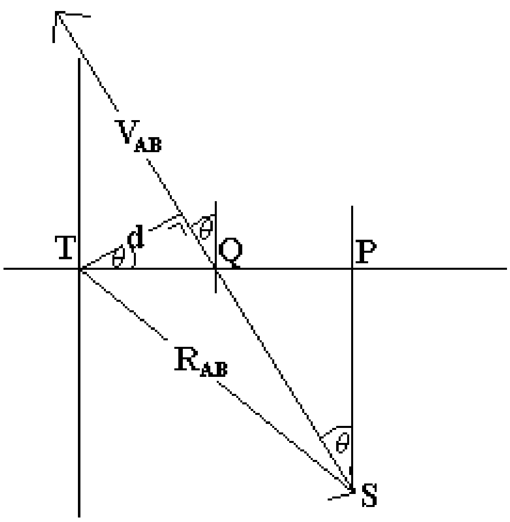


Figure 4.8: An annotation of Figure 4.7. The θ at S defines θ: the angles at R and T are found to be θ also.

d may now be found in three steps.

(i) Find $|PQ|$.

$$\tan\theta = \frac{-\mathbf{V}_{ABx}}{\mathbf{V}_{ABy}} = \frac{|PQ|}{|PS|}$$

But $|PS| = |\mathbf{R}_{ABy}|$; hence $|PQ|$ may be found.

(ii) Find $|TQ|$.

$$|TQ| = |TP| - |PQ|$$

But $|TP| = |\mathbf{R}_{ABx}|$; hence $|TQ|$ may be found.

(iii) Find d

$$\cos\theta = \frac{d}{|TQ|}$$
$$\Rightarrow d = |TQ|\cos\theta$$

but $\cos\theta$ may be calculated from $\tan\theta$. Hence d may be found.

2. Alternatively consider the following fact:

4.6.1 Theorem

The distance perpendicular distance d, from a point $p(x_1, y_1)$ to a Line $L \equiv ax + by + c = 0$ is given by

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \tag{4.7}$$

4.6.2 Corollary

The shortest distance between a point $p(x_1, y_1)$ and a line $L \equiv ax + by + c = 0$ is given by (4.7)

Clearly in this context our problem is finding the perpendicular distance from a point $p(x_1, y_1) = (0, 0)$ to the line given by $\mathbf{V}_{AB}$. The slope of the relevant line is the slope of $-\mathbf{V}_{AB}$;

$$m = \text{ slope of } -\mathbf{V}_{AB}$$
$$\Rightarrow m = \text{ slope of } -\mathbf{V}_{ABx}\mathbf{i} - \mathbf{V}_{ABy}\mathbf{j}$$
$$\Rightarrow m = \frac{-\mathbf{V}_{ABy}}{-\mathbf{V}_{ABx}}$$
$$\Rightarrow m = \frac{\mathbf{V}_{ABy}}{\mathbf{V}_{ABx}} = -\frac{1}{\tan\theta}$$

and $\mathbf{R}_{AB} = (\mathbf{R}_{ABx}, \mathbf{R}_{ABy})$ is a point on the line:

$$L \equiv y - y_1 = m(x - x_1)$$
$$\Rightarrow L \equiv y - \mathbf{R}_{ABy} = -\frac{1}{\tan\theta}(x - \mathbf{R}_{ABx})$$
$$\Rightarrow L \equiv \tan\theta y - \tan\theta\mathbf{R}_{ABy} = -x + \mathbf{R}_{ABx}$$
$$\Rightarrow L \equiv x + \tan\theta\, y + (-\mathbf{R}_{ABx} - \tan\theta\mathbf{R}_{ABy})$$
$$\Rightarrow d = \frac{|\mathbf{R}_{ABx} + \tan\theta\mathbf{R}_{ABy}|}{\sqrt{1 + \tan^2\theta}}$$
$$\underset{1+\tan^2=\sec^2}{\Rightarrow} d = \frac{|\mathbf{R}_{ABx} + \tan\theta\mathbf{R}_{ABy}|}{\sqrt{\sec^2\theta}}$$
$$\underset{\sec\theta\geq 0}{\Rightarrow} d = \frac{|\mathbf{R}_{ABx} + \tan\theta\mathbf{R}_{ABy}|}{\sec\theta}$$
$$\underset{\sec=1/\cos}{\Rightarrow} d = |\cos\theta\mathbf{R}_{ABx} + \sin\theta\mathbf{R}_{ABy}|$$

Worked Example 4.3 (Method 2) finds the equation of the line in a slightly different way by setting up the question a little differently (see below). However this equivalent method does not have $p(x_1, y_1) = (0, 0)$ which simplifies the calculation of d. Both ways are equivalent and there is much of a muchness between them. Do note however that the calculation done here works in full generality. Ordinarily numbers will be used rather than algebraic factors; however it is comforting to know the methods work in full generality.

Worked Example 4.3 illustrates both these approaches well in a slightly different but equivalent way.

Such questions on the shortest distance between two objects have come up at the LC level every single year since 1999.

4.7 Rivers & Winds

Consider an object in a constant flow F (river or wind). Suppose the velocity of the flow is $\mathbf{V}_F$. Consider an object capable of moving with velocity $\mathbf{V}_{PF}$ in (water/ air) in the absence of flow. What is the velocity of P in the flow F? If the flow is zero, the velocity of P is $\mathbf{V}_{PF}$. Hence the velocity of P relative to F is $\mathbf{V}_{PF}$. But

$$\mathbf{V}_{PF} = \mathbf{V}_P - \mathbf{V}_F \tag{4.8}$$

$$\Rightarrow \mathbf{V}_P = \mathbf{V}_{PF} + \mathbf{V}_F \tag{4.9}$$

Chapter 5

Newton's Laws and Connected Particles

5.1 Theory

5.1.1 Definition

A *force*, $\mathbf{F}$, is what causes a body to accelerate.

5.1.2 Definition

The *mass*, m, of a body is the quantity of matter in it

Remark

Mass is not the same as volume. Two bodies of equal volume do not necessarily have the same mass. A ball of lead has more mass than a ball of wood - lead has more matter than wood.

5.1.3 Definition

The *weight* of a body is the force on a body due to gravity.

Remark

If $\mathbf{g}$ is the acceleration due to gravity in the frame of reference of a body of mass m, the weight on the body is

$$\mathbf{w} = m\mathbf{g} \tag{5.1}$$

Mass is not the same as weight. The weight of a 1 kg body is greater on Earth than Moon - as the gravitational force of Earth is stronger. Its mass, however, is the same on Earth and Moon.

5.1.4 Definition

The momentum, $\mathbf{p}$, of a body is the product of its mass and velocity:

$$\mathbf{p} = m\mathbf{v} \tag{5.2}$$

Remark

Momentum is a vector in the same direction as that of the velocity. Momentum has no special unit. It is measured in kg m/s.

5.1.5 Definition

A physical quantity A is *proportional* to a physical quantity B, $A \propto B$ if

$$A = kB \tag{5.3}$$

for some $k \in \mathbb{R}$.

5.1.6 Statement of Newton's Laws

1. *Every particle remains at rest or moves with constant velocity unless or until acted upon by an external force*
2. *The rate of change of momentum is proportional to the applied force, and takes place in the direction in which the force is applied*
3. *For every action there is an equal and opposite reaction*

Remarks

1. This is equivalent to the definition of force: in the absence of force there is no acceleration.
2. The applied force is the sum of all external forces on the body:

$$\mathbf{F} = \sum \mathbf{F}_{\text{ext}} \tag{5.4}$$

Consider an object of mass m travelling with velocity $\mathbf{u}$. It is acted upon by a force, $\mathbf{F}$, for t seconds and ends up with a velocity of $\mathbf{v}$. According to the Second Law, the force is proportional to the rate of change of momentum. But the momentum before is $m\mathbf{u}$ and the momentum after is $m\mathbf{v}$ so the change in momentum is $m\mathbf{v} - m\mathbf{u}$. This took t seconds, so the rate of change of momentum is

$$\frac{m\mathbf{v} - m\mathbf{u}}{t}$$

So

$$\mathbf{F} \propto \frac{m\mathbf{v} - m\mathbf{u}}{t} = m\left(\frac{\mathbf{v} - \mathbf{u}}{t}\right)$$

But $(\mathbf{v} - \mathbf{u})/t = \mathbf{a}$:

$$\therefore\ \mathbf{F} = km\mathbf{a}$$
$$\Rightarrow \mathbf{F} = km\mathbf{a}$$

The unit of force, the *Newton*, N, is chosen such that the constant of proportionality is $k = 1$; i.e. a force of 1 N will give a body of mass 1 kg an acceleration of 1 m/s^2. It follows that

$$\sum \mathbf{F}_{\text{ext}} = m\mathbf{a} \tag{5.5}$$

Note that if $\mathbf{F} = 0$ then $\mathbf{a} = 0$. Hence the First Law is a corollary of the first.

5.1.7 Definition

Tension is a force exerted through strings and on objects through strings.

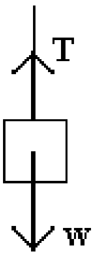


Figure 5.1: The string exerts an upwards force on the body (otherwise the body would fall under the force of its weight). The only forces on the body are its weight, **w**, and the tension, T, in the string. Tension acts *along* the line of the string, *away* from the object.

5.1.8 Definition

A *normal force*, R, is a force transmitted from one object to another by direct contact.

5.1.9 Definition

Friction, F, is a force that opposed the relative motion between two surfaces in contact.

Remark

Consider a body of mass m resting on a surface, pulled along by a string. The forces on the body are:

(i) the tension in the string, $\mathbf{T}$

(ii) the weight $m\mathbf{g}$

(iii the normal reaction R between the surface and the object

(iv) the friction force F between the surface and the bottom of the object

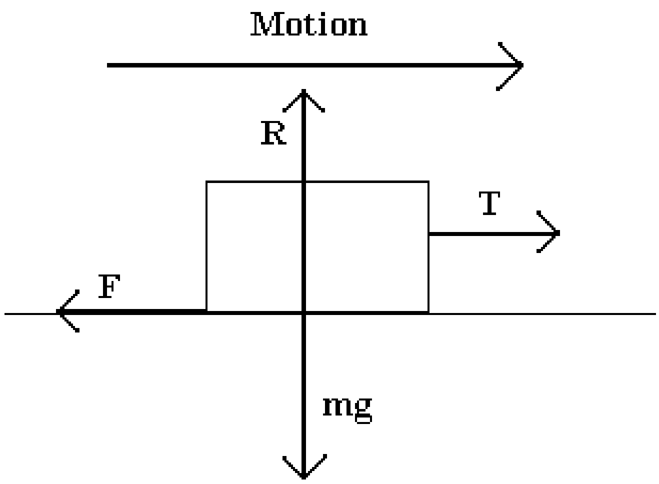
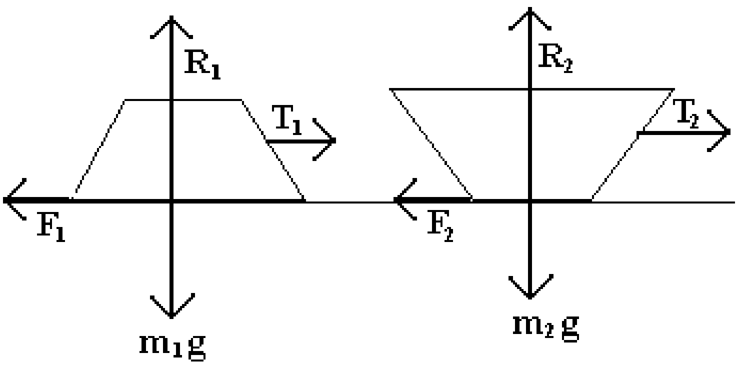


Figure 5.2: Notice that the friction force F is in the opposite direction to the motion; opposing it.

Now suppose that two different sized bodies of the same kind of material are moving along the surface, pulled by two strings:



Experiment has shown:

$$\frac{F_1}{R_1} = \frac{F_2}{R_2} \tag{5.6}$$

This common ratio is called the *coefficient of friction*, μ. This coefficient depends *only* on the materials of the surfaces whose relative motion the friction is opposing - not on the shape or size of the bodies to which the surfaces belong. So when a body is moving across of a surface, where the normal force on the body is R, and the coefficient of friction between the material of the body and the surface is μ, the friction force is given by:

$$F = \mu R \tag{5.7}$$

Consider the body in the *force diagram* Figure 5.2. Since the object does not move up or down, the force in the $\hat{j}$ direction is 0:

$$R - mg = 0$$
$$\Rightarrow R = mg$$

But

$$F = \mu R$$
$$\Rightarrow F = \mu mg$$

So when the object is moving a friction force of magnitude μmg opposes the motion. What if there is no horizontal force on the object? Friction only exists when there is a force in the opposite direction, and in its absence the object stays at rest.
If there is a force on the object which is less than or equal to μR then the particle will not move: the force of friction will be strong enough to oppose this force (motion) but once the force (motion) is opposed, the friction doesn't get any stronger.
Finally if there is a force on the object which is great than μR then the friction force takes on the constant value μR. In this sense μR is called the *limiting friction*:

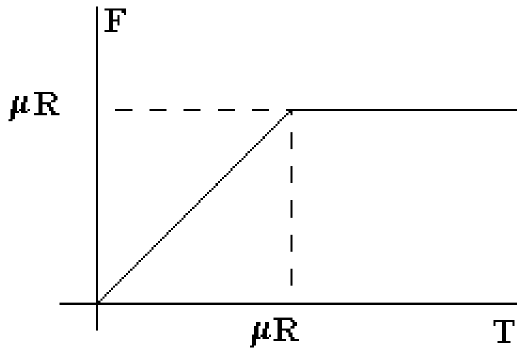
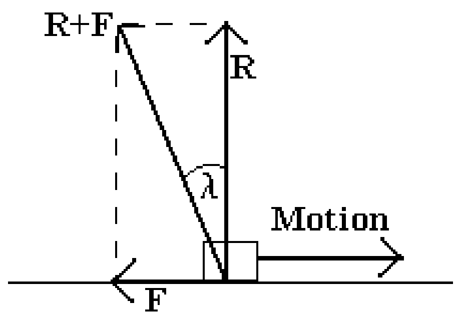


Figure 5.3: A graph of the friction force, F, vs an applied force, T. Note when the applied force is less than the limiting friction, μR, the friction force equals the applied force. When the applied force exceeds the limiting friction, the friction force takes the constant value μR.

5.1.10 Definition

The *angle of friction*, λ, is the angle between $\mathbf{R}$ and $\mathbf{R}+\mathbf{F}$:



From the diagram it is clear that

$$\tan\lambda = \frac{F}{R} = \mu$$

Chapter 6

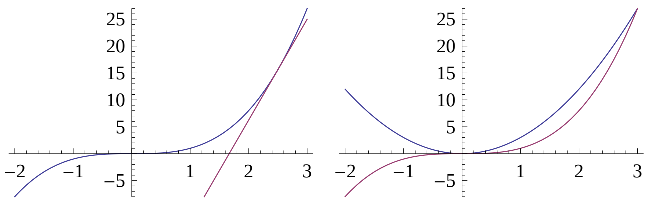
Differential Equations

6.1 Introduction

We are familiar with equations in the set of real numbers:

$$x^2 - 3x + 2 = 0 \tag{6.1}$$

We have techniques of algebra and analysis that tell us that the solutions to this equation are 2, $1 \in \mathbb{R}$. A differential equation is another creature altogether. Consider now briefly *the derivative* of a function. The derivative of a function, $f(x)$ is a function whose value at x_0 is the slope of the tangent to the curve at $(x_0, f(x_0))$:



A differential equation is an equation whose solution is a function rather than a number, and the terms of the equation will be functions and derivatives. For example:

$$\frac{dy}{dx} = 2x\sqrt{1 - y^2} \tag{6.2}$$

asks for which function $y = f(x)$, is the slope of the tangent at (x, y) given by $2x\sqrt{1 - y^2}$?

6.2 Differentiation, Integration and the Fundamental Theorem of Calculus

In this section the derivative and definite integral are introduced in their correct setting. At Leaving Cert level, integration is introduced merely as the inverse of differentiation. This approach simplifies things but greater understanding comes out of a proper treatment. Historically integration was developed separately of differentiation and the link between them was later discovered. The link between them; namely that integration is indeed the inverse of differentiation, is known as the *Fundamental Theorem of Calculus.* The topics of coordinate geometry, limits and functions should be studied in more depth prior to a thorough study calculus, but this brief chapter is merely intended as an exposition to aid understanding.

6.2.1 Differentiation

In the figure below, the line from a to b is called a *secant* line.

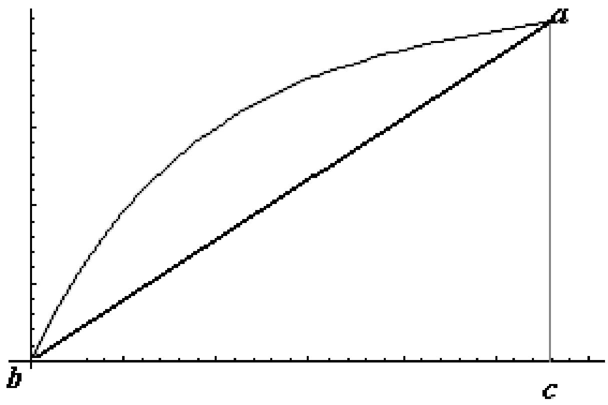


Figure 6.1: Secant Line.

Introduce the idea of slope. The slope of a line is something intuitive. A steep hill has a greater slope than a gentle rolling hill. The slope of the secant line is simply the ratio of how much the line travels vertically as the line travels horizontally. Denote slope by m:

$$m = \frac{|ac|}{|bc|}. \tag{6.3}$$

What about the slope of the curve? From a to b it is continuously changing. Maybe at one point its slope is equal to that of the secant but that doesn't tell much. It could be estimated, however, using a ruler the slope at any point. It would be the tangent, as shown:

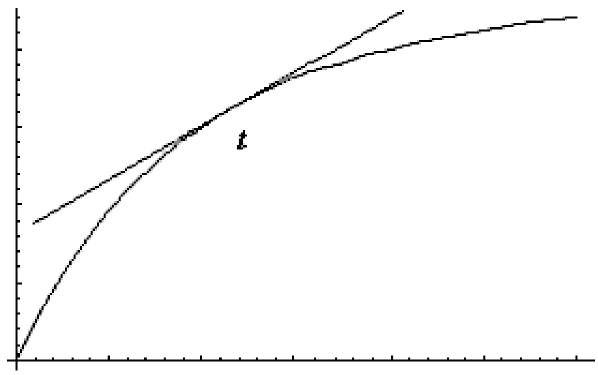


Figure 6.2: Tangent Line

The above line *is* the slope of the curve at t. Construct a secant line:
Now with respect to analytic geometry, with a function $f(x)$, the slope of this secant is given by:

$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$
$$\Rightarrow m = \frac{f(x+h) - f(x)}{h}$$

It is apparent that the secant line has a slope that is close, in value, to that of the tangent line. Let h become smaller and smaller:

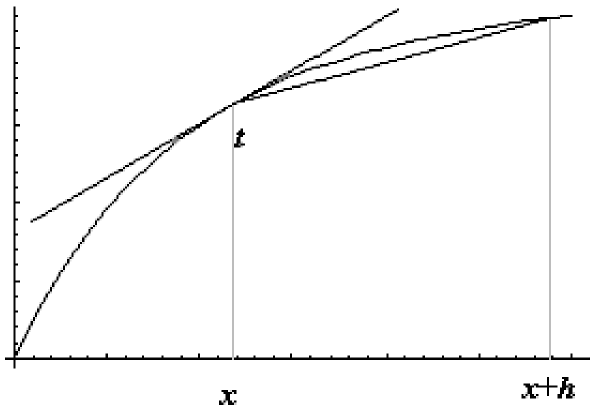


Figure 6.3: Secant line and Tangent line

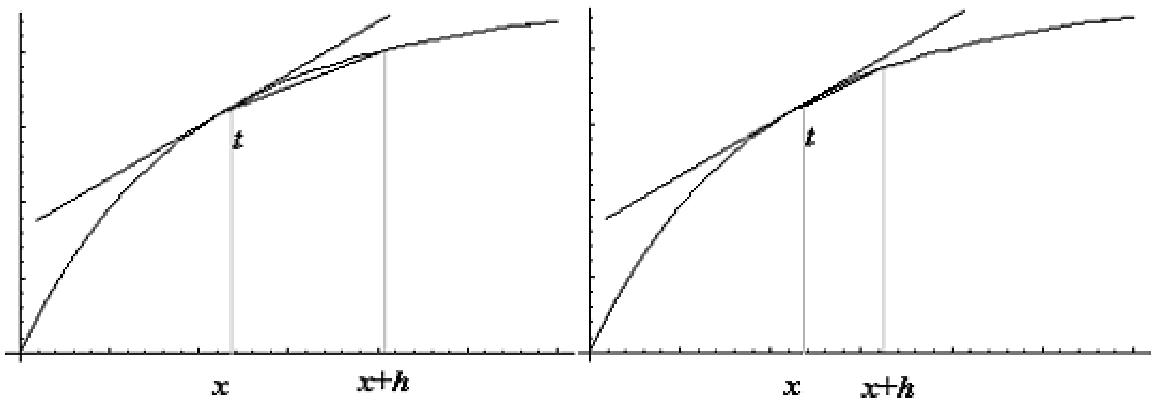


Figure 6.4: Secant line approaching slope of Tangent line

The slope of the secant line is almost identical to that of our tangent. Let $h \to 0$. Of course, if $h = 0$ there is no secant. But if h got *so close to* 0 *as doesn't matter* then there would be a secant and hence a slope. With respect to analytic geometry, with a function $f(x)$, the slope of this secant, which is indistinguishable from that of the tangent to the point, is given by:

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{6.4}$$

This m is the *derivative of f(x).* This gives the slope of the curve at *every* point on the curve.

Notation

There are two significant branches of notation used to denote the derivative of f. The difference is just notation. They are different ways of writing down the same thing. This section is included to fight against simple misunderstandings.

Newtonian Notation

The function is denoted $f(x)$ and the graph is the set of points $(x, f(x))$:

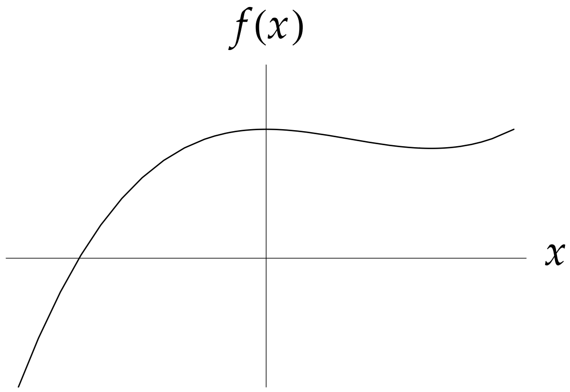


Figure 6.5: Newtonian notation for functions

The derivative of $f(x)$ is denoted $f'(x)$. Other names for the derivative of $f(x)$ include:

- the differentiation of $f(x)$
- the derived function for $f(x)$
- the slope of the tangent at $(x, f(x))$
- the gradient
- $\frac{df}{dx}$

Leibniz Notation

The function is denoted $y = f(x)$ (e.g. $y = x^2$); and the graph is the set of points (x, y):

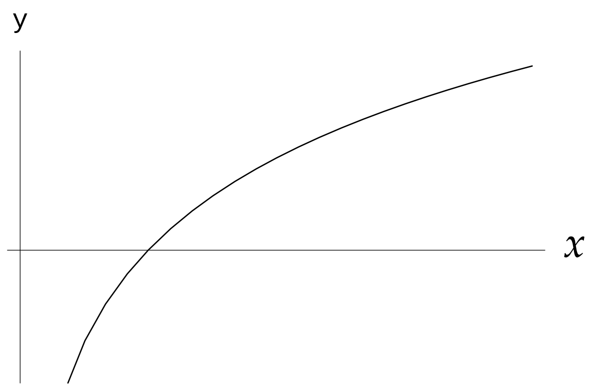


Figure 6.6: Leibniz notation for functions

In this notation, y is equivalent to $f(x)$. However, the notation for the derivative of y is:

$$\frac{dy}{dx}. \tag{6.5}$$

It must be understood that if $y = f(x)$; then

$$f'(x) \equiv \frac{dy}{dx}, \tag{6.6}$$

and there is no notion of canceling the ds; it is just a notation. It is an illuminating one because if the second graph of figure 6.4 is magnified about the secant:

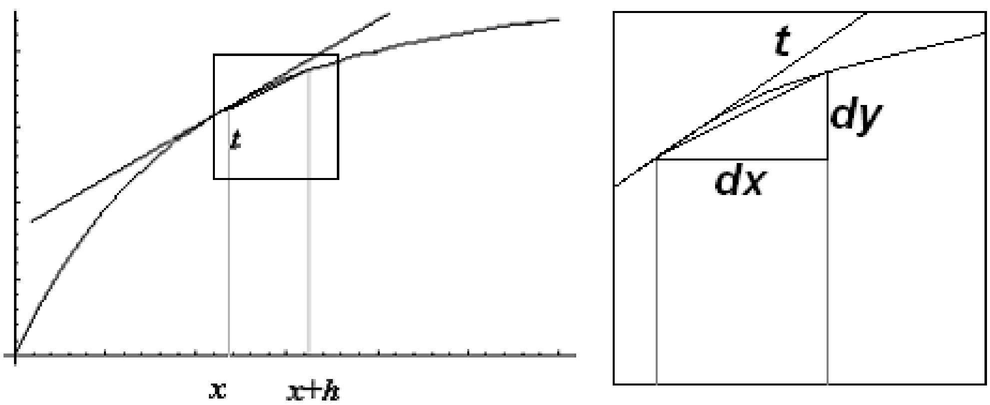


Figure 6.7: Leibniz notation for the derivative

If dy is associated with a small variation in $y \sim f(x+h) - f(x)$; and dx associated with a small variation in $x \sim h$; then dy/dx makes sense.

To reiterate if $y = f(x)$; then dy/dx is the same thing as:

- the derivative of $f(x)$
- the differentiation of $f(x)$
- the derived function for $f(x)$
- the slope of the tangent at $(x, f(x))$
- the gradient
- $f'(x)$

So to emphasise $f(x)$ and y are freely interchangeable; as are $f'(x)$ and dy/dx.

6.2.2 Integration

What is the area of the shaded region under the curve $f(x)$?

Start by subdividing the region into n strips S_1, S_2, ..., S_n of equal width as Figure 6.8.

The width of the interval $[a, b]$ is $b - a$ so the width of each of the n strips is

$$\Delta x = \frac{b-a}{n}.$$

Approximate the ith strip S_i by a rectangle with width Δx_i and height $f(x_i)$, which is the value of f at the right endpoint. Then the area of the ith rectangle is $f(x_i)\ \Delta x_i$:

The area of the original shaded region is approximated by the sum of these rectangles:

$$A \approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x. \tag{6.7}$$

This approximation becomes better and better as the number of strips increases, that is, as $n \to \infty$. Therefore the area of the shaded region is given by the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots f(x_n)\,\Delta x]. \tag{6.8}$$

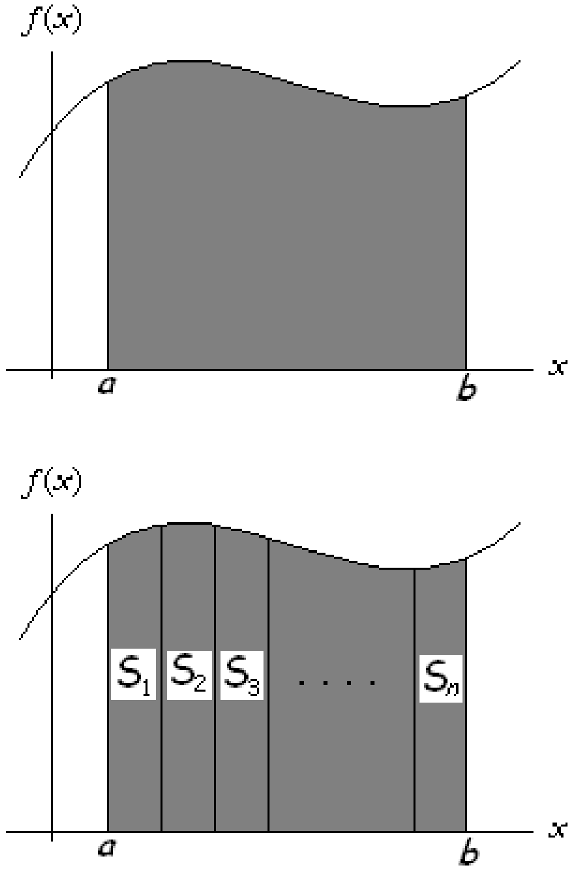


Figure 6.8:

Definition

If $f(x)$ is a function defined in $[a, b]$ and x_i, Δx are as defined above, then the *definite integral of f from a to b* is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x. \tag{6.9}$$

So an integral is an infinite sum. Associate $\int \sim \lim_{n\to\infty} \sum_n$. Again dx is associated with a small variation in $x \sim \Delta x$.

6.2.3 Fundamental Theorem of Calculus

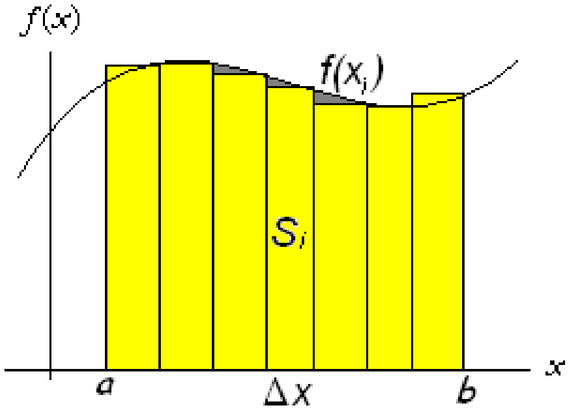
Rough Version

If f is a function with derivative f' then

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Proof

Identical to that of Theorem 2.2.1 replacing $s(t)$ by $f(x)$ and $v(t)$ by $f'(x)$. □



The Indefinite Integral

In Leaving Cert where integrals are defined as the inverse of derivatives, an indefinite integral defines integration.

Definition

If $f(x)$ is a function and it differential with respect to x is $f'(x)$, then

$$\int f'(x)\,dx = f(x) + c \tag{6.10}$$

where c is called the *constant of integration*.

Note the constant of integration. It's inclusion is vital because if $f(x)$ is a function with derivative $f'(x)$ then $f(x) + c$ also has derivative $f'(x)$ as:

$$\frac{d}{dx}(f(x) + c) = \underbrace{\frac{df}{dx}}_{=f'(x)} + \underbrace{\frac{d}{dx}c}_{=0},$$
$$\Rightarrow \frac{d}{dx}(f(x) + c) = f'(x).$$

Geometrically a curve $f(x)$ with slope $f'(x)$ has the same slope as a curve that is shifted upwards; $f(x) + c$. Note that the constant of integration can be disregarded for the indefinite integral. Suppose the integrand is $f'(x)$ and the anti-derivative is $f(x) + c$. Then:

$$\int_a^b f'(x)\,dx = (f(b) + c) - (f(a) + c),$$
$$\Rightarrow \int_a^b f'(x)\,dx = f(b) - f(a).$$

The cs cancel!

6.2.4 Conclusion

Finding the derivative of a function f at x is finding the slope of the tangent to the curve at x. Integration meanwhile measures the area between two points $x = a$ and $x = b$. The Fundamental Theorem of Calculus states however that differentiation and integration are intimately related; that is given a function f:

$$\frac{d}{dx}\int f(x)\,dx = f(x),$$
$$\int \frac{d}{dx} f(x)\,dx = f(x) + c.$$

i.e. differentiation and integration are essentially inverse

6.3 Differentiation

6.3.1 Proposition

$$\frac{d}{dx}x^n = nx^{n-1} \tag{6.11}$$
$$\frac{d}{dx}\sin x = \cos x \tag{6.12}$$
$$\frac{d}{dx}\cos x = -\sin x \tag{6.13}$$

Proof

By using (6.4):

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

□

6.3.2 Proposition

(D1) ***The Sum/ Difference Rule***
If $u(x)$ and $v(x)$ are two functions and if

$$f(x) = u(x) \pm v(x), \textit{ then}$$
$$\frac{df}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}. \tag{6.14}$$

(D2) ***The Product Rule***
If $u(x)$ and $v(x)$ are two functions and if

$$f(x) = u(x).v(x), \textit{ then}$$
$$\frac{df}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}. \tag{6.15}$$

(D3) ***The Quotient Rule***
If $u(x)$ and $v(x)$ are two functions and if

$$f(x) = \frac{u(x)}{v(x)}, \textit{ then}$$
$$\frac{df}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}. \tag{6.16}$$

(D4) ***The Scalar Rule***
If $g(x)$ is a function and if $f(x) = k\,g(x)$, where k is a constant ($k \in \mathbb{R}$) then

$$\frac{df}{dx} = k.\frac{dg}{dx}. \tag{6.17}$$

Proof

By using (6.4). □

6.3.3 Corollary

$$\frac{d}{dx}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1 \qquad (6.18)$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad (6.19)$$

6.3.4 The Chain Rule

The Chain Rule allows us to differentiate a composite function. If $u(x)$ and $v(x)$ are functions and $f(x) = u(v(x)) = u \circ v(x)$ then:

$$\frac{df}{dx} = \frac{du}{dv}\frac{dv}{dx}. \qquad (6.20)$$

6.3.5 Inverting

Ordinarily in LC maths, $y = f(x)$. That is y is a function of x. However, for some functions $y = f(x)$ there is an inverse: $x = f^{-1}(y)$. For such functions, $y = f(x)$, $x = g(y)$ may be written; where $gf(x) = x$. Then x can be differentiated with respect to y. Suppose

$$\frac{dx}{dy} = u(y)$$

some function of y. Then

$$\frac{dy}{dx} = \frac{1}{dx/dy},$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{u(y)}.$$

Now substitute $f(x)$ for y in $u(x)$. This method of *inverting* is used to differentiate both $\ln x$ and the inverse trigonometric functions $\sin^{-1} x$ and $\tan^{-1} x$ - whose inverses are, respectively, $x = e^y$, $x = \sin y$ and $x = \tan y$.

6.3.6 Corollary

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad (6.21)$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad (6.22)$$

6.3.7 The Exponential Function & The Natural Logarithm

The Exponential Function

The Exponential Function may be defined in many different ways. Possibly the best way to think of it at LC level is to consider the following:

As it turns out there is a very special number, $e \approx 2.718$ given by:

$$e := \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \sum_{i=0}^{\infty}\frac{1}{i!} \qquad (6.23)$$

such that the function e^x and its derivative are equal; that is:

$$\frac{d}{dx}e^x = e^x$$

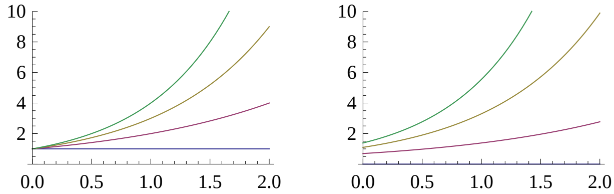


Figure 6.9: A plot of the functions a^x and their derivatives for $a = 1, 2, 3, 4$. Observe the similarity between a^x and da^x/dx.

By the Fundamental Theorem of Calculus:

$$\int e^x \, dx = e^x + c \tag{6.24}$$

Remark

Later it will be seen in Further Calculus and Series that functions have a power series expansion. The power series expansion of e^x is:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{6.25}$$

The general term in this series is:

$$t_n = \frac{x^n}{n!}$$

By the Sum Rule, the differentiation of a sum is the sum of the derivatives. Hence consider

$$\frac{d}{dx} t_n = \frac{n}{n!} x^{n-1} = \frac{n}{n(n-1)(n-2)\cdots(2)(1)} x^{n-1} = \frac{x^{n-1}}{(n-1)!} = t_{n-1}.$$

$$\Rightarrow \frac{d}{dx} e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\Rightarrow \frac{d}{dx} e^x = e^x.$$

The Natural Logarithm

The natural logarithm is the *inverse* of the exponential function. Suppose $e^a = b$; then $\ln b = a$. So e^x and ln are inverse in the sense:

$$e^x = y \Leftrightarrow \ln y = x \tag{6.26}$$

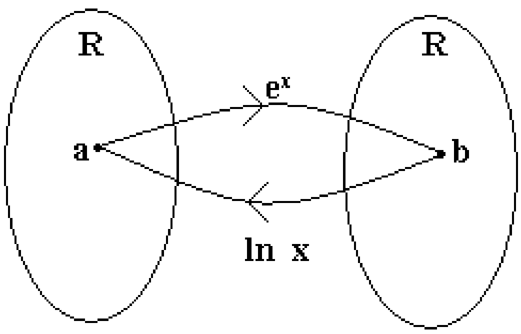


Figure 6.10: The exponential and natural logarithm functions are inverse

From our tables we can see that

$$\frac{d}{dx}\ln x = \frac{1}{x} \tag{6.27}$$

However we can derive this as follows. Suppose

$$\begin{aligned} y &= \ln x \\ \Rightarrow e^y &= x \\ \underset{dx}{\Rightarrow} e^y \frac{dy}{dx} &= 1 \\ \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} &= \frac{1}{x} \end{aligned}$$

Obviously then

$$\int \frac{1}{x}\,dx \equiv \int \frac{dx}{x} = \ln|x| + c \tag{6.28}$$

The absolute value sign is there because ln is only defined for strictly positive numbers. Note if we try to use the formula:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \tag{6.29}$$

for

$$\begin{aligned} \int \frac{1}{x}\,dx &= \int x^{-1}\,dx \\ \Rightarrow \int \frac{1}{x}\,dx &= \frac{x^0}{0} + c \end{aligned}$$

which is undefined.

6.4 Integration

From the Fundamental Theorem of Calculus

$$\int f'(x)\,dx = f(x) + c \tag{6.30}$$

Thus:

$f(x)$	$\int f(x)$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
e^x	$e^x + c$
$\sec^2 x$	$\tan x + c$
$\frac{1}{x}$	$\ln\|x\| + c$

Also because

$$\begin{aligned} \frac{d}{dx}\sin nx &= n\cos nx \text{ , and} \\ \frac{d}{dx}\cos nx &= -n\sin nx \\ \Rightarrow \int \cos nx\,dx &= \frac{\sin nx}{n} + c \text{ and} \\ \Rightarrow \int \sin nx\,dx &= -\frac{\cos nx}{n} + c \end{aligned}$$

Also, let $a > 0$;

$$\begin{aligned}
\frac{d}{dx}\frac{1}{a}\arctan\frac{x}{a} &= \frac{1}{a}\frac{1}{1+x^2/a^2}.\frac{1}{a} \\
\Rightarrow \frac{d}{dx}\frac{1}{a}\arctan\frac{x}{a} &= \frac{1}{\cancel{a}}\frac{1}{(a^2+x^2)/\cancel{a^2}}.\frac{1}{\cancel{a}} \\
\Rightarrow \frac{d}{dx}\frac{1}{a}\arctan\frac{x}{a} &= \frac{1}{a^2+x^2} \\
\Rightarrow \int \frac{1}{a^2+x^2}\,dx &= \frac{1}{a}\arctan\frac{x}{a} + c
\end{aligned}$$

Also

$$\begin{aligned}
\frac{d}{dx}\arcsin\frac{x}{a} &= \frac{1}{\sqrt{1-x^2/a^2}}\frac{1}{a} \\
\Rightarrow \frac{d}{dx}\arcsin\frac{x}{a} &= \frac{1}{\sqrt{(a^2-x^2)/a^2}}\frac{1}{a} \\
\Rightarrow \frac{d}{dx}\arcsin\frac{x}{a} &= \sqrt{\frac{\cancel{a^2}}{a^2-x^2}}\frac{1}{\cancel{\sqrt{a^2}}} \\
\Rightarrow \int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \arcsin\frac{x}{a}
\end{aligned}$$

6.4.1 Proposition

- $$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx \qquad (6.31)$$

- $$\int k\,f(x)\,dx = k\int f(x)\,dx\ ,\ \textit{where } k \in \mathbb{R} \qquad (6.32)$$

Proof

□

Remarks

There is no direct analogue of the product, quotient nor chain rule for integration. Although the substitution method below is like a chain rule for integrals.

6.4.2 The Substitution Method for Evaluating Integrals

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \tag{6.33}$$

where $u = g(x)$

Proof (Non-Examinable)

$$\begin{aligned} u &= g(x) \\ \Rightarrow \frac{du}{dx} &= g'(x) \\ \Rightarrow \frac{dx}{du} &= \frac{1}{g'(x)} \\ \Rightarrow dx &= \frac{du}{g'(x)} \end{aligned}$$

So

$$\int f(g(x))g'(x)dx = \int f(u)\cancel{g'(x)}\frac{du}{\cancel{g'(x)}} = \int f(u)du$$

LIATE

If we cannot see a $g(x)$, $g'(x)$ pattern we can use the LIATE rule. Choose u according to the most complicated expression in the following hierarchy:

- L ogarithms
- I nverses (inverse sine, tan)
- A lgebraic (polynomials in x)
- T rigonometric
- E xponential

In general this works well (also works for Integration by Parts).

6.5 Second Derivatives

In a sense, differentiation is a *function on the set of functions*[1], $C(\mathbb{R})$:

$$D : C(\mathbb{R}) \to C(\mathbb{R})\,,\ f \mapsto f'(x) \tag{6.34}$$

So the derivative of a function is not a number: but another function. If we differentiate $f'(x)$ we will get another function:

$$D(f'(x)) = f''(x) \tag{6.35}$$

This $f''(x)$ is the *second derivative of* $f(x)$. In Leibniz Notation:

$$\frac{d^2 f}{dx^2} \equiv f''(x) \tag{6.36}$$

6.6 Laws of Logs

In this chapter the particular integral:

$$\int \frac{dx}{x} = \ln|x| + c \tag{6.37}$$

will frequently arise. It would be wise therefore to state here the definition and laws of logs:

[1]strictly the set of differentiable functions but we won't worry about these technicalities

6.6.1 Definition

Let $a,\, x \in \mathbb{R};\, a,\, x > 0$.

$$\log_a x = p \Leftrightarrow a^p = x. \tag{6.38}$$

6.6.2 Proposition

Let $a,\, b,\, x,\, y \in \mathbb{R};\, a,\, b,\, x,\, y > 0,\, n \in \mathbb{R}$.

(L1)

$$\log_a x + \log_a y = \log_a xy$$

(L2)

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

(L3)

$$n \log_a x = \log_a x^n$$

(L4)

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof

(L1) Let $\log_a x = p$ and $\log_a y = q$:

$$\begin{aligned}
\Rightarrow a^p = x \text{ , and } a^q &= y \\
\Rightarrow xy &= a^p a^q \\
\Rightarrow xy &= a^{p+q} \\
\Rightarrow \log_a xy = p + q &= \log_a x + \log_a y
\end{aligned}$$

□

(L2) Let $\log_a x = p$ and $\log_a y = q$:

$$\begin{aligned}
\Rightarrow a^p = x \text{ , and } a^q &= y \\
\Rightarrow \frac{x}{y} &= \frac{a^p}{a^q} \\
\Rightarrow \frac{x}{y} &= a^{p-q} \\
\Rightarrow \log_a \left(\frac{x}{y}\right) = p - q &= \log_a x - \log_a y
\end{aligned}$$

□

(L3) Let $\log_a x = p$

$$\begin{aligned}
\Rightarrow a^p &= x \\
\Rightarrow x^n = a^{pn} &= a^{np} \\
\Rightarrow \log_a x^n = np &= n \log_a x
\end{aligned}$$

□

(L4) Let $\log_a x = p$:

$$\begin{aligned}
\Rightarrow x &= a^p \\
\Rightarrow \log_b x &= \log_b a^p \\
\Rightarrow \log_b x &= (\log_a x)(\log_b a) \\
\Rightarrow \log_a x &= \frac{\log_b x}{\log_b a}
\end{aligned}$$

□

6.7 First Order Separable Differential Equation

A first order separable differential equation has the form:

$$f(x)g(y)\frac{dy}{dx} = p(x)q(y) \tag{6.39}$$

The key is to put all the x-terms on one side and the y-terms on the other:

$$\frac{g(y)}{q(y)}\,dy = \frac{p(x)}{f(x)}\,dx$$

Integrate both sides:

$$\begin{aligned}\int \frac{g(y)}{q(y)}\,dy &= \int \frac{p(x)}{f(x)}\,dx\\ \Rightarrow F(y) + c_1 &= G(x) + c_2\\ \Rightarrow F(y) &= G(x) + \underbrace{(c_2 - c_1)}_{:=c}\end{aligned}$$

Now solve for y:

$$y = F^{-1}\left(G(x) + c\right) = H(x) + C \tag{6.40}$$

Now this answer is not unique due to the constant of integration term. A boundary condition will be given:

$$y(x_0) = y_0 \tag{6.41}$$

Plugging this in:

$$\begin{aligned}y_0 &= H(x_0) + C\\ \Rightarrow C &= y_0 - H(x_0)\\ \Rightarrow y &= H(x) + y_0 - H(x,0)\end{aligned}$$

6.8 Second Order Separable Differential Equation

A second order separable differential equation has the form:

$$f\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = g\left(\frac{dy}{dx}\right) \tag{6.42}$$

Let

$$\begin{aligned}v &= \frac{dy}{dx}\\ \Rightarrow f(v)\frac{dv}{dx} &= g(v)\\ \Rightarrow \frac{f(v)}{g(v)}\,dv &= dx\\ \Rightarrow \int \frac{f(v)}{g(v)}\,dv &= \int dx\\ \Rightarrow F(v) &= x + c\\ \Rightarrow v = F^{-1}(x + C_1) &= H(x) + C_1\end{aligned}$$

But this is just the first order separable differential equation:

$$\frac{dy}{dx} = H(x) + C_1 \tag{6.43}$$

$$\begin{aligned}\Rightarrow dy &= (H(x) + C_1)\,dx\\ \Rightarrow \int dy &= \int (H(x) + C_1)\,dx\end{aligned}$$

$$\Rightarrow y = P(x) + C_1x + C_2 \tag{6.44}$$

Again this answer is not unique due to the constant of integration terms. A boundary condition will be given:

$$y(x_0) = y_0 \tag{6.45}$$
$$\frac{dy}{dx}(x_1) = y_1 \tag{6.46}$$

Plugging this in (6.43):

$$y_1 = H(x_1) + C_1$$
$$\Rightarrow C_1 = y_1 - H(x_1)$$

Plugging this into (6.44):

$$y_0 = P(x_0) + (y_1 - H(x_1))x_0 + C_2$$
$$\Rightarrow C_2 = y_0 - P(x_0) - (y_1 - H(x_1))x_0$$
$$\Rightarrow y = P(x) + (y_1 - H(x_1))x + y_0 - P(x_0) - (y_1 - H(x_1))x_0$$

6.9 Second Order Separable Differential Equation requiring the Chain Rule

A second order separable differential equation requiring the chain rule has the form:

$$f(y)\frac{d^2y}{dx^2} = g(y) \tag{6.47}$$

Let

$$v = \frac{dy}{dx}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\cdot\frac{dy}{dx} = v\frac{dv}{dy}$$
$$\Rightarrow f(y)v\frac{dv}{dy} = g(y)$$

which is a first order separable differential equation and so solvable as before for:

$$v = F(y) + C_1$$

Again there will be boundary conditions:

$$y(x_0) = y_0 \tag{6.48}$$
$$\frac{dy}{dx}(y_1) = v_1 \tag{6.49}$$

the second of which will allow us to calculate C_1. Now we solve:

$$v = F(y) + C_1$$
$$\Rightarrow \frac{dy}{dx} = F(y) + C_1$$
$$\Rightarrow \int \frac{dy}{F(y) + C_1} = \int dx$$
$$\Rightarrow P(y) = Q(x) + C_2$$

which with the first boundary condition will determine y uniquely when we solve for y.